

12/2/2019 Lecture 28 - Basis of \mathbb{R}^n & linearly dependent *

Recall that B basis of $V \Leftrightarrow \begin{cases} B \text{ linearly independent} \\ \text{span}(B) = V \end{cases}$

$$\forall u \in V: \exists! \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}: u = \lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_n u_n$$

(with basis $B = \{u_1, u_2, \dots, u_n\}$)

① B basis of $V \supseteq A \Rightarrow A$ linearly dependent
 $|A| > |B|$

② B_1 basis of $V \supseteq B_2$ basis of $V \Rightarrow |B_1| = |B_2|$

► $\dim(V) = \text{size of any basis } B \text{ of } V$

► $\dim(\mathbb{R}^n) = n$

$\dim(M_{nm}(\mathbb{R})) = nm$

$\dim(M_n(\mathbb{R})) = n^2$

Theorem: given vector space $(V, +, \cdot)$ and $A \subseteq V$, then
 $\begin{cases} A \text{ linearly independent} \\ |A| = \dim(V) \end{cases} \Leftrightarrow A \text{ basis of } V$

Proof (\Leftarrow) obvious, by definition

(\Rightarrow) Assume that $A = \{x_1, x_2, \dots, x_n\}$ linearly independent and $\dim(V) = n$. Sufficient to show that $\text{span}(A) = V$

Since $A \subseteq V \Rightarrow \text{span}(A) \subseteq V$ Eq(1)
 $\subseteq V$ vector space

let $u \in V$ be given. Want to show that $u \in \text{span}(A)$

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Proof of theorem: given vector space $(V, +, \cdot)$ and $A \subseteq V$

$$\begin{cases} \subseteq A \text{ linearly independent} \\ |A| = \dim(V) \end{cases} \Leftrightarrow A \text{ basis of } V$$

Proof continued to show for $u \in V$, $u \in \text{span}(A)$

we distinguish between the following cases

Case 1 Assume that $u \in A \Rightarrow \exists a \in [n]: u = x_a \in A \subseteq \text{span}(A)$
 $\Rightarrow u \in \text{span}(A)$

Case 2 Assume that $u \notin A$. then

$$\begin{aligned} \cancel{|A \cup \{u\}|} &= |\{x_1, x_2, \dots, x_n, u\}| = n+1 > n = \dim(V) \\ \Rightarrow |A \cup \{u\}| > \dim(V) &\Rightarrow A \cup \{u\} \text{ linearly dependent} \Rightarrow \\ &A \text{ linearly independent} \end{aligned}$$

$\Rightarrow u \in \text{span}(A)$, [we show that for all $u \in V$, $u \in \text{span}(A)$]

we conclude that

$$(\forall u \in V: u \in \text{span}(A)) \Rightarrow V \subseteq \text{span}(A) \text{ Eq(2)}$$

from Eq(1) and Eq(2) $\text{span}(A) = V$ $\Rightarrow A$ basis of V
 A linearly independent

Examples Show that $B = \{(a, a+1), (a+1, a+2)\}$ is basis of \mathbb{R}^2
for all $a \in \mathbb{R}$

Solution ~~Let $a \in \mathbb{R}$ be given~~

Define $x = (a, a+1)$ and $y = (a+1, a+2)$, then

$$\det(x, y) = \begin{vmatrix} a & a+1 \\ a+1 & a+2 \end{vmatrix} = a(a+2) - (a+1)^2 = a^2 + 2a - (a^2 + 2a + 1)$$

$$= \underline{a^2} + \underline{2a} - \underline{a^2} - \underline{2a} - 1 = -1 \neq 0 \Rightarrow B = \{x, y\} \text{ linearly independent} \Rightarrow$$
$$|B| = 2 = \dim(\mathbb{R}^2)$$

$\Rightarrow B$ basis of \mathbb{R}^2

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~~Basis~~ Basis in \mathbb{R}^n examples

Example $B = \{(3a-1, a), (3a, a+1)\}$ Find all $a \in \mathbb{R}$ such that B basis of \mathbb{R}^2

Solution define $x = (3a-1, a)$ and $y = (3a, a+1)$, then

$$\det([x \ y]) = \begin{vmatrix} 3a-1 & 3a \\ a & a+1 \end{vmatrix} = (3a-1)(a+1) - (3a)(a) =$$

$$= \underline{3a^2} + \underline{3a} - \underline{a} - \underline{1} - \underline{3a^2} = 2a - 1 \neq 0$$

Since $|B| = 2 = \dim(\mathbb{R}^2)$ it follows that B basis of $\mathbb{R}^2 \Leftrightarrow$
 $\Leftrightarrow B$ linearly independent $\Leftrightarrow \det([x \ y]) \neq 0 \Leftrightarrow 2a - 1 \neq 0$
 $\Leftrightarrow a \neq 1/2 \Leftrightarrow a \in \mathbb{R} - \{1/2\}$
 we conclude that B basis of $\mathbb{R}^2 \Leftrightarrow a \in \mathbb{R} - \{1/2\}$

Example Let $B = \{x, y\}$ be a basis of \mathbb{R}^2 , define $u = x+y$ and $v = 2x$
 Show that $B_2 = \{u, v\}$ basis of \mathbb{R}^2

Solution since

$B = \{x, y\}$ basis of $\mathbb{R}^2 \Rightarrow \{x, y\}$ linearly independent
 $\Rightarrow \forall a, b \in \mathbb{R}: (ax + by = \mathbf{0} \Rightarrow a = b = 0)$ Eq(1)

► sufficient to show that $\forall a, b \in \mathbb{R}: (au + bv = \mathbf{0} \Rightarrow a = b = 0)$

Let $a, b \in \mathbb{R}$ be given and assume that $au + bv = \mathbf{0}$, then

$au + bv = \mathbf{0} \Rightarrow a(x+y) + b(2x-y) = \mathbf{0} \Rightarrow ax + ay + b2x + (-by) = \mathbf{0}$
 \Rightarrow rearrange $\Rightarrow (a+2b)x + (a-b)y = \mathbf{0} \Rightarrow$ Eq(1) $\begin{cases} a+2b=0 \\ a-b=0 \end{cases} \Rightarrow$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Eq(2) \Rightarrow has obvious solution $a=b=0$

by cramer rule, since $\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = (1)(-1) - (1)(2) = -1 - 2 = -3 \neq 0$

\Rightarrow Eq(2) has a unique solution $\Rightarrow a=b=0$

we conclude that $\forall a, b \in \mathbb{R}: (au + bv = \mathbf{0} \Rightarrow a = b = 0) \Rightarrow$

$\Rightarrow \{u, v\}$ linearly independent $\Rightarrow \{u, v\}$ basis of \mathbb{R}^2
 $| \{u, v\} | = 2 = \dim(\mathbb{R}^2)$

12/2/2019 Homework 51-59 Basis

Problem Let $(V, +, \cdot)$ be vector space and let $B \subseteq V$

Want $\dim(\text{span}(B))$?

\rightarrow IF B linearly independent $\} \Rightarrow B$ basis of $\text{span}(B)$
 $\text{span}(B) = \text{span}(B)$

$\Rightarrow \Rightarrow \underline{\dim(\text{span}(B)) = |B|}$

What if B linearly dependent?

Theorem $\{x_1, x_2, \dots, x_p\}$ linearly independent $\} \Rightarrow$
 $\forall u \in \{x_{p+1}, x_{p+2}, \dots, x_n\} : \{x_1, \dots, x_p, u\}$ linearly dependent $\}$
 $\Rightarrow \text{span}\{x_1, x_2, \dots, x_n\} = \text{span}\{x_1, x_2, \dots, x_p\}$
then it follows that
 $\dim(\text{span}\{x_1, \dots, x_n\}) = p$

Project - how to computerize this

Basis can write belonging condition for space