Optimization Applications

- Recall that

$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ 

$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$ 

Optimization in $[a,b]$

- The problem is to optimize a function $f(x)$ in an interval $[a,b]$.
- Let $x_0 \in [a,b]$. We say that
  a) x_0 is <u>absolute max</u> of f in $[a,b]$ iff
  $$a \leq x_0 \leq b \Rightarrow f(x) \leq f(x_0)$$
  b) x_0 is <u>absolute min</u> of f in $[a,b]$ iff
  $$a \leq x \leq b \Rightarrow f(x) \geq f(x_0)$$

- <u>Methodology</u>:

1. Solve the equation
$$f'(x)$$
and accept only the solutions that lie in the interval $[a,b]$.
2. The <u>critical points</u> of the problem are the accepted solutions AND the endpoints $x=a$ and $x=b$.

- 3 Evaluate the function f at the critical points
- 4 By the extremum value theorem the critical point where f has the largest value is the absolute ~~local~~ max and the critical point where f has the smallest value is the absolute min.
- 5 If you need to identify if the remaining critical points are local min or local max, you must build a sign chart and include the interval constraint in the sign chart.

examples

a) $f(x) = \dfrac{1-x}{3+x}$ at $x \in [0,3]$

b) $f(x) = \dfrac{x}{x^2+2}$ at $x \in [-1,4]$

Optimization in (a,b)

Methodology: Same as in optimization in $[a,b]$ above. However we DO NOT include $x=a$ or $x=b$ as critical points.

AND a table of signs may be needed to confirm if a point is min or max, IF you only have ONE critical point.

<u>examples</u>

a) $f(x) = x^3 - 3x + 1$ at $x \in (0, 2)$

b) $f(x) = xe^{2-x^2}$ at $x \in (0, +\infty)$

<u>Application Problems</u>

To solve applied min/max problems we work as follows:

- 1 Read the problem very carefully to understand it.
- 2 Read the problem again and draw diagrams where you identify and label all the <u>relevant quantities</u>.
- 3 Translate the problem to mathematical statements that show how the <u>relevant quantities</u> are related to each other.
- 4 Recast the question of the problem as an optimization problem using the mathematical statements from step 3. This requires you to define the function that must be optimized and to identify its domain.

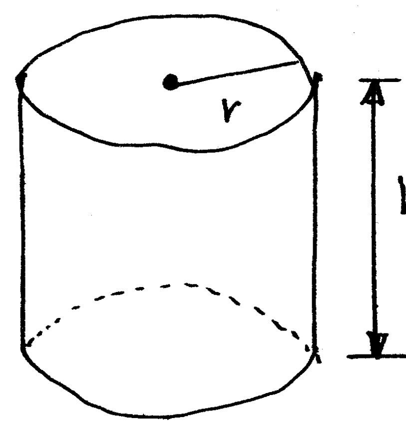
• 5 Solve the optimization problem.

examples

a) A cylindrical can with volume $V = 58 \text{ in}^3$ must be at least 1 in high and 2 in in diameter. What dimensions use the least amount of material?

Solution

• Diagram



• Volume $V = \pi r^2 h$
• Material used:

$$A = A_{top} + A_{bottom} + A_{around} =$$

$$= \pi r^2 + \pi r^2 + 2\pi r h$$

$$= 2\pi r^2 + 2\pi r h$$

• Constraints $r \geq 1$ and $h \geq 1$.
• Want to minimize A with $V = 58$ fixed.

- <u>Formulation</u>

$$V = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2}$$

thus

$$A(r) = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2} =$$

$$= 2\pi r^2 + \frac{2V}{r} = \frac{2\pi r^3 + 2V}{r}$$

Also $h \geq 1 \Leftrightarrow \frac{V}{\pi r^2} \geq 1 \Leftrightarrow \pi r^2 \leq V \Leftrightarrow$

$$\Leftrightarrow r^2 \leq \frac{V}{\pi} \Leftrightarrow r \leq \sqrt{\frac{V}{\pi}}$$

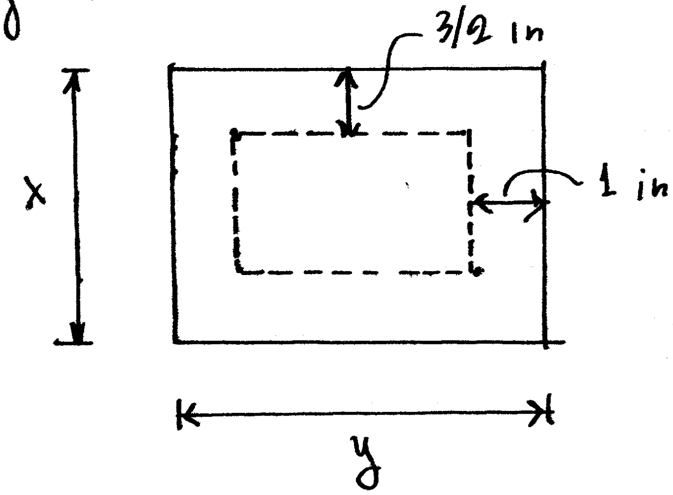
Thus: $1 \leq r \leq \sqrt{\frac{V}{\pi}}$.

Optimization problem:

Minimize: $\boxed{\begin{array}{l} A(r) = \frac{2\pi r^3 + 2V}{r} \\ \text{with } r \in \left[1, \sqrt{\frac{V}{\pi}}\right] \end{array}}$

6) A book has 36 in^2 of printed matter with 1 in side margin and 3/2 in top-bottom margin. Find the page dimensions using the minimum amount of paper.

- Diagram



- Amount of paper used: $A = xy$.
- Area of printed matter:

$$36 = (x - 2\cdot(3/2))(y - 2\cdot 1) =$$
$$= (x-3)(y-2) \qquad (1)$$

- Solve for y:

$$(1) \Leftrightarrow y - 2 = \frac{36}{x-3} \Leftrightarrow y = 2 + \frac{36}{x-3} =$$
$$= \frac{2(x-3)+36}{x-3} = \frac{2x-6+36}{x-3} = \frac{2x+30}{x-3}$$

consequently:

$$A = xy = x \cdot \frac{2x+30}{x-3} = \frac{x(2x+30)}{x-3}$$

Constraints:

$$\begin{cases} x-3 > 0 \\ y-2 > 0 \end{cases} \Leftrightarrow \begin{cases} x > 3 \\ y > 2. \end{cases}$$

For $y > 2 \Leftrightarrow \dfrac{2x+30}{\underbrace{x-3}_{+}} > 2 \Leftrightarrow 2x+30 > 2(x-3)$

$\Leftrightarrow 2x+30 > 2x-6 \Leftrightarrow 30 > -6 \leftarrow$ always true.

- Formulation:

Minimize $\boxed{A(x) = \dfrac{x(2x+30)}{x-3} \quad \text{with } x > 3.}$