EQUATIONS AND INEQUALITIES

▼ Terminology

- An <u>equation</u> is an expression of the form $f(x)=g(x)$ which may or may not be true for some values of x.
- A <u>solution set</u> S of an equation $f(x)=g(x)$ is the set of all real numbers $x \in \mathbb{R}$ for which the equation is true.

▸ <u>example</u>: For the equation $x^2=4$, the solution set is $S=\{-2,2\}$.

- An <u>identity</u> is an equation $f(x)=g(x)$ with solution set $S=\mathbb{R}$. (i.e. the equation is <u>always true</u> for all real numbers $x\in\mathbb{R}$).
- An equation $f(x)=g(x)$ is <u>inconsistent</u> if it's solution set is $S=\emptyset$. (i.e. the equation is <u>always false</u> for all real numbers $x\in\mathbb{R}$. OR equivalently, the equation is <u>never true</u> for any real number $x\in\mathbb{R}$).

▸ <u>examples</u>

a) The equation $(x+1)^2=x^2+2x+1$ is an identity.

b) The equation $x^2+4=0$ is inconsistent.
(Because $x^2+4 \geq 0+4=4>0$ for all $x\in\mathbb{R}$).

Basic Logic Notation

- A predicate $p(x)$ is a statement about x which is either true or false depending on the value of x.

▸ example : Any equation is also a predicate. For example $p(x): 2x+1=3x$ is a predicate.

Logical or / and / implications

$p(x) \wedge q(x)$: $p(x)$ and $q(x)$ are both true.

$p(x) \vee q(x)$: At least one of $p(x)$ or $q(x)$ is true ($p(x)$ OR $q(x)$).

$p(x) \Rightarrow q(x)$: If $p(x)$ is true, then $q(x)$ is true.

$p(x) \Leftrightarrow q(x)$: $p(x)$ is true if and only if $q(x)$ is true. (i.e. if $p(x)$ is true then $q(x)$ is true AND if $q(x)$ is true then $p(x)$ is true).

▸ examples

a) $x>2 \Rightarrow x>1$ TRUE

b) $x>1 \Rightarrow x>2$ FALSE

c) $x>2 \Leftrightarrow x>1$ FALSE

d) $2x=2 \Leftrightarrow x=1$ TRUE.

Basic properties of equations.

1) Let $x, y, a \in \mathbb{R}$. Then
$x = y \Leftrightarrow x + a = y + a$
(i.e.: we can add a number to both sides of an equation).

2) $x + a = y \Leftrightarrow x = y - a$
(i.e.: we can move a term on the other side of the equation but we must change its sign.)

3) Let $x, y, a \in \mathbb{R}$ and assume that $a \neq 0$. Then
$x = y \Leftrightarrow ax = ay$.
(i.e.: we can multiply a non-zero number to both sides of an equation)
- Note this property requires that $a \neq 0$. !!

4) Let $x, y \in \mathbb{R}$. Then
$xy = 0 \Leftrightarrow x = 0 \vee y = 0$
(i.e. A product xy is zero if and only if $x = 0$ or $y = 0$).

5) Let $x, y \in \mathbb{R}$. Then
$x^2 = y^2 \Leftrightarrow x = y \vee x = -y$
(i.e. $x^2 = y^2$ if and only if $x = y$ or $x = -y$).
- Note that
  $x = y \Rightarrow x^2 = y^2$ is TRUE, but
  $x^2 = y^2 \Rightarrow x = y$ is FALSE!

6) Let $x, y \in \mathbb{R}$. Then
$$x^2 + y^2 = 0 \Leftrightarrow x = 0 \wedge y = 0$$

▼ Solving an equation

To solve an equation $f(x) = g(x)$, we use the above properties 1-6 to construct an argument of the form:

$$\begin{aligned} f(x) = g(x) &\Leftrightarrow f_1(x) = g_1(x) \Leftrightarrow \\ &\Leftrightarrow f_2(x) = g_2(x) \Leftrightarrow \\ &\Leftrightarrow \cdots \Leftrightarrow \\ &\Leftrightarrow x = x_1 \vee x = x_2 \vee \cdots \vee x = x_n \end{aligned}$$

It follows that the solution set is
$$S = \{x_1, x_2, \ldots, x_n\}$$

- It is very important that EVERY step in the argument must be valid in BOTH directions (i.e. $\Leftrightarrow$ instead of only $\Rightarrow$ or $\Leftarrow$)
- When $\Rightarrow$ fails (but $\Leftarrow$ works): Every number you find is a solution but you may have more solutions out there that you have failed to find.
- When $\Leftarrow$ fails (but $\Rightarrow$ works): All of your solutions are among the numbers you found, but some of your numbers may not satisfy the equations (extraneous "solutions").

V Polynomial Equations

- A polynomial equation is an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with $a_0, a_1, \ldots, a_n \in \mathbb{R}$.

- n = degree of the equation.

① Linear Equations $\longrightarrow$ $\boxed{\begin{array}{l} ax+b=0 \\ \text{with } a,b \in \mathbb{R} \end{array}}$

Solution:

- $a \neq 0 \Rightarrow$ unique solution $x = -b/a$
- $a = 0 \wedge b \neq 0 \Rightarrow$ inconsistent $(S = \emptyset)$
- $a = 0 \wedge b = 0 \Rightarrow$ identity $(S = \mathbb{R})$.

EXAMPLES

a) $\dfrac{x-2}{3} - \dfrac{x+1}{2} = \dfrac{1-3x}{6} \Leftrightarrow$

$\Leftrightarrow 6\left[\dfrac{x-2}{3} - \dfrac{x+1}{2}\right] = 6 \cdot \dfrac{1-3x}{6} \Leftrightarrow$

$\Leftrightarrow 2(x-2) - 3(x+1) = 1 - 3x \Leftrightarrow$

$\Leftrightarrow 2x - 4 - 3x - 3 = 1 - 3x \Leftrightarrow$

$\Leftrightarrow (2-3+3)x = 4+3+1 \Leftrightarrow$

$\Leftrightarrow 2x = 8 \Leftrightarrow x = 4 \longleftarrow \underline{S = \{4\}}.$

b) $\frac{x+1}{2} = x - \frac{2x+3}{4} \Leftrightarrow 2(x+1) = 4x - (2x+3) \Leftrightarrow$

$\Leftrightarrow 2x+2 = 4x-2x-3 \Leftrightarrow (2-4+2)x = -2-3 \Leftrightarrow$

$\Leftrightarrow 0x = -5 \leftarrow$ inconsistent $\rightarrow$ $\underline{S=\emptyset}$

c) $3-5x-2(4-5x) = -4(x-1)+3(3x-2)-3 \Leftrightarrow$

$\Leftrightarrow 3-5x-8+10x = -4x+4+9x-6-3 \Leftrightarrow$

$\Leftrightarrow (-5+10+4-9)x = -3+8+4-6-3 \Leftrightarrow$

$\Leftrightarrow 0x = 0 \leftarrow$ identity $\rightarrow$ $\underline{S=\mathbb{R}}$.

EXERCISE

① Solve the equations

a) $5x-2(3-x) = -6-3(-x-1)$

b) $-3+2(5+4x) = 9-3(4-2x)$

c) $2x - \frac{3-2x}{6} = 1 - \frac{5-x}{4}$

d) $\frac{2x}{5} - \frac{x-3}{15} = -1 - \frac{x+1}{10}$

e) $x + \frac{3-x}{3} = 1 + \frac{2x}{3}$

f) $\frac{x-5}{2} + \frac{14}{4} = \frac{7x}{2} - 3(x-3)$

g) $\frac{x+2}{6} - \frac{5-x}{2} = \frac{2x-7}{6} + \frac{x-3}{3}$.

② Completed square equations

$$\boxed{(ax+b)^2 = c}$$

a) If $c>0$, then

$(ax+b)^2 = c \Leftrightarrow ax+b=\sqrt{c} \vee ax+b=-\sqrt{c}$

$\Leftrightarrow \dots$

b) If $c=0$, then

$(ax+b)^2=0 \Leftrightarrow ax+b=0 \Leftrightarrow \dots$

c) If $c<0$, then

$(ax+b)^2=c$ is <u>inconsistent in $\mathbb{R}$.</u>

EXAMPLES

a) $(2x-1)^2-5=0 \Leftrightarrow (2x-1)^2=5 \Leftrightarrow$

$\Leftrightarrow 2x-1=\sqrt{5} \vee 2x-1=-\sqrt{5} \Leftrightarrow$

$\Leftrightarrow 2x=1+\sqrt{5} \vee 2x=1-\sqrt{5} \Leftrightarrow$

$\Leftrightarrow x=\dfrac{1+\sqrt{5}}{2} \vee x=\dfrac{1-\sqrt{5}}{2}$

b) $(3x+2)^2=0 \Leftrightarrow 3x+2=0 \Leftrightarrow 3x=-2 \Leftrightarrow x=-\dfrac{2}{3}$

c) $(5x+3)^2+2=0 \Leftrightarrow (5x+3)^2=-2<0$

thus inconsistent in $\mathbb{R}$.

EXERCISES

② Solve the equations

a) $(x+1)^2 = 5$

b) $(2x+3)^2 - 7 = 0$

c) $(3x-1)^2 + 2 = 5$

d) $(5x-2)^2 + 1 = -2$

e) $(3x+2)^2 = 0$

f) $(3-2x)^2 = 3$

(3) Quadratic Equations $\to$ $ax^2+bx+c=0$ with $a,b,c \in \mathbb{R}$ and $a \neq 0$.

- 1 Calculate the discriminant

$$\Delta = b^2 - 4ac$$

- 2 Distinguish among the following cases:

a) $\Delta > 0 \Rightarrow$ Two solutions

$$x_1 = \frac{-b+\sqrt{\Delta}}{2a} \vee x_2 = \frac{-b-\sqrt{\Delta}}{2a}$$

b) $\Delta = 0 \Rightarrow$ One solution $x = -\frac{b}{2a} = x_1 = x_2$

c) $\Delta < 0 \Rightarrow$ Equation is inconsistent in $\mathbb{R}$.
and has 2 solutions in $\mathbb{C}$: $x_{1,2} = (-b \pm i\sqrt{-\Delta})/2a$

$\to$ If the quadratic has two solutions x_1 and x_2 or one double solution (when $\Delta = 0 \Rightarrow x_1 = x_2 = -b/2a$) then they satisfy:

$$\boxed{x_1 + x_2 = -\frac{b}{a}}$$

$$\boxed{x_1 x_2 = \frac{c}{a}}$$

Proof

$$x_1 + x_2 = \frac{-b+\sqrt{\Delta}}{2a} + \frac{-b-\sqrt{\Delta}}{2a} =$$

$$= \frac{-b+\sqrt{\Delta}-b-\sqrt{\Delta}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$

$$x_1 x_2 = \frac{-b+\sqrt{\Delta}}{2a} \cdot \frac{-b-\sqrt{\Delta}}{2a} = \frac{(-b)^2-(\sqrt{\Delta})^2}{4a^2} =$$

$$= \frac{b^2-\Delta}{4a^2} = \frac{b^2-(b^2-4ac)}{4a^2} =$$

$$= \frac{b^2-b^2+4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a} \quad \square$$

Truncated Forms

1) $ax^2 = 0 \Leftrightarrow x = 0$

2) $ax^2 + bx = 0 \Leftrightarrow x(ax+b) = 0$

$\Leftrightarrow x = 0 \vee ax + b = 0$

(linear equations)

3) $ax^2 + c = 0 \Leftrightarrow x^2 = -\frac{c}{a}$

$$\Leftrightarrow \begin{cases} x = \pm\sqrt{-c/a}, & \text{if } -c/a \geq 0 \\ \text{inconsistent in } \mathbb{R}, & \text{if } -c/a < 0. \end{cases}$$

EXAMPLES

a) $2x^2+x-6=0$

Solution

$$\Delta = b^2-4ac = 1^2-4\cdot 2\cdot(-6) = 1+48 = 49 = 7^2 \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{-b\pm\sqrt{\Delta}}{2a} = \frac{-1\pm 7}{2\cdot 2} = \frac{-1\pm 7}{4} =$$

$$= \begin{cases} -8/4 = -2 \\ 6/4 = 3/2. \end{cases}$$

b) $x^2-6x+9=0$

Solution

$$\Delta = b^2-4ac = (-6)^2-4\cdot 1\cdot 9 = 36-36 = 0 \Rightarrow$$

$\Rightarrow$ unique solution $x = \dfrac{-b}{2a} = \dfrac{-(-6)}{2\cdot 1} = 3$

c) $2x^2-5x+4=0$

Solution

$$\Delta = b^2-4ac = (-5)^2-4\cdot 2\cdot 4 = 25-32 = -7 < 0 \Rightarrow$$

$\Rightarrow$ inconsistent in $\mathbb{R}$.

d) $3x^2-2=0 \Leftrightarrow 3x^2=2 \Leftrightarrow x^2=\frac{2}{3} \Leftrightarrow$

$\Leftrightarrow x=\pm\frac{\sqrt{2}}{\sqrt{3}}=\frac{\pm\sqrt{6}}{3}$

e) $2x^2+5x=0 \Leftrightarrow x(2x+5)=0 \Leftrightarrow$

$\Leftrightarrow x=0 \vee 2x+5=0 \Leftrightarrow$

$\Leftrightarrow x=0 \vee 2x=-5 \Leftrightarrow$

$\Leftrightarrow x=0 \vee x=-5/2.$

EXERCISES

③ Solve the equations

a) $2x^2-3x+1=0$
b) $x^2-4x+4=0$
c) $x^2+2x+4=0$
d) $2x^2-x-3=0$
e) $x^2=3x$
f) $x(2x+1)=x+4$
g) $x(x+1)=4$
h) $x^2-6x+9=0$
i) $x(2x+1)-x^2=0$
j) $(x-1)(x+2)=4$

④ Polynomial equations of high order

Form: $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$
with $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$,
$a_n \neq 0$, $n \in \mathbb{N}$ with $n \geq 3$.

Such equations can be solved by factoring:

- •1 Factor the equation to 1st and 2nd order factors.
- •2 Use the property:

$$a_1 a_2 a_3 \cdots a_n = 0 \Leftrightarrow a_1 = 0 \vee a_2 = 0 \vee \cdots \vee a_n = 0$$

- •3 Solve the resulting equations.

EXAMPLES

a) $(x+3)^2(x^2-4)^3(x^2-1)=0 \Leftrightarrow$

$\Leftrightarrow x+3=0 \vee x^2-4=0 \vee x^2-1=0 \Leftrightarrow$

$\Leftrightarrow x=-3 \vee x^2=4 \vee x^2=1 \Leftrightarrow$

$\Leftrightarrow x=-3 \vee x=2 \vee x=-2 \vee x=1 \vee x=-1.$

b) $x^3=10-2(x-1)^2 \Leftrightarrow$

$\Leftrightarrow x^3=10-2(x^2-2x+1) \Leftrightarrow$

$\Leftrightarrow x^3=10-2x^2+4x-2 \Leftrightarrow$

$\Leftrightarrow x^3+2x^2-4x-8=0 \Leftrightarrow$

$\Leftrightarrow x^2(x+2)-4(x+2)=0 \Leftrightarrow$

$\Leftrightarrow (x^2-4)(x+2)=0 \Leftrightarrow x^2-4=0 \vee x+2=0 \Leftrightarrow$

$\Leftrightarrow x=2 \vee x=-2 \vee x=-2 \Leftrightarrow x=2 \vee x=-2.$

c) $(x+2)(x^2-4)+(x-2)(x^2+5x+6)=0 \Leftrightarrow$

$\Leftrightarrow (x+2)(x+2)(x-2)+(x-2)(x+2)(x+3)=0 \Leftrightarrow$

$\Leftrightarrow (x+2)(x-2)[(x+2)+(x+3)]=0 \Leftrightarrow$

$\Leftrightarrow (x+2)(x-2)(2x+5)=0 \Leftrightarrow$

$\Leftrightarrow x+2=0 \vee x-2=0 \vee 2x+5=0 \Leftrightarrow$

$\Leftrightarrow x=-2 \vee x=2 \vee x=-5/2.$

d) $(x^3-8)(x^2+4x+4)+(x^2-4)(x^2+5x+6)=0 \Leftrightarrow$

$\Leftrightarrow (x-2)(x^2+2x+4)(x+2)^2+(x-2)(x+2)(x+2)(x+3)=0$

$\Leftrightarrow (x-2)(x+2)^2[(x^2+2x+4)+(x+3)]=0 \Leftrightarrow$

$\Leftrightarrow (x-2)(x+2)^2(x^2+2x+4+x+3)=0 \Leftrightarrow$

$\Leftrightarrow (x-2)(x+2)^2(x^2+3x+7)=0 \Leftrightarrow$

$\Leftrightarrow x-2=0 \lor x+2=0 \lor x^2+3x+7=0 \qquad (1)$

To solve $x^2+3x+7=0$:

$\Delta = b^2-4ac = 3^2-4\cdot1\cdot7 = 9-28<0 \Rightarrow$

$\Rightarrow$ no real solutions

It follows that

$(1) \Leftrightarrow x-2=0 \lor x+2=0 \Leftrightarrow$

$\Leftrightarrow x=2 \lor x=-2$

thus $S=\{-2,2\}$.

Note that we use equation labelling to interrupt the main line of our argument, to solve all quadratic factors, and then restart and finish it.

EXERCISES

④ Solve the equations

a) $(2x+1)(x-2)(x+3)=0$

b) $(x-1)(x+1)^2(1-3x)=0$

c) $(3x-1)(x+2)(x^2+1)=0$

d) $(2x-1)^3(2x^2+1)(x^2-1)=0$

⑤ Solve the equations

a) $5x^3-20x=0$

b) $x^3=x^2+6x$

c) $(3x-1)(x-2)^2=9(3x-1)$

d) $x^3-x^2-x+1=0$

e) $(x^2-4)^2-(x+2)^2(5x-4)=0$

f) $3(x-1)^2-2(x-1)(x+1)=(x+1)^2$

g) $(x-3)(2x+1)^2-(x^2-9)(x+3)=0$

h) $x^5+x^4+x^3+x^2+x+1=0$

i) $(x+1)^4-x^4=4x^3$.

Rational Equations

- A rational equation is an equation that has an unknown x in the denominator of at least one fraction.

Solution

1. Find the LCM (Least Common Multiple) of the denominators
2. From the condition $LCM(x) \neq 0$ find the domain $A \subseteq \mathbb{R}$ of the equation.
3. Multiply both sides of the equation with the LCM and solve the resulting polynomial equation
4. Accept the solutions that belong to the domain A and reject any solutions that do not belong to the domain A

EXAMPLES

a) $$\frac{4x}{x^2-x} = \frac{4}{x^2-1} - \frac{x}{x+1} \quad (1)$$

Require:

$$\begin{cases} x^2-x \neq 0 \\ x^2-1 \neq 0 \\ x+1 \neq 0 \end{cases} \Leftrightarrow \begin{cases} x(x-1) \neq 0 \\ (x-1)(x+1) \neq 0 \\ x+1 \neq 0 \end{cases} \Leftrightarrow \begin{cases} x \neq 0 \\ x \neq 1 \\ x \neq -1 \end{cases}$$

thus domain: $A = \mathbb{R} - \{0, -1, 1\}$.

$$(1) \Leftrightarrow \frac{4}{x-1} = \frac{4}{x^2-1} - \frac{x}{x+1} \leftarrow \text{LCM} = x^2-1 = (x-1)(x+1)$$

$\Leftrightarrow 4(x+1) = 4 - x(x-1) \Leftrightarrow$

$\Leftrightarrow 4x+4 = 4 - x^2 + x \Leftrightarrow 4x = -x^2 + x \Leftrightarrow$

$\Leftrightarrow x^2 + (4-1)x = 0 \Leftrightarrow x^2 + 3x = 0 \Leftrightarrow$

$\Leftrightarrow x(x+3) = 0 \Leftrightarrow x = 0 \vee x+3 = 0 \Leftrightarrow$

$\Leftrightarrow x = 0 \vee x = -3$

Reject $x = 0$ since $0 \notin A$

Accept $x = -3$ since $-3 \in A$.

Solution set: $S = \{-3\}$.

b) $$\frac{x-14}{x^2-4} + \frac{3}{x-2} = \frac{4}{x+2} \quad (1)$$

Require:

$$\begin{cases} x^2-4 \neq 0 \\ x-2 \neq 0 \\ x+2 \neq 0 \end{cases} \Leftrightarrow \begin{cases} (x-2)(x+2) \neq 0 \\ x-2 \neq 0 \\ x+2 \neq 0 \end{cases} \Leftrightarrow \begin{cases} x \neq 2 \\ x \neq -2 \end{cases}$$

thus domain : $A = \mathbb{R} - \{2, -2\}$.

$LCM = x^2 - 4 = (x-2)(x+2)$

$(1) \Leftrightarrow (x-14) + 3(x+2) = 4(x-2) \Leftrightarrow$

$\Leftrightarrow x - 14 + 3x + 6 = 4x - 8 \Leftrightarrow$

$\Leftrightarrow 4x - 8 = 4x - 8 \Leftrightarrow 0x = 0 \leftarrow$ identity.

Solution set: $\underline{S = \mathbb{R} - \{-2, 2\}}$.

EXERCISES

(10) Solve the equations

a) $\dfrac{x}{x-3}+3=\dfrac{3}{x-3}$

b) $\dfrac{1}{x+1}+\dfrac{1}{x-1}=\dfrac{2x}{x^2-1}$

c) $\dfrac{1}{2-x}+\dfrac{2}{x+1}+\dfrac{3}{x^2-x-2}=0$

d) $\dfrac{1}{x}-\dfrac{x}{1-x}=\dfrac{6x+5}{x^2-x}$

e) $\dfrac{13}{x+1}-\dfrac{1}{1-x}=\dfrac{5x-3}{x^2-1}$

f) $\dfrac{1}{x+1}+\dfrac{1}{x+2}+\dfrac{1}{x-1}+\dfrac{1}{x-2}=0$

g) $\dfrac{2}{x(x+2)}=\dfrac{-1}{x^2+5x+6}$

h) $\dfrac{x+1}{x-2}+\dfrac{x-1}{x+2}=\dfrac{2x^2+4}{x^2-4}$

Inequalities - Terminology.

- An <u>inequality</u> is an expression of the form $f(x) < g(x)$ or $f(x) \leq g(x)$ or $f(x) > g(x)$ or $f(x) \geq g(x)$. which is either true or false depending on the value of the variable x.
- The <u>solution set S</u> of an inequality is the set of all real numbers $x \in \mathbb{R}$ for which the inequality is true.

Intervals

The solution sets of inequalities are written as unions of intervals, which are defined as follows:

$x \in [a,b] \Leftrightarrow a \leq x \leq b$	$x \in [a,+\infty) \Leftrightarrow a \leq x$
$x \in (a,b] \Leftrightarrow a < x \leq b$	$x \in (a,+\infty) \Leftrightarrow a < x$
$x \in [a,b) \Leftrightarrow a \leq x < b$	$x \in (-\infty,b] \Leftrightarrow x \leq b$
$x \in (a,b) \Leftrightarrow a < x < b$	$x \in (-\infty,b) \Leftrightarrow x < b$

The set of all real numbers can also be written as
$\mathbb{R} = (-\infty, +\infty)$.

Basic properties of inequalities

1) Let $x, y, a \in \mathbb{R}$. Then

$$x > y \Leftrightarrow x + a > y + a$$
$$x \geqslant y \Leftrightarrow x + a \geqslant y + a$$
$$x < y \Leftrightarrow x + a < y + a$$
$$x \leqslant y \Leftrightarrow x + a \leqslant y + a$$

(i.e.: We can add any number to both sides of an inequality)

2) Let $x, y \in \mathbb{R}$ and $p \in (0, +\infty)$. Then

$$x > y \Leftrightarrow px > py \quad \Big| \quad x < y \Leftrightarrow px < py$$
$$x \geqslant y \Leftrightarrow px \geqslant py \quad \Big| \quad x \leqslant y \Leftrightarrow px \leqslant py$$

(i.e: We can multiply a positive number to both sides of an inequality.)

3) Let $x, y \in \mathbb{R}$ and $n \in (-\infty, 0)$. Then

$$x > y \Leftrightarrow nx < ny \quad \Big| \quad x < y \Leftrightarrow nx > ny$$
$$x \geqslant y \Leftrightarrow nx \leqslant ny \quad \Big| \quad x \leqslant y \Leftrightarrow nx \geqslant ny$$

(i.e: We can multiply a negative number to both sides of an inequality, but then the direction of the inequality must be reversed).

↳ The general strategy for solving inequalities is to first move every term to the same side, then <u>simplify</u> and <u>factor</u> the resulting expression.

Polynomial inequalities

1) Linear Inequalities $\longrightarrow$ $ax+b \gtrless 0$

Consider, for example, the inequality $ax+b>0$.

a) For $a>0$:

$ax+b>0 \Leftrightarrow ax>-b \Leftrightarrow x>-b/a$.

thus $S=(-b/a, +\infty)$.

b) For $a<0$:

$ax+b>0 \Leftrightarrow ax>-b \Leftrightarrow x<-b/a$

thus $S=(-\infty, -b/a)$.

c) For $a=0$: the inequality is an identity or it is inconsistent, which we determine on a case by case basis.

EXAMPLES

a) $3(x-2)-5(x+1) \geqslant 3-2(3-x) \Leftrightarrow$

$\Leftrightarrow 3x-6-5x-5 \geqslant 3-6+2x \Leftrightarrow$

$\Leftrightarrow -2x-11 \geqslant -3+2x \Leftrightarrow -2x-2x \geqslant 11-3 \Leftrightarrow$

$\Leftrightarrow -4x \geqslant 8 \underset{(!!)}{\Leftrightarrow} x \leqslant \dfrac{8}{-4} \Leftrightarrow x \leqslant -2$

therefore $S=(-\infty, -2]$

- Note that because of $\leqslant$, -2 is included in S.

b) $x+3-\frac{3x-5}{2} > 2-\frac{x}{2} \Leftrightarrow$

$\Leftrightarrow 2(x+3)-(3x-5) > 4-x \Leftrightarrow$

$\Leftrightarrow 2x+6-3x+5 > 4-x \Leftrightarrow -x+11 > 4-x$

$\Leftrightarrow 0x > 4-11 \Leftrightarrow 0x > -7 \leftarrow$ always true.

therefore $S = \mathbb{R}$.

(i.e. the inequality is an identity; it is satisfied by all real numbers $x \in \mathbb{R}$).

c) $\frac{x-3}{4} - \frac{x+5}{2} < -1 - \frac{10+x}{4} \Leftrightarrow$

$\Leftrightarrow (x-3)-2(x+5) < -4-(10+x) \Leftrightarrow$

$\Leftrightarrow x-3-2x-10 < -4-10-x \Leftrightarrow$

$\Leftrightarrow -x-13 < -14-x \Leftrightarrow 0x < 13-14 \Leftrightarrow 0x < -1. \leftarrow$ always false

therefore $S = \emptyset$.

(i.e. the inequality is inconsistent. It can never be satisfied).

↳ Note that when the inequality has a fractions, we first eliminate all fractions by multiplying both sides of the inequality with a large enough positive number.

EXERCISES

(13) Solve the following inequalities

1) $-3x+1>0$

2) $0x>4$

3) $0x<4$

4) $0x\geqslant 2$

5) $0x>-4$

6) $0x<-4$

7) $0x>0$

8) $0x\geqslant 0$

9) $0x\leqslant 2$

10) $0x\leqslant 0$

11) $-x-2<0$

12) $1>3x$

13) $4(2x-1)\leqslant x-2$

14) $3(2x+7)-4(15-x)\leqslant 29+12x$

15) $2(4x+5)-3(x+3)\leqslant -5x-9(1-x)$

16) $-6(x-2)-(5-3x)<9(x+3)-2x$

17) $2(x+1)\geqslant 4-(x+3)-3(2-x)$

18) $13-3(x-2)<4(x+3)-7(x-3)$

19) $1-\dfrac{3-x}{3}\geqslant\dfrac{19}{21}-\dfrac{1-x}{7}$

20) $\dfrac{x-3}{2}-\dfrac{x-5}{4}>1-\dfrac{4-x}{3}$

21) $\dfrac{x+1}{3}-\dfrac{5x-16}{6}\geqslant\dfrac{x+8}{12}$

22) $\dfrac{10x-1}{24}-\dfrac{2x-1}{8}<\dfrac{2x+5}{4}-\dfrac{x+3}{2}$

23) $\dfrac{2}{5}-\dfrac{3-x}{2}<\dfrac{x-1}{10}-\dfrac{3-2x}{5}$

24) $\dfrac{x+1}{16}-\dfrac{1+x}{2}\geqslant\dfrac{x-1}{16}-\dfrac{2x+1}{4}$

2) Quadratic Inequalities $\to ax^2+bx+c \gtrless 0$

- •1 Calculate the discriminant

$$\Delta = b^2 - 4ac$$

and the two zeroes $x_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

if they exist in $\mathbb{R}$.

- •2 The expression ax^2+bx+c has the same sign as "a" for all values of x EXCEPT when $\Delta > 0$ and $x_1 < x < x_2$. We use this to construct a sign chart.
- •3 From the sign chart we deduce the solution.

EXAMPLES

a) Solve $-3x^2+6x+2 \geq 0$. (1)

$$\Delta = 6^2 - 4\cdot(-3)\cdot 2 = 36+24 = 60 =$$

$$= 3\cdot 2\cdot 5\cdot 2 = 2^2\cdot 15 \Rightarrow$$

$\Rightarrow$ two zeroes:

$$x_{1,2} = \frac{-6 \pm 2\sqrt{15}}{2\cdot(-3)} = \frac{-3 \pm \sqrt{15}}{-3} =$$

$$= 1 \pm \frac{\sqrt{15}}{3}$$

x		$1-\frac{\sqrt{15}}{3}$		$1+\frac{\sqrt{15}}{3}$	
$-3x^2+6x+2$	$-$	0	$+$	0	$-$

$(1) \Leftrightarrow \quad x \in \left[1-\frac{\sqrt{15}}{3},\ 1+\frac{\sqrt{15}}{3}\right]$

b) $-3x^2+x-2>0 \qquad (1)$

$\Delta = 1^2-4\cdot(-3)(-2) = 1-24 = -23<0 \Rightarrow$
$\Rightarrow -3x^2+x-2<0, \ \forall x \in \mathbb{R}$
$\Rightarrow$ (1) inconsistent

c) $x^2+4x+4 \leq 0 \qquad (1)$

$\Delta = 16-4\cdot1\cdot4 = 16-16 = 0 \Rightarrow x_{1,2} = \frac{-4}{2\cdot1} = -2$

x		-2	
x^2+4x+4	$+$	0	$+$

$(1) \Leftrightarrow \underline{x=-2}$!!

3) <u>Higher-Order Inequalities</u>

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \gtrless 0$$

- Move everything to LHS and factor to linear and quadratic factors.

- 2 Find the zeroes of every factor
- 3 Make a sign chart for each factor and for their product
- 4 See where the inequality is satisfied.

EXAMPLES

a) Solve $-3x(2x-1)^3(5-x)^2 < 0$ (1)

Zeroes: $0, 1/2, 5$

x		0		$1/2$		5	
$-3x$	$+$	0	$-$		$-$		$-$
$(2x-1)^3$	$-$		$-$	0	$+$		$+$
$(5-x)^2$	$+$		$+$		$+$	0	$+$
$f(x)$	$-$	0	$+$	0	$-$	0	$-$

(1) $\Leftrightarrow$ $x \in (-\infty, 0) \cup (1/2, 5) \cup (5, +\infty)$

b) Solve $x^2(x-2) - 2x(x-2)^2 \leq 0$ (1)

(1) $\Leftrightarrow$ $x(x-2)[x-2(x-2)] \leq 0 \Leftrightarrow$

$\Leftrightarrow$ $x(x-2)(x-2x+4) \leq 0$

$\Leftrightarrow$ $x(x-2)(4-x) \leq 0$ (2).

x		0		2		4	
x	$-$	0	$+$		$+$		$+$
$x-2$	$-$		$-$	0	$+$		$+$
$4-x$	$+$		$+$		$+$	0	$-$
	$+$	0	$-$	0	$+$	0	$-$

$(2) \Leftrightarrow \quad x \in [0,2] \cup [4,+\infty)$.

EXERCISES

(14) Solve the inequalities

a) $3x^2-6x+1 \geq 0$

b) $4x^2-1<0$

c) $2x^2-20x+50 \leq 0$

d) $-2x^2+x-1 \geq 0$

e) $x^2-3x+4 \geq 0$

f) $x^2-4<0$

g) $x^2-5 \leq 0$

h) $x^2+3>0$

i) $4x^2+25 \leq 0$

j) $\dfrac{2x^2+3x}{4} \geq \dfrac{x^2}{3}-\dfrac{2x}{5}$

(15) Solve the inequalities

a) $(3x-1)(x-1)^3(2-x)(2x+5)^4 \geq 0$

b) $5x(x^2-4x+3)(x^2-10x+25)(x^2+x+1) \leq 0$

c) $(2x^4-x^2)(x^2-3)^2(2-x)^3<0$

d) $(2x^2-5x-3)^2(x^3-x^2-x)>0$

(16) Solve the inequalities

a) $x^3+4x>5x^2$

b) $x^3+x \leq x^2+1$

c) $x^3<8$

d) $2x(x+1)^2-3x^2(x+1) \leq 0$

e) $(2x+1)^4(2x-1)^3>(2x+1)^3(2x-1)^4$

f) $x(x+1)^2(x+2)-(x^2+2x)(x^2+3x+2) \geq 0$

g) $3(2x+3)^2(x-1)^3+2(2x+3)^3(x-1)^2 \leq 0$

Rational Inequalities

Form: $\dfrac{P(x)}{Q(x)} \gtrless 0$

with P, Q polynomials.

Method: The method entails the same steps as with polynomial inequalities. However, the zeroes of numerator factors must be distinguished from the zeroes of denominator factors.

- Denominator zeroes are shown with the $\not\phi$ symbol instead of ϕ in the last entry of your sign table because at these zeroes, the expression is undefined.
- Denominator zeroes are to be excluded from the solution set.

examples

1) $\dfrac{x-5}{x-3} \geq \dfrac{x-2}{x-1} \quad (1)$

Solution:

$$(1) \Leftrightarrow \frac{x-5}{x-3} - \frac{x-2}{x-1} \geq 0 \Leftrightarrow \frac{(x-5)(x-1)-(x-2)(x-3)}{(x-3)(x-1)} \geq 0$$

$$\Leftrightarrow \frac{(x^2-6x+5)-(x^2-5x+6)}{(x-3)(x-1)} \geq 0$$

$$\Leftrightarrow \frac{(-6+5)x+(5-6)}{(x-3)(x-1)} \geq 0$$

$$\Leftrightarrow \frac{-x-1}{(x-3)(x-1)} \geq 0. \quad (2)$$

Zeroes: $-1, 3, 1$

x		-1		1		3	
$-x-1$	+	0	−		−		−
$x-3$	−		−		−	0	+
$x-1$	−		−	0	+		+
$f(x)$	+	0	−	‖	+	‖	−

$(2) \Leftrightarrow x \in (-\infty, -1] \cup (1, 3)$

Note that -1 is a zero of $f(x)$ but 1 and 3 are not, so they are not included in the solution.

CAUTION: If the fraction has cancellations then you must find the domain of the inequality before solving it:

example:
$$\frac{(x+1)(x^2+4x+4)}{(x^2+5x+6)} \geq 0. \qquad (1)$$

$$(1) \Leftrightarrow \frac{(x+1)(x+2)^2}{(x+2)(x+3)} \geq 0 \Leftrightarrow \frac{(x+1)(x+2)}{x+3} \geq 0$$

▶ Domain:

$x^2+5x+6 \neq 0 \Leftrightarrow \underline{x \in \mathbb{R} - \{-2, -3\} = A}$

x		-3		-2		-1	
$x+1$	$-$		$-$		$-$	0	$+$
$x+2$	$-$		$-$	0	$+$		$+$
$x+3$	$-$	0	$+$		$+$		$+$
$f(x)$	$-$	$\nparallel$	$+$	$\nparallel$	$-$	0	$+$

thus

$(1) \Leftrightarrow \underline{x \in (-3, -2) \cup [-1, +\infty)}$

-2 looks like a numerator zero but it cannot solve the original inequality because the domain
$$A = \mathbb{R} - \{-2, -3\}$$
of the inequality EXCLUDES -2 !!

EXERCISES

(17) Solve the inequalities:

a) $\dfrac{2-x}{3x+1} \geq 0$ b) $\dfrac{-(1-x)(3+x)(-3+x)}{(x+2)^2(x+1)^3} \geq 0$

c) $\dfrac{-x^2(3-x)(x^2+3x+2)(x^2-3)}{3(x+1)} \geq 0$

(18) Solve the inequalities

a) $\dfrac{2x-1}{x^2+4x+3} \leq \dfrac{1}{5}$

b) $\dfrac{x+1}{x-1} < \dfrac{2x+3}{x+1}$

c) $\dfrac{x^2+14}{x^2+6x+8} \leq 1$

d) $\dfrac{x+1}{x^2+x-2} \leq \dfrac{x}{x^2-1}$

e) $\dfrac{(x+1)^3-1}{(x-1)^3-1} \leq 1$

f) $\dfrac{x-10}{x^2+5} < \dfrac{1}{2}$

g) $\dfrac{6x^2-3x+8}{x^2-5x+6} \leq 6$

h) $\dfrac{x}{1+x^2} > 10$

Absolute Values

- Let $x \in \mathbb{R}$ be given. We define the <u>absolute value</u> $|x|$ of x as:

$$\boxed{|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}}$$

▸ <u>examples</u> : $|3| = 3$, $|-7| = 7$, $|0| = 0$

Properties of absolute value

$\lvert x\rvert \geq 0$	$\lvert x\rvert - \lvert y\rvert \leq \lvert x+y\rvert \leq \lvert x\rvert + \lvert y\rvert$
$\lvert -x\rvert = \lvert x\rvert$	$\lvert x\rvert - \lvert y\rvert \leq \lvert x-y\rvert \leq \lvert x\rvert + \lvert y\rvert$
$-\lvert x\rvert \leq x \leq \lvert x\rvert$	$\lvert xy\rvert = \lvert x\rvert\lvert y\rvert$, $\left\lvert \frac{x}{y} \right\rvert = \frac{\lvert x\rvert}{\lvert y\rvert}$
$\lvert x\rvert^2 = x^2$	

Equations with absolute values

Let $a, x \in \mathbb{R}$, and $p \in [0, +\infty)$, and $n \in (-\infty, 0)$. Then:

1) $|x| = |a| \Leftrightarrow x = a \vee x = -a$
2) $|x| = p \Leftrightarrow x = p \vee x = -p$
3) $|x| = n$ is inconsistant.

We use the above 3 properties to solve equations with absolute values as in the following examples.

EXAMPLES

a) $(x^2-|x|)(3|x|+1)=0 \quad (1)$

<u>Solution</u>

Let $y=|x| \Rightarrow x^2=|x|^2=y^2$, thus

$(1) \Leftrightarrow (y^2-y)(3y+1)=0 \Leftrightarrow y(y-1)(3y+1)=0 \Leftrightarrow$

$\Leftrightarrow y=0 \lor y-1=0 \lor 3y+1=0 \Leftrightarrow$

$\Leftrightarrow y=0 \lor y=1 \lor y=-1/3 \Leftrightarrow$

$\Leftrightarrow |x|=0 \lor |x|=1 \lor |x|=-1/3 \Leftrightarrow$

$\Leftrightarrow x=0 \lor x=1 \lor x=-1$

thus $S=\{0,1,-1\}$.

Note that $|x|=-1/3$ has no solutions since $-1/3<0$.

b) $|x+3|+2=0 \quad (1)$

<u>Solution</u>

$(1) \Leftrightarrow |x+3|=-2<0 \leftarrow$ inconsistent.

Thus $S=\emptyset$.

c) $|2x-1|-5=0 \quad (1)$

<u>Solution</u>

$(1) \Leftrightarrow |2x-1|=5 \Leftrightarrow 2x-1=\pm 5 \Leftrightarrow 2x=1\pm 5$

$\Leftrightarrow x=\dfrac{1\pm 5}{2}=\begin{cases} 6/2=3 \\ -4/2=-2 \end{cases}$, thus $S=\{3,-2\}$.

d) $|2x+3| = |x+9|$

Solution

$|2x+3| = |x+9| \Leftrightarrow 2x+3 = x+9 \vee 2x+3 = -(x+9) \Leftrightarrow$

$\Leftrightarrow (2-1)x = 9-3 \vee 2x+3 = -x-9 \Leftrightarrow$

$\Leftrightarrow x = 6 \vee 2x+x = -3-9 \Leftrightarrow x = 6 \vee 3x = -12$

$\Leftrightarrow x = 6 \vee x = -4.$

thus $S = \{6, -4\}$.

e) $|x-4| = 5-2x \quad (1)$

Solution

Require $5-2x \geq 0 \Leftrightarrow 5 \geq 2x \Leftrightarrow x \leq 5/2$

thus domain: $A = (-\infty, 5/2]$.

$(1) \Leftrightarrow x-4 = 5-2x \vee x-4 = -(5-2x) \Leftrightarrow$

$\Leftrightarrow x+2x = 4+5 \vee x-4 = -5+2x \Leftrightarrow$

$\Leftrightarrow 3x = 9 \vee x-2x = 4-5 \Leftrightarrow$

$\Leftrightarrow x = 3 \vee -x = -1 \Leftrightarrow$

$\Leftrightarrow x = 3 \vee x = 1.$ ← accept $x=1$, reject $x=3$

For equations of the form $|f(x)| = g(x)$ we require

$g(x) \geq 0 \Leftrightarrow x \in A$

and reject solutions that do not belong to A.

f) $|x+3| - |2-x| = x+5 \quad (1)$

Solution

x		-3		2	
$x+3$	$-$	0	$+$		$+$
$2-x$	$+$		$+$	0	$-$

Distinguish 3 cases:

Case 1: If $x \in (-\infty, -3)$ then

$|x+3| = -(x+3)$ and $|2-x| = 2-x$.

$(1) \Leftrightarrow -(x+3) - (2-x) = x+5 \Leftrightarrow$

$\Leftrightarrow -x-3-2+x = x+5 \Leftrightarrow$

$\Leftrightarrow -x-5 = 5 \Leftrightarrow x = -5-5 = -10 \leftarrow$ accepted

$-10 \in (-\infty, -3)$.

Case 2: If $x \in [-3, 2)$ then

$|x+3| = x+3$ and $|2-x| = 2-x$.

$(1) \Leftrightarrow (x+3) - (2-x) = x+5 \Leftrightarrow$

$\Leftrightarrow x+3-2+x = x+5 \Leftrightarrow 2x+1 = x+5 \Leftrightarrow$

$\Leftrightarrow 2x - x = 5-1 \Leftrightarrow x = 4 \leftarrow$ rejected

$4 \notin [-3, 2)$.

Case 3: If $x \in [2, +\infty)$ then

$|x+3| = x+3$ and $|2-x| = -(2-x)$

$(1) \Leftrightarrow (x+3) + (2-x) = x+5 \Leftrightarrow$

$\Leftrightarrow x+3+2-x = x+5 \Leftrightarrow 5 = x+5 \Leftrightarrow x = 0$

rejected

$0 \notin [2, +\infty)$.

Thus $S = \{-10\}$.

EXERCISES

25 Solve the equations

a) $\dfrac{2+|-5x|}{|x|-1} = 3$

b) $\dfrac{3+|x|}{|2x|+1} = 4$

c) $2x^2 + 5|x| + 7 = 0$

d) $(|2x|-3)(|x^3|-x^2) = 0$

26 Solve the equations

a) $|2x| = |x-1|$

b) $|3x-2| = |2-x|$

c) $|x^2-1| = |2-x|$

d) $|2x-3| = x$

e) $|2x-1| = 4$

f) $|2x^2-5x-1| = 4x$

g) $|x^2-1| = 2x+1$

h) $x^3-x^2+|x-1| = 0$

27 Solve the equations

a) $|3x| + |2-x| - x + 1 = 0$

b) $|x-3| - 3|x-1| + |x| = 5$

c) $2|x+1| - 3|x-1| = 1$

d) $|x^2-4x+3| - 2|3-x^2| = 1$

Inequalities with absolute values

The solution of inequalities with absolute values is based on the following properties:

1) If $a > 0$, then

$|x| \leq a \Leftrightarrow -a \leq x \leq a$

$|x| < a \Leftrightarrow -a < x < a$

$|x| \geq a \Leftrightarrow x \geq a \vee x \leq -a$

$|x| > a \Leftrightarrow x > a \vee x < -a$

2) If $a < 0$, then

$$\left.\begin{array}{l} |x| \leq a \leftarrow \text{Inconsistent} \\ |x| < a \leftarrow \text{Inconsistent} \\ |x| \geq a \leftarrow \text{Identity} \\ |x| > a \leftarrow \text{Identity} \end{array}\right\} \text{because } |x| \geq 0$$

3) For the case $a = 0$:

$|x| \leq 0 \Leftrightarrow x = 0$

$|x| < 0 \leftarrow$ Inconsistent

$|x| \geq 0 \leftarrow$ Identity

$|x| > 0 \Leftrightarrow x \neq 0$

We apply these properties as in the following examples.

EXAMPLES

a) $|2x-3| \leq 5$

Solution

$$|2x-3| \leq 5 \Leftrightarrow -5 \leq 2x-3 \leq 5 \Leftrightarrow \begin{cases} 2x-3 \leq 5 \\ -5 \leq 2x-3 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2x \leq 8 \\ 2x \geq -5+3 = -2 \end{cases} \Leftrightarrow \begin{cases} x \leq 4 \\ x \geq -1 \end{cases} \Leftrightarrow x \in [-1, 4]$$

thus $S = [-1, 4]$

b) $|1-2x| > 7$

Solution

$$|1-2x| > 7 \Leftrightarrow 1-2x > 7 \vee 1-2x < -7 \Leftrightarrow$$

$$\Leftrightarrow 1-7 > 2x \vee 1+7 < 2x \Leftrightarrow -6 > 2x \vee 8 < 2x$$

$$\Leftrightarrow x < -3 \vee x > 4$$

thus $S = (-\infty, -3) \cup (4, +\infty)$.

c) $|x-5| < -2$

Solution

Since $\forall x \in \mathbb{R} : |x-5| \geq 0$, it follows that $|x-5| < -2$ is inconsistent. Thus $S = \emptyset$.

d) $|21-4x| \geq -3$

<u>Solution</u>

$\forall x \in \mathbb{R}: |21-4x| \geq 0$ thus
$\forall x \in \mathbb{R}: |21-4x| \geq -3$ thus
solution set $S = \mathbb{R}$.

(!) e) $|x-2| \geq |x+3|$

<u>Solution</u>

$|x-2| \geq |x+3| \Leftrightarrow (x-2)^2 \geq (x+3)^2 \Leftrightarrow$
$\Leftrightarrow x^2-4x+4 \geq x^2+6x+9 \Leftrightarrow$
$\Leftrightarrow -4x+4 \geq 6x+9 \Leftrightarrow -4x-6x \geq -4+9 \Leftrightarrow$
$\Leftrightarrow -10x \geq 5 \Leftrightarrow 10x \leq -5 \Leftrightarrow x \leq -\frac{1}{2}$
thus $S = (-\infty, -1/2]$.

↳ For inequalities of the form $|f(x)| \lesseqgtr |g(x)|$ we can raise squares because BOTH sides of the inequality are guaranteed to be positive.

f) $|x-3| > 2x+1$ (1)

↳ We CANNOT square both sides because we do NOT know whether $2x+1$ is positive or negative.

Solution

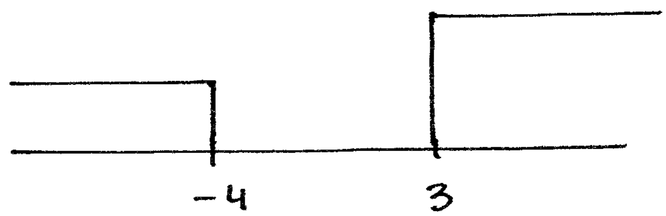
Distinguish two cases:

Case 1: $x-3 \geq 0 \Leftrightarrow x \geq 3 \Leftrightarrow x \in [3,+\infty)$

Then $|x-3| = x-3$.

$(1) \Leftrightarrow x-3 > 2x+1 \Leftrightarrow x-2x > 3+1 \Leftrightarrow -x > 4$

$\Leftrightarrow x < -4 \Leftrightarrow x \in (-\infty,-4)$.



thus $S_1 = (-\infty,-4) \cap [3,+\infty) = \emptyset$.

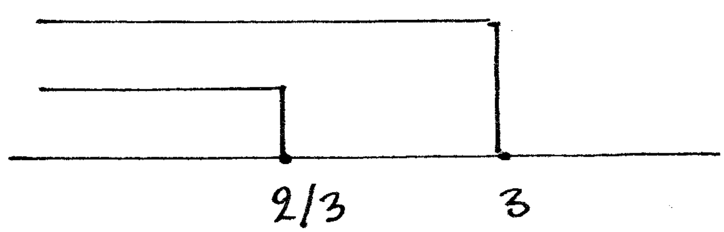
Case 2: $x-3 < 0 \Leftrightarrow x < 3 \Leftrightarrow x \in (-\infty,3)$

then $|x-3| = -(x-3)$.

$(1) \Leftrightarrow -(x-3) > 2x+1 \Leftrightarrow -x+3 > 2x+1 \Leftrightarrow$

$\Leftrightarrow -x-2x > 1-3 \Leftrightarrow -3x > -2 \Leftrightarrow 3x < 2 \Leftrightarrow$

$\Leftrightarrow x < 2/3 \Leftrightarrow x \in (-\infty, 2/3)$.



thus $S_2 = (-\infty, 2/3) \cap (-\infty,3) = (-\infty, 2/3)$.

It follows that the solution set is:

$S = S_1 \cup S_2 = \emptyset \cup (-\infty, 2/3)$

$= (-\infty, 2/3)$.

g) $|x+3| - |1-x| - 2x > 7 \qquad (1)$

Solution

x		-3		1	
$x+3$	$-$	0	$+$		$+$
$1-x$	$+$		$+$	0	$-$

Distinguish three cases:

Case 1 : For $x \in (-\infty, -3)$, we have

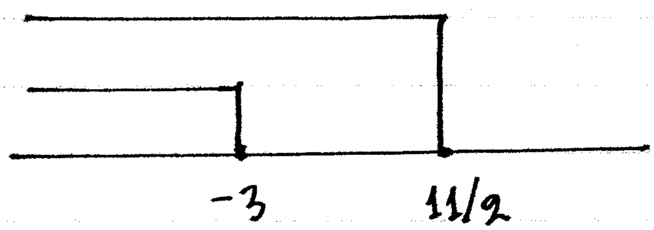
$|x+3| = -(x+3)$ and $|1-x| = 1-x$

$(1) \Leftrightarrow -(x+3) - (1-x) - 2x > 7 \Leftrightarrow$

$\Leftrightarrow -x-3-1+x \quad -2x > 7 \Leftrightarrow$

$\Leftrightarrow -2x-4 > 7 \Leftrightarrow -2x > 7+4 \Leftrightarrow -2x > 11$

$\Leftrightarrow x < 11/2.$



thus $S_1 = (-\infty, 11/2) \cap (-\infty, -3) = (-\infty, -3)$.

Case 2: For $x \in [3, 1)$, we have

$|x+3| = x+3$ and $|1-x| = 1-x$

$(1) \Leftrightarrow (x+3) - (1-x) - 2x > 7 \Leftrightarrow$

$\Leftrightarrow x+3-1+x-2x > 7 \Leftrightarrow$

$\Leftrightarrow 0x+2 > 7 \Leftrightarrow 0x > 7-2 \Leftrightarrow 0x > 5 \leftarrow$ inconsistent

Thus: $S_2 = \emptyset \cap [-3, 1) = \emptyset$.

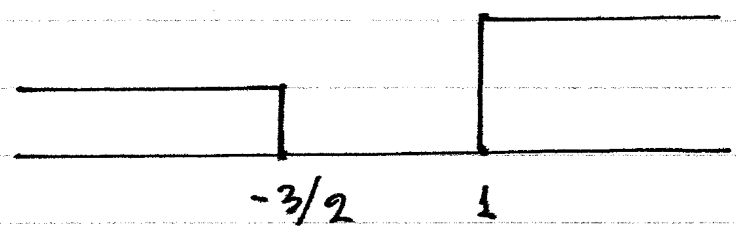
<u>Case 3:</u> For $x \in [1, +\infty)$, we have
$|x+3| = x+3$ and $|1-x| = -(1-x)$

$(1) \Leftrightarrow (x+3) + (1-x) - 2x > 7 \Leftrightarrow$

$\Leftrightarrow x+3+1-x-2x > 7 \Leftrightarrow$

$\Leftrightarrow -2x+4 > 7 \Leftrightarrow -2x > 7-4 \Leftrightarrow -2x > 3 \Leftrightarrow$

$\Leftrightarrow x < -3/2$



thus $S_3 = (-\infty, -3/2) \cap [1, +\infty) = \emptyset$.
It follows that the solution set is
$S = S_1 \cup S_2 \cup S_3 = (-\infty, -3) \cup \emptyset \cup \emptyset =$
$= (-\infty, -3)$.

EXERCISES

(28) Solve the inequalities

a) $|x| < 5$
b) $|x| \geq 3$
c) $|x| < -1$
d) $|x| \geq 0$
e) $|x| < 0$
f) $|x| > 0$
g) $|x| > -2$
h) $|x| \geq -3$
i) $3|x| - 2 > |x| + 8$
j) $2(|x| - 1) \geq 3|x| - 2$
k) $2|x| - |-2x| \geq 3 - |3x|$
l) $\dfrac{2 + |x|}{|x| + 1} > 2$

(29) Solve the inequalities

a) $|-3x| \geq -2$
b) $|-2x| \geq |2x|$
c) $|1 + 2x| \leq 3$
d) $|x - 3| > 1$
e) $|x - 30| - 6 > 0$
f) $|3x + 2| - 4 \leq 0$
g) $|x - 5| - 4 \leq 0$
h) $|2x - 3| + 3 < 0$
i) $-3|7 - x| + 5 \leq 0$
j) $|x^2 + 3| \geq 4x$

(30) Solve the inequalities

a) $|2x| + |x + 1| < 3$
b) $|x + 1| - |x - 1| \geq 1/2$
c) $|2x - 1| + |x - 3| + 3 < -|x + 1|$
d) $|x^2 - 1| - 3x \geq 0.$
e) $|2x + 3| - |x + 5| < 0$
f) $|x + 1| - 3x > 0$
g) $|x^2 - 1| \leq |2x + 1|.$