Multivariate Calculus

Functions of several variables

- A function of two independent variables ~~is~~ f is a rule (mapping) that maps a pair (x,y) of two independent variables onto a dependent variable $z = f(x,y)$ which is unique for each choice of (x,y).

Partial Derivatives

- Let $z = f(x,y)$. The partial derivatives of f with respect to x and y are defined as

$$f_x(x,y) = \frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$f_y(x,y) = \frac{\partial f(x,y)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

- Method: It is possible to use the rules of differentiation to evaluate $\partial f / \partial x$ if we treat y as a constant.

Likewise, we may evaluate $\partial f/\partial y$ if we treat x as a constant.

examples

1) $f(x,y) = \ln(x^4 + xy) \leftarrow \begin{matrix} \partial f/\partial x \\ \partial f/\partial y \end{matrix}$

2) $f(x,y) = x^2 e^{x+y} \leftarrow \begin{matrix} \partial f/\partial x \\ \partial f/\partial y \end{matrix}$

Second order partial derivatives

- Let $z = f(x,y)$. The second order partial derivatives are defined in terms of the previously defined partial derivatives as follows:

$$f_{xx}(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial x}\right)$$

$$f_{yy}(x,y) = \frac{\partial^2 f(x,y)}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial y}\right)$$

$$f_{xy}(x,y) = \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right)$$

$$f_{yx}(x,y) = \frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right).$$

- Under the condition that f_{xy} and f_{yx} are continuous, we have

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x}$$

<u>example</u>: For $f(x,y) = 9xy^2 - x^3 - y^3$
find
f_{xx}, f_{yy}, f_{xy}

▼ Extrema of Multivariate Functions

- Let $f(x,y)$ be a function and consider a point (x_0,y_0) in its domain. We say that

a) f has local maximum at (x_0,y_0) iff there is a circular region B centered around (x_0,y_0) such that
$$(x_0,y_0) \in B \Rightarrow f(x,y) \leq f(x_0,y_0)$$

b) f has a local minimum at (x_0,y_0) iff there is a circlular region B centered around (x_0,y_0) such that
$$(x_0,y) \in B \Rightarrow f(x,y) \geq f(x_0,y_0).$$

- If $z = f(x,y)$ has a local min or max at (a,b) then, provided the derivatives exist,
$$\begin{cases} \dfrac{\partial f(a,b)}{\partial x} = 0 \\[2ex] \dfrac{\partial f(a,b)}{\partial y} = 0 \end{cases} \qquad (1)$$
All points that satisfy (1) are called <u>critical points.</u>

- A saddle point is a critical point which is not a local max or a local min.

- Let $z = f(x,y)$ be a function. Its <u>Jacobian</u> is defined as

$$M = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left[\frac{\partial^2 f}{\partial x \partial y}\right]^2$$

If (a,b) is a critical point of f then

a) $M(a,b) > 0$ and $f_{xx}(a,b) < 0 \Rightarrow$
$\Rightarrow (a,b)$ local max.

b) $M(a,b) > 0$ and $f_{xx}(a,b) > 0 \Rightarrow$
$\Rightarrow (a,b)$ local min

c) $M(a,b) < 0 \Rightarrow (a,b)$ saddle point.

- <u>Remark</u>: For the case $M(a,b) = 0$ we do not know if (a,b) is local min or max or saddle point. In this case, an alternative method is to reduce the problem to a 1-variable problem.

<u>examples</u>

a) $f(x,y) = x^2 + y^2 - 2x - 6y + 14$ (local min at $(1,3)$)

b) $f(x,y) = x^3 + y^3 - 3xy$ (local min at $(1,1)$ saddle point at $(0,0)$)

c) $f(x,y) = e^{x^2+y^2}$. (local min at $(0,0)$)