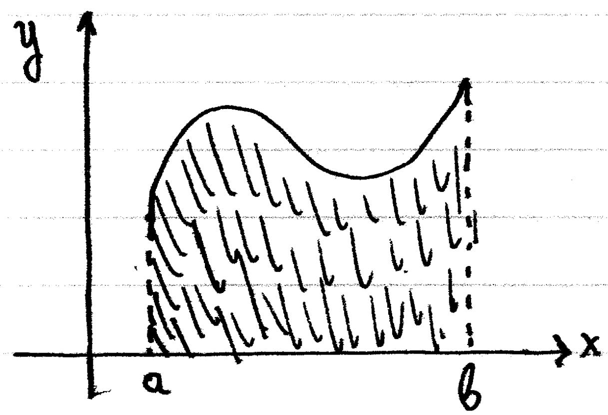
Integral Calculus

Definition of the Riemann integral



The problem is to calculate the area A between the x-axis, the lines $(\ell_1): x = a$ and $(\ell_2): x = b$ and the curve $(c): y = f(x)$.

The solution of the problem, according to Riemann is as follows:

- 1 Divide the interval $[a,b]$ to n equal intervals $[x_{k-1}, x_k]$ with
$$x_k = a + (b-a)(k/n), \ \forall k \in [n]$$
with $[n] = \{0, 1, 2, \dots, n\}$.
- 2 Let m_k and M_k be the min and max value of f in the interval $[x_{k-1}, x_k]$:

$$m_k = \min_{x \in [x_{k-1}, x_k]} f(x)$$

$$M_k = \max_{x \in [x_{k-1}, x_k]} f(x)$$

• 3 We form the Riemann sums

$$L_n = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$$

$$U_n = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$$

Obviously the area A will satisfy

$\forall n \in \mathbb{N} : L_n \leq A \leq U_n \quad (1)$

• 4 We prove that $\lim L_n = \lim U_n = l$ which combined with (1) implies that

$$\boxed{\lim L_n = \lim U_n = A}$$

If the limits $\lim L_n$ and $\lim U_n$ converge and coincide, we say that

f integrable at $[a, b]$

and write

$$\boxed{\lim L_n = \lim U_n = \int_a^b f(x)\,dx}$$

This definition assumes that $a<b$. For convenience we generalize by defining:

$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

→ From the definition it follows that the integral can be calculated as the limit of the following sequence:

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty}\left[\frac{b-a}{n}\sum_{k=1}^{n} f\left(a+k\,\frac{b-a}{n}\right)\right]$$

▶ Basic Sums

$$S_1(n) = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$S_2(n) = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$S_3(n) = 1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4} = [S_1(n)]^2.$$

example : $\int_0^a x^2\,dx = \frac{a^3}{3}$.

Properties of the integral

① f continuous at $[a,b] \Rightarrow f$ integrable at $[a,b]$

② Let f, g integrable at $[a,b]$

Then

a) $$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

b) $$\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx \quad , \forall \lambda \in \mathbb{R}$$

c) $$\gamma \in [a,b] \Rightarrow \int_a^b f(x)\,dx = \int_a^\gamma f(x)\,dx + \int_\gamma^b f(x)\,dx$$

③ Let f integrable at $[a,b]$.

a) $$\big(\forall x \in [a,b] : f(x) \geqslant 0\big) \Rightarrow \int_a^b f(x)\,dx \geqslant 0$$

b) $$\big(\forall x \in [a,b] : f(x) \leqslant g(x)\big) \Rightarrow \int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx$$

c) $$\left| \int_a^b f(x)\,dx \right| \leqslant \int_a^b |f(x)|\,dx$$

Fundamental theorem of calculus.

- Let f be a function.
  If $F'(x) = f(x)$ then we say that F is the <u>antiderivative</u> of f.

- If F, G are both antiderivatives of f then there is a $c \in \mathbb{R}$ such that
  $$F(x) = G(x) + c$$

- <u>Fundamental theorem of calculus:</u>

  If F is the antiderivative of f then
  $$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

  Thus, to evaluate a definite integral of f it is sufficient to find the antiderivative of f.

- This motivates the definition of the <u>indefinite integral</u>.
  $$\int f(x)\,dx = F(x) + c \qquad \text{with } F'(x) = f(x).$$

Integration formulas

1) $$\int x^a\,dx = \begin{cases} \dfrac{x^{a+1}}{a+1} + c & , \text{ if } a \neq -1 \\ \ln|x| + c & , \text{ if } a = -1 \end{cases}$$

- Special cases

a) $\int dx = x + c \qquad (a = 0)$

b) $\int \dfrac{dx}{\sqrt{x}} = 2\sqrt{x} + c \qquad (a = -1/2)$

examples

1) $I = \displaystyle\int_1^2 (2x+1)(x-1)\,dx$

2) $I = \displaystyle\int \frac{x^2+1}{\sqrt{x}}\,dx$

3) $I = \displaystyle\int \frac{x\sqrt{x}}{\sqrt[3]{x}}\,dx$

4) $I = \displaystyle\int_1^3 \frac{(x+1)^2}{x}\,dx$

2) $\int e^{ax}\,dx = \dfrac{e^{ax}}{a} + c$, for $a \neq 0$.

- For $a = 0 \Rightarrow e^{ax} = e^{0} = 1 \Rightarrow \int e^{ax}\,dx = \int dx =$

$= x + c$.

<u>examples</u>

$$I = \int_0^2 e^{3x}\,dx$$

$$I = \int_0^2 e^{-x}(1 + e^{x})\,dx$$

$$I = \int \frac{(e^{x}+1)^2}{e^{x}}\,dx$$

$$I = \int \frac{xe^{-x}+1}{x}\,dx$$

Method of substitution

The method of substitution is based on the identity:

$$\boxed{\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(y)\, dy}$$

which is derived from the chain rule and the fundamental theorem of calculus.

Proof

Let F be the antiderivative of f. Then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_a^b F'(g(x))\, g'(x)\, dx =$$

$$= \int_a^b [F(g(x))]'\, dx =$$

$$= F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} F(y)\, dy.$$

Methodology: Definite Integrals $\int_a^b f(x)\,dx$

- 1 Identify the required substitution
$$y = g(x)$$
on a case by case basis
- 2 Calculate the differential
$$dy = g'(x)\,dx$$
- 3 Calculate the new limits of integration $g(a)$ and $g(b)$.

- 4 Change the integral in terms of y and change the limits of integration.

example: $I = \int_1^2 \sqrt{2x+3}\,dx$

Let $y = 2x+3 = g(x) \Rightarrow \begin{cases} dy = 2dx \Rightarrow dx = (1/2)\,dy \\ g(1) = 2\cdot 1 + 3 = 5 \\ g(2) = 2\cdot 2 + 3 = 7 \end{cases}$

$$\Rightarrow I = \int_5^7 \sqrt{y}\,(1/2)\,dy = \int_5^7 y^{1/2}\cdot(1/2)\,dy =$$

$$= \left[\frac{y^{3/2}}{3/2}(1/2)\right]_5^7 = \left[\frac{y\sqrt{y}}{3}\right]_5^7 =$$

$$= \frac{7\sqrt{7} - 5\sqrt{5}}{3}$$

<u>Methodology</u>: Indefinite Integrals $I = \int f(x)\,dx$

- 1 Identify the required substitution
  $$y = g(x)$$
- 2 Calculate the differential
  $$dy = g'(x)\,dx$$
- 3 Rewrite the integral in terms of y and then perform the integral.
- 4 For <u>indefinite integrals</u>: you obtain an answer in terms of an auxiliary variable. You must rewrite the final answer in terms of x. ← Backsubstitution

<u>example</u>: $I = \int 3x e^{x^2}\,dx$

Let $y = x^2 \Rightarrow dy = 2x\,dx \Rightarrow 3x\,dx = (3/2)\,dy \Rightarrow$

$$\Rightarrow I = \int e^y (3/2)\,dy = (3/2)e^y + c =$$

↑ Backsubstitution

$$= (3/2)e^{x^2} + c$$

Substitution Forms

1) Form $I = \int [f(x)]^a f'(x)\,dx$

- let $y = f(x)$

examples : a) $I = \int_0^1 (x^2+1)^5\, x\,dx$

b) $I = \int x^3 \sqrt{x^4+1}\,dx$

- Linear substitutions $y = ax+b$ always work provided they simplify the integral

c) $I = \int_0^2 2x\,(3x+1)^7\,dx$

2) Form $I = \int e^{f(x)} f'(x)\,dx$

- Let $y = f(x)$

examples : a) $I = \int_0^2 x^2 e^{-x^3}\,dx$

b) $I = \int \dfrac{3x}{e^{x^2}}\,dx$

3) Form $I = \int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$

▶ Or, let $y = f(x)$.

e.g: a) $I = \int_1^3 \frac{2x+5}{x^2+5x+12}\, dx$

b) $I = \int_1^4 \frac{dx}{x \ln x}$

c) $I = \int_1^2 \frac{e^x}{e^x+1}\, dx$

4) Form $I = \int \frac{f(\ln x)}{x}\, dx$

▶ Let $y = \ln x$

e.g. : a) $I = \int \frac{(\ln x)^2 + 1}{x \ln x}\, dx$

b) $I = \int_1^3 \frac{x + \ln(3x^2)}{x}\, dx$

5) Form $I = \int f(e^x)\, e^x \, dx$

- Let $y = e^x$

<u>example</u> : $I = \int_0^2 \frac{(3e^{2x}+1)\, e^x}{e^{3x}+e^x}\, dx$

↳ Backsubstitution

We apply this method to integrals of the form

$$I = \int f(x, \sqrt[n]{ax+b})\,dx \qquad \text{OR}$$

$$I = \int f\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)dx$$

The idea is to employ the substitution theorem in reverse.

Methodology

- 1 Let $y = \sqrt[n]{ax+b}$ $\left(\text{or } y = \sqrt[n]{\frac{ax+b}{cx+d}}\right)$
- 2 Solve for x: $x = g(y)$.
- 3 Calculate $dx = g'(y)\,dy$.
- 4 If the integral is definite, compute the new limits of integration.
- 5 Rewrite the integral in terms of y and proceed to evaluate it.

• 6 If the integral is <u>indefinite</u> rewrite the final answer in terms of x.

<u>examples</u>

a) $I = \int 2x\sqrt{3x-2}\,dx$

b) $I = \int_0^2 x^2\sqrt{x+2}\,dx$

c) $I = \int \dfrac{x-1}{\sqrt{2x+1}}\,dx$