FUNCTIONS

Definition of functions

- Let $A \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$. An intuitive, but not rigorous definition of the concept of a real-valued function $f: A \longrightarrow \mathbb{R}$ is that it is a "rule" (or algorithm) that "maps" every element of A into some element of $\mathbb{R}$, which is unique.
- Thus, for any $x \in A$, the function f maps x to the unique element $f(x) \in \mathbb{R}$.
- $A =$ the domain of the function f.

↳ To properly define a function f, we must give:
a) The domain A
b) The formula $y = f(x)$.
When the domain is not given, we assume by default the widest possible subset of $\mathbb{R}$ for which the formula $y = f(x)$ yields a real number as the default domain.

▸ **Method:** To find the widest possible domain, we must introduce and solve constraints to prevent:
a) Division by zero
b) Taking the square root of a negative number.

EXAMPLES

a) For $f(x) = x^2 + 3x + 1$, evaluate $f(1+\sqrt{2})$ and $f(2a-1)$.

Solution

$$f(1+\sqrt{2}) = (1+\sqrt{2})^2 + 3(1+\sqrt{2}) + 1 =$$
$$= (1 + 2\sqrt{2} + 2) + 3(1+\sqrt{2}) + 1 =$$
$$= 1 + 2\sqrt{2} + 2 + 3 + 3\sqrt{2} + 1 = 7 + 5\sqrt{2}$$
$$f(2a-1) = (2a-1)^2 + 3(2a-1) + 1 =$$
$$= 4a^2 - 4a + 1 + 6a - 3 + 1 =$$
$$= 4a^2 + 2a - 1.$$

b) Find the default domain for $f(x) = x^3(x^2+1)^4$.

Solution

No constraints, thus $A = \mathbb{R}$.

↳ For polynomial functions of the form
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
there are no constraints, therefore the default domain is always $A = \mathbb{R}$.

c) Find the default domain for $f(x) = \dfrac{x^2-9}{x^2+4x+3}$

<u>Solution</u>

We require $x^2+4x+3 \neq 0$

Solve: $x^2+4x+3 = 0 \Leftrightarrow (x+1)(x+3) = 0 \Leftrightarrow$

$\Leftrightarrow x+1=0 \vee x+3=0 \Leftrightarrow x=-1 \vee x=-3.$

It follows that $A = \mathbb{R} - \{-1,-3\}.$

$= (-\infty,-3) \cup (-3,-1) \cup (-1,+\infty).$

Note that this function can be simplified to:

$$f(x) = \frac{x^2-9}{x^2+4x+3} = \frac{(x-3)(x+3)}{(x+3)(x+1)} = \frac{x-3}{x+1}$$

However, the default domain of the simplified formula is wider: $A = \mathbb{R} - \{-1\}$. Thus, to find the correct default domain, you must NOT try to simplify or otherwise modify the formula for $f(x)$ before writing down the constraints.

d) Find the default domain of

$$f(x) = \sqrt{\frac{2x+1}{1+3x}} \quad \text{and} \quad g(x) = \frac{\sqrt{2x+1}}{\sqrt{1+3x}}$$

<u>Solution</u>

- For $f(x)$

Require $\dfrac{2x+1}{1+3x} \geq 0$. (1)

x		$-1/2$		$-1/3$	
$2x+1$	$-$	0	$+$		$+$
$1+3x$	$-$		$-$	0	$+$
ineq	$+$	0	$-$	$\parallel$	$+$

Thus (1) $\Leftrightarrow x \in (-\infty, -1/2] \cup (-1/3, +\infty)$
and therefore $A = (-\infty, -1/2] \cup (-1/3, +\infty)$.

- For $g(x)$

Require: $\begin{cases} 2x+1 \geq 0 \\ 1+3x > 0 \end{cases} \Leftrightarrow \begin{cases} 2x \geq -1 \\ 3x > -1 \end{cases} \Leftrightarrow \begin{cases} x \geq -1/2 \\ x > -1/3 \end{cases}$

$\Leftrightarrow x > -1/3$

therefore $A = (-1/3, +\infty)$.

EXERCISES

① Given the function $f(x) = x^2+3x+7$, evaluate and simplify: a) $f(\sqrt{2})$, b) $f(1-2\sqrt{3})$, c) $f(\sqrt{2}+\sqrt{3})$,
d) $f(2a-3)$

② Given the function $f(x) = \dfrac{2x+1}{3x-2}$, evaluate and simplify: a) $f(\sqrt{3})$, b) $f(2+3\sqrt{5})$, c) $f(3a+1)$.

③ Find the default domain for the following functions.

a) $f(x) = x^2(x+1)^3$

b) $f(x) = \dfrac{-3}{x-1}$

c) $f(x) = \dfrac{x-2}{x^2-4}$

d) $f(x) = \dfrac{3x^2}{x^2-4x}$

e) $f(x) = \dfrac{2x^2-3}{x^2+5x+6}$

$$\left.\begin{array}{l} \text{f) } f(x) = \dfrac{x^2-4}{x^2-5x+6} \\[2ex] \text{g) } f(x) = \dfrac{x+2}{x-3} \end{array}\right\} \text{compare}$$

h) $f(x) = \sqrt{3-x}$

i) $f(x) = \sqrt{x^2-x-6}$

j) $f(x) = \sqrt{x^3+5x^2-6x}$

k) $f(x) = \sqrt{x^4-x^3+x-1}$

$$\left.\begin{array}{l} \text{l) } f(x) = \sqrt{\dfrac{x+2}{x-3}} \\[2ex] \text{m) } f(x) = \dfrac{\sqrt{x+2}}{\sqrt{x-3}} \end{array}\right\} \text{compare}$$

n) $f(x) = \sqrt{\dfrac{x^2-4}{x^2-9}}$

(4) Find the default domain for the following functions:

a) $f(x) = \dfrac{-3|x-1|}{|x+4|}$

b) $f(x) = \dfrac{x-1}{|x-2|-2}$

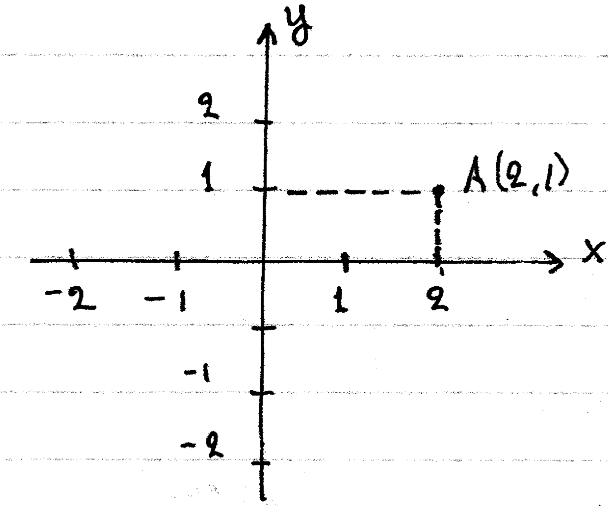
c) $f(x) = \sqrt{|3x-1|-2}$

d) $f(x) = \dfrac{\sqrt{x}}{|x|}$

e) $f(x) = \sqrt{\dfrac{|x|-2}{|x|+1}}$

f) $f(x) = \dfrac{x-1}{\sqrt{2-|x+2|}}$

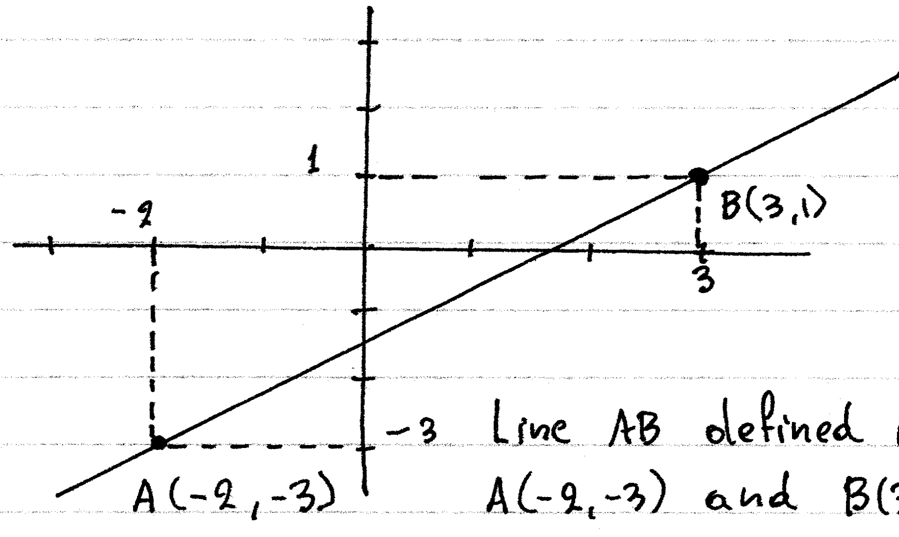
Coordinate systems



- To draw a coordinate system:
  - 1 Draw the x-axis and the y-axis perpendicular to each other.
  - 2 Indicate with an arrow the positive direction on the x-axis and y-axis.
  - 3 And and label tickmarks to the x-axis and y-axis.

- To add a point to the coordinate system (e.g. A(2,1) in the figure above)
  - 1 Draw the point on the plane as a dot.
  - 2 Label the point with its assigned letter and coordinates (e.g. A(2,1)).
  - 3 Project the point to both the x-axis and y-axis using dotted lines.

- To draw a line (AB) defined by points A and B.
  - 1 Draw the coordinate system.
  - 2 Add the points A and B.
  - 3 Connect the points with a line.



- Let $A(x_A, y_A)$ and $B(x_B, y_B)$ be two points. We define the slope $\lambda(AB)$ between A and B as:

$$\boxed{\lambda(AB) = \frac{y_B - y_A}{x_B - x_A}}$$

- Remarks

a) It is not necessary to label all tickmark on the x-axis and y-axis. However, the projections of points on the plane to the axises, MUST be labeled.
b) Points that happen to be on the x-axis or y-axis do not require the dotted projections to the axises.
c) You MUST use a ruler to draw these figures properly.
d) Be sure your axis are as long as necessary and be careful in how you place your labels to avoid solid and dotted lines.

EXERCISES

(5) Draw the line (AB) defined by the points A and B below and calculate the slope λ_{AB} between A and B.

a) $A(-1,1)$ and $B(0,3)$

b) $A(-1,2)$ and $B(1,2)$

c) $A(-3,-2)$ and $B(1,2)$

d) $A(1,-2)$ and $B(-3,1)$

e) $A(0,0)$ and $B(3,1)$

f) $A(3,0)$ and $B(0,-1)$

g) $A(3,4)$ and $B(1,1)$

h) $A(3,-1)$ and $B(1,2)$

i) $A(2,1)$ and $B(5,1)$

j) $A(-1,-2)$ and $B(2,-2)$.

Graphing functions

- Let f be a function. The graph $(c): y = f(x)$ consists of all points $M(x, y)$ whose coordinates satisfy the equation $y = f(x)$.
- In other words:

$$\boxed{M(x,y) \in (c) \Leftrightarrow y = f(x)}$$

Linear function $f(x) = ax + b$

- The graph $(c): y = ax + b$ is a straight line, so to plot it on a coordinate system all we need is to find 2 points A and B. It is easier if you choose integer values for x for which the resulting y is also an integer, whenever possible.

Remark : For any two points $A(x_1, f(x_1))$ and $B(x_2, f(x_2))$, the slope $\lambda(AB)$ is constant and given by

$$\boxed{A, B \in (c): y = ax + b \Rightarrow \lambda(AB) = a}$$

Proof

$$\lambda(AB) = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{(ax_2+b)-(ax_1+b)}{x_2-x_1} =$$

$$= \frac{a(x_2-x_1)}{x_2-x_1} = a \qquad \square$$

<u>Remark</u> : For $a>0$, $f(x)$ increases with increasing x
For $a<0$, $f(x)$ decreases with decreasing x.

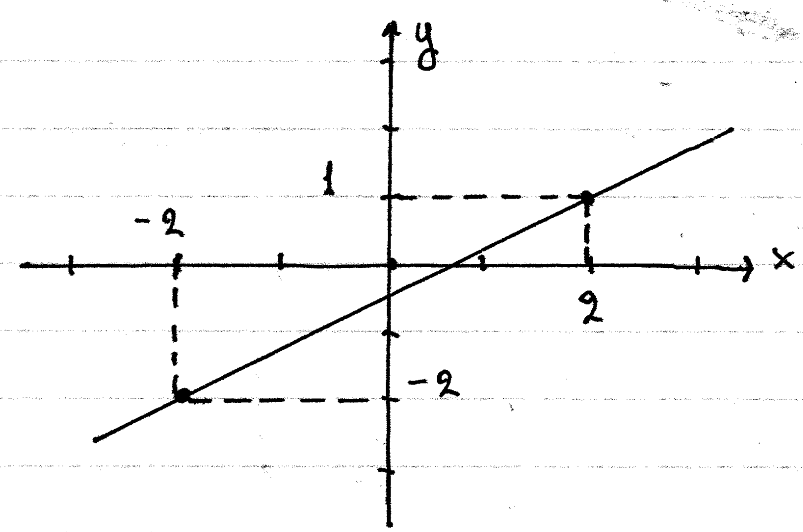
EXAMPLE

Graph the function $f(x) = \dfrac{3x-2}{4}$

<u>Solution</u>

$f(2) = (6-2)/4 = 1$
$f(-2) = (-6-2)/4 =$
$= -8/4 = -2$

x	$f(x)$
2	1
-2	-2



Lines, in general

- Every line can be represented as

$$(\ell): Ax + By + C = 0$$

with $A, B, C \in \mathbb{R}$ and $|A| + |B| > 0$. Then, the slope λ of the line (ℓ) is given by:

$$\boxed{\lambda = \frac{-A}{B}}$$

- The equation of a line can be found as follows:

1) Given a point and slope

$$\boxed{\left.\begin{array}{l} M(x_0, y_0) \in (\ell) \\ \lambda \text{ slope of } (\ell) \end{array}\right\} \Rightarrow (\ell): y - y_0 = \lambda(x - x_0)}$$

2) Given two points

$$\boxed{\left.\begin{array}{l} A(x_1, y_1) \in (\ell) \\ B(x_2, y_2) \in (\ell) \end{array}\right\} \Rightarrow (\ell): \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}}$$

3) Vertical lines

$$\boxed{\left.\begin{array}{l} A(x_0, y_0) \in (\ell) \\ (\ell) \text{ vertical} \end{array}\right\} \Rightarrow (\ell): x = x_0}$$

EXAMPLES

a) Find the equation of the line defined by $A(3,1)$ and $B(-1,-2)$.

Solution

$$\left.\begin{array}{l} A(3,1)\in(\ell) \\ B(-1,-2)\in(\ell) \end{array}\right\} \Rightarrow (AB): \frac{y-1}{x-3} = \frac{(-2)-1}{(-1)-3} \Leftrightarrow$$

$$\Leftrightarrow \frac{y-1}{x-3} = \frac{-3}{-4} \Leftrightarrow \frac{y-1}{x-3} = \frac{3}{4} \Leftrightarrow$$

$$\Leftrightarrow 4(y-1) = 3(x-3) \Leftrightarrow 4y-4 = 3x-9 \Leftrightarrow$$

$$\Leftrightarrow 3x-4y-9+4=0 \Leftrightarrow 3x-4y-5=0.$$

thus $(AB): 3x-4y-5=0$.

b) Find the equation of the line with slope $\lambda=3$ passing through $A(3,4)$.

Solution

$$\left.\begin{array}{l} A(3,4)\in(\ell) \\ \lambda=3 \end{array}\right\} \Rightarrow (\ell): y-4=3(x-3) \Leftrightarrow y-4=3x-9 \Leftrightarrow$$

$$\Leftrightarrow 3x-y-9+4=0 \Leftrightarrow 3x-y-5=0$$

thus: $(\ell): 3x-y-5=0$.

c) Find the equation of a vertical line passing through the point $A(2,1)$

Solution

$$\left.\begin{array}{l} A(2,1)\in(\ell) \\ (\ell) \text{ vertical} \end{array}\right\} \Rightarrow (\ell): x=2.$$

▶ <u>Relative position between two lines</u>

Consider two lines (ℓ_1) and (ℓ_2) given by:

$$(\ell_1): y = a_1 x + b_1$$
$$(\ell_2): y = a_2 x + b_2.$$

It can be shown that:

$$\boxed{\begin{array}{l} (\ell_1) \parallel (\ell_2) \Leftrightarrow a_1 = a_2 \\ (\ell_1) \perp (\ell_2) \Leftrightarrow a_1 a_2 = -1 \end{array}}$$

EXAMPLE

Consider the lines (ℓ_1) and (ℓ_2) given by

$$(\ell_1): (a+3)x - (a+1)y = 2$$
$$(\ell_2): (a-2)x - (a+2)y = 6$$

a) Find all $a \in \mathbb{R}$ such that $(\ell_1) \parallel (\ell_2)$

b) Find all $a \in \mathbb{R}$ such that $(\ell_1) \perp (\ell_2)$

<u>Solution</u>

Assume that $a+1 \neq 0 \land a+2 \neq 0 \Leftrightarrow a \neq -1 \land a \neq -2$

Slope of (ℓ_1): $\lambda_1 = \dfrac{a+3}{a+1}$

and

slope of (ℓ_2): $\lambda_2 = \dfrac{a-2}{a+2}$

a) $(\ell_1) \parallel (\ell_2) \Leftrightarrow \lambda_1 = \lambda_2 \Leftrightarrow \dfrac{a+3}{a+1} = \dfrac{a-2}{a+2} \Leftrightarrow$

$\Leftrightarrow (a+3)(a+2) = (a+1)(a-2) \Leftrightarrow a^2+5a+6 = a^2-a-2 \Leftrightarrow$

$\Leftrightarrow 5a+6 = -a-2 \Leftrightarrow 5a+a = -6-2 \Leftrightarrow 6a = -8 \Leftrightarrow$

$\Leftrightarrow a = -8/6 = -4/3.$

thus $(\ell_1) /\!/ (\ell_2) \Leftrightarrow a = -4/3.$

b) $(\ell_1) \perp (\ell_2) \Leftrightarrow \lambda_1 \lambda_2 = -1 \Leftrightarrow \dfrac{a+3}{a+1} \dfrac{a-2}{a+2} = -1 \Leftrightarrow$

$\Leftrightarrow \dfrac{a^2+a-6}{a^2+3a+2} = -1 \Leftrightarrow a^2+a-6 = -a^2-3a-2 \Leftrightarrow$

$\Leftrightarrow 2a^2+4a-4=0 \Leftrightarrow a^2+2a-2=0$

$\Delta = 4+8 = 12 = 4 \cdot 3 \Rightarrow a_{1,2} = \dfrac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$

thus: $(\ell_1) \perp (\ell_2) \Leftrightarrow a = -1+\sqrt{3} \vee a = -1-\sqrt{3}.$

EXERCISES

(6) Graph the following functions.

a) $f(x) = (1/2)x$

b) $f(x) = -x$

c) $f(x) = -x+1$

d) $f(x) = 2x-1$

e) $f(x) = -3x+2$

f) $f(x) = (2x+1)/5$

g) $f(x) = (1/3)x-2$

h) $f(x) = (3x-12)/7$

i) $f(x) = 3$

j) $f(x) = (3x-10)/5$

(7) Find the equation of the line $(\ell): Ax+By+C=0$ such that

a) (ℓ) goes through $A(1,3)$, $B(2,5)$

b) (ℓ) goes through $A(2,3)$, $B(2,4)$

c) (ℓ) goes through $A(1,4)$, $B(4,1)$

d) (ℓ) goes through $A(-2,3)$, $B(-3,-2)$

e) (ℓ) goes through $A(2,3)$ with slope -1

f) (ℓ) goes through $A(-1,2)$ with slope 4

g) $(\ell) \parallel (\ell_1): 3x+2y+4=0$ and goes through $A(-1,5)$

h) $(\ell) \parallel (\ell_2): x-3y+1=0$ and goes through $A(3,-2)$

i) $(\ell) \perp (\ell_1): 2x+5y+4=0$ and goes through $A(-2,3)$

j) $(\ell) \perp (\ell_1): 3x-y+2=0$ and goes through $A(1,-1)$

(8) Find all $a \in \mathbb{R}$ such that $(\ell_1): ax+5y+7=0$ is parallel to $(\ell_2): 2(a-1)x-3y+9=0$.

(9) Find all $a \in \mathbb{R}$ such that $(\ell_1): (a+3)x + y - 7 = 0$ is perpendicular to $(\ell_2): (1-a)x - 4y + 12 = 0$

(10) Find all $a \in \mathbb{R}$ such that $(\ell_1): (a-1)x - y + (a-2) = 0$ is parallel to $(\ell_2): (3a-7)x - y - 2a + 5 = 0$. Then find a line (ℓ) that is perpendicular to (ℓ_1) and (ℓ_2) going through the point $A(-1,1)$.

(11) Find all $a \in \mathbb{R}$ such that the line

$(\ell): (a-2)x - (a-1)y + (3a-2)(a-1) = 0$

is

a) Parallel to the x-axis.

b) Parallel with $(\ell_1): 4x - y + 3 = 0$

c) ~~Pepe~~ Perpendicular to $(\ell_1): 2x + y - 5 = 0$.

Quadratic function $f(x) = ax^2 + bx + c$

- The graph of the quadratic function $f(x) = ax^2 + bx + c$ is a curve called <u>parabola</u> which has the following properties:

a) Its <u>vertex</u> is the point $A(-b/2a, -\Delta/4a)$ with $\Delta = b^2 - 4ac$ the discriminant of the quadratic $ax^2 + bx + c$.

i) For $a > 0 \Rightarrow$ the vertex A is a minimum

ii) For $a < 0 \Rightarrow$ the vertex A is the maximum.

b) It has <u>axis of symmetry</u> the line $(\ell): x = -b/2a$

c) Intersects the y-axis at $C(0, c)$

d) Intersects the x-axis at

i) $A_1(x_1, 0)$ and $A_2(x_2, 0)$ with x_1, x_2 the zeroes of the quadratic, when $\Delta > 0$.

ii) Tangent with the x-axis at the vertex, when $\Delta = 0$.

iii) No intersection when $\Delta < 0$.

e) For $a > 0$: the parabola opens up

$a < 0$: the parabola opens down.

- To justify the above claims, we rewrite the quadratic in the completed square form:

$$\boxed{f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a}}$$

Proof

$$f(x) = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) =$$

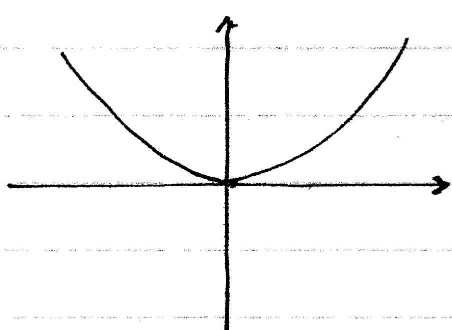
$$= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right] =$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right] =$$

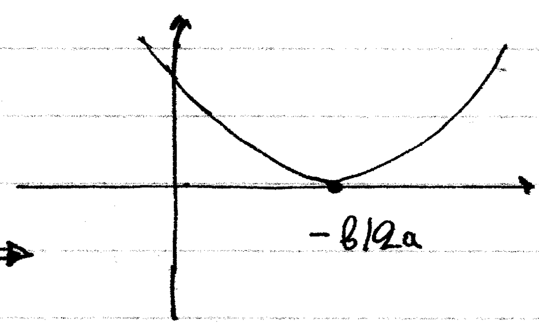
$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] =$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right] =$$

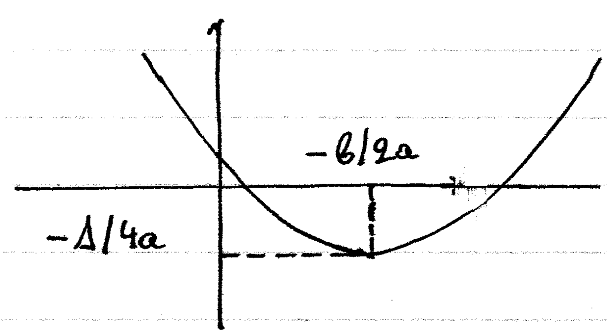
$$= a(x + b/2a)^2 - \Delta/4a. \qquad \square$$



$f(x) = ax^2$



$f(x) = a(x + b/2a)^2$



$f(x) = ax^2 + bx + c = a(x + b/2a)^2 - \Delta/4a$

- To plot the graph of a quadratic function, we first determine the coordinates of the vertex, and the points where the graph intersects the x-axis (if they exist) and the y-axis. If these are not sufficient, we find additional points by evaluating the function.

EXAMPLE

Graph the function $f(x) = x^2 + 5x + 6$.

Solution

Discriminant $\Delta = b^2 - 4ac = 5^2 - 4 \cdot 1 \cdot 6 = 25 - 24 = 1$

$$\left.\begin{array}{l} \dfrac{-b}{2a} = \dfrac{-5}{2 \cdot 1} = \dfrac{-5}{2} \\[2ex] \dfrac{-\Delta}{4a} = \dfrac{-1}{2 \cdot 1} = \dfrac{-1}{2} \end{array}\right\} \Rightarrow \text{Vertex } A(-5/2, -1/2)$$

x-axis intercepts:

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-5 \pm \sqrt{1}}{2 \cdot 1} = \begin{cases} -6/2 = -3 \\ -4/2 = -2 \end{cases}$$

thus $(-3, 0)$, $(-2, 0)$.

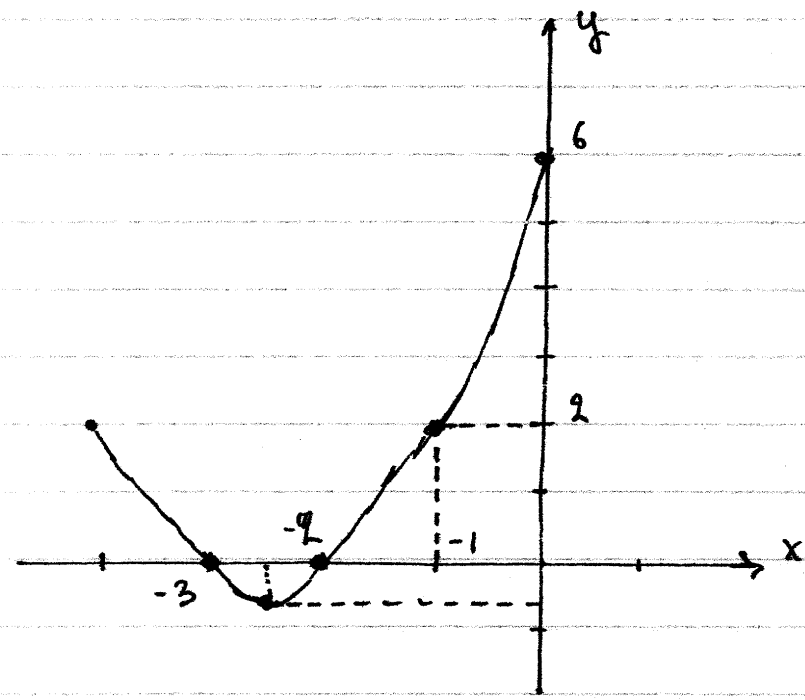
y-axis intercept: $(0, 6)$

Also need at $x = -1$: $f(-1) = (-1)^2 + 5 \cdot (-1) + 6 =$

$= 1 - 5 + 6 = 2$

Thus, to summarize:

x	-3	$-5/2$	-2	-1	0
y	0	$-1/2$	0	2	6


y
6
2
x
-3
-2
-1

EXERCISES

12 Graph the following functions.

a) $f(x) = 2x^2$

b) $f(x) = -3x^2$

c) $f(x) = x^2/2$

d) $f(x) = -x^2/3$

e) $f(x) = 3x^2+1$

f) $f(x) = -2x^2+3$

g) $f(x) = x^2-2$

h) $f(x) = x^2-2x$

i) $f(x) = -2x^2+3x$

j) $f(x) = 2x^2+4x+2$

k) $f(x) = x^2-4x+3$

l) $f(x) = -x^2+x+2$

m) $f(x) = -2x^2-6x-4$

n) $f(x) = x^2+x+1$

o) $f(x) = x^2+3x+4$

p) $f(x) = -x^2+2x-3$