

12/2/2019 Lecture 28 - Basis of  $\mathbb{R}^n$  & linearly dependent \*

Recall that  $B$  basis of  $V \Leftrightarrow \begin{cases} B \text{ linearly independent} \\ \text{span}(B) = V \end{cases}$

$$\forall u \in V: \exists! \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}: u = \lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_n u_n$$

(with basis  $B = \{u_1, u_2, \dots, u_n\}$ )

①  $B$  basis of  $V \supseteq A \Rightarrow A$  linearly dependent  
 $|A| > |B|$

②  $B_1$  basis of  $V \supseteq B_2$  basis of  $V \Rightarrow |B_1| = |B_2|$

►  $\dim(V) = \text{size of any basis } B \text{ of } V$

►  $\dim(\mathbb{R}^n) = n$

$\dim(M_{nm}(\mathbb{R})) = nm$

$\dim(M_n(\mathbb{R})) = n^2$

Theorem: given vector space  $(V, +, \cdot)$  and  $A \subseteq V$ , then  
 $\begin{cases} A \text{ linearly independent} \\ |A| = \dim(V) \end{cases} \Leftrightarrow A \text{ basis of } V$

Proof ( $\Leftarrow$ ) obvious, by definition

( $\Rightarrow$ ) Assume that  $A = \{x_1, x_2, \dots, x_n\}$  linearly independent and  $\dim(V) = n$ . Sufficient to show that  $\text{span}(A) = V$

Since  $A \subseteq V \Rightarrow \text{span}(A) \subseteq V$  Eq(1)  
 $\subseteq V$  vector space

let  $u \in V$  be given. Want to show that  $u \in \text{span}(A)$

12/2/2019 Basis in  $\mathbb{R}^n$  linear independence examples

Proof of theorem: given vector space  $(V, +, \cdot)$  and  $A \subseteq V$

$$\begin{cases} A \text{ linearly independent} \\ |A| = \dim(V) \end{cases} \Leftrightarrow A \text{ basis of } V$$

Proof continued to show for  $u \in V$ ,  $u \in \text{span}(A)$

we distinguish between the following cases

Case 1 Assume that  $u \in A \Rightarrow \exists a \in [n]: u = x_a \in A \subseteq \text{span}(A)$   
 $\Rightarrow u \in \text{span}(A)$

Case 2 Assume that  $u \notin A$ . then

$$\begin{aligned} & \cancel{|A \cup \{u\}|} = |\{x_1, x_2, \dots, x_n, u\}| = n+1 > n = \dim(V) \\ \Rightarrow & |A \cup \{u\}| > \dim(V) \Rightarrow A \cup \{u\} \text{ linearly dependent} \Rightarrow \\ & A \text{ linearly independent} \end{aligned}$$

$\Rightarrow u \in \text{span}(A)$  [we show that for all  $u \in V$ ,  $u \in \text{span}(A)$ ]

we conclude that

$$(\forall u \in V: u \in \text{span}(A)) \Rightarrow V \subseteq \text{span}(A) \text{ Eq(2)}$$

from Eq(1) and Eq(2)  $\text{span}(A) = V$   $\Rightarrow A$  basis of  $V$   
 $A$  linearly independent

Examples Show that  $B = \{(a, a+1), (a+1, a+2)\}$  is basis of  $\mathbb{R}^2$   
for all  $a \in \mathbb{R}$

Solution ~~Let  $a \in \mathbb{R}$  be given~~

Define  $x = (a, a+1)$  and  $y = (a+1, a+2)$ , then

$$\det(x, y) = \begin{vmatrix} a & a+1 \\ a+1 & a+2 \end{vmatrix} = a(a+2) - (a+1)^2 = a^2 + 2a - (a^2 + 2a + 1)$$

$$= \underline{a^2} + \underline{2a} - \underline{a^2} - \underline{2a} - 1 = -1 \neq 0 \Rightarrow B = \{x, y\} \text{ linearly independent} \Rightarrow$$
$$|B| = 2 = \dim(\mathbb{R}^2)$$

$\Rightarrow B$  basis of  $\mathbb{R}^2$

12/2/2019

~~Basis~~ <sup>Basis</sup> in  $\mathbb{R}^n$  examples

Example  $B = \{(3a-1, a), (3a, a+1)\}$  Find all  $a \in \mathbb{R}$  such that  $B$  basis of  $\mathbb{R}^2$

Solution define  $x = (3a-1, a)$  and  $y = (3a, a+1)$ , then  

$$\det([x \ y]) = \begin{vmatrix} 3a-1 & 3a \\ a & a+1 \end{vmatrix} = (3a-1)(a+1) - (3a)(a) =$$
  

$$= \underline{3a^2} + \underline{3a} - \underline{a} - \underline{1} - \underline{3a^2} = 2a - 1 \neq 0$$

Since  $|B| = 2 = \dim(\mathbb{R}^2)$  it follows that  $B$  basis of  $\mathbb{R}^2 \Leftrightarrow$   
 $\Leftrightarrow B$  linearly independent  $\Leftrightarrow \det([x \ y]) \neq 0 \Leftrightarrow 2a - 1 \neq 0$   
 $\Leftrightarrow a \neq 1/2 \Leftrightarrow a \in \mathbb{R} - \{1/2\}$   
 we conclude that  $B$  basis of  $\mathbb{R}^2 \Leftrightarrow a \in \mathbb{R} - \{1/2\}$

Example Let  $B = \{x, y\}$  be a basis of  $\mathbb{R}^2$ , define  $u = x+y$  and  $v = 2x$   
 Show that  $B_2 = \{u, v\}$  basis of  $\mathbb{R}^2$

Solution since

$B = \{x, y\}$  basis of  $\mathbb{R}^2 \Rightarrow \{x, y\}$  linearly independent  
 $\Rightarrow \forall a, b \in \mathbb{R}: (ax + by = \mathbf{0} \Rightarrow a = b = 0)$  Eq(1)

► sufficient to show that  $\forall a, b \in \mathbb{R}: (au + bv = \mathbf{0} \Rightarrow a = b = 0)$

Let  $a, b \in \mathbb{R}$  be given and assume that  $au + bv = \mathbf{0}$ , then

$au + bv = \mathbf{0} \Rightarrow a(x+y) + b(2x-y) = \mathbf{0} \Rightarrow ax + ay + b2x + (-by) = \mathbf{0}$   
 $\Rightarrow$  rearrange  $\Rightarrow (a+2b)x + (a-b)y = \mathbf{0} \Rightarrow$  Eq(1)  $\begin{cases} a+2b=0 \\ a-b=0 \end{cases} \Rightarrow$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  Eq(2)  $\Rightarrow$  has obvious solution  $a=b=0$

by cramer rule, since  $\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = (1)(-1) - (1)(2) = -1 - 2 = -3 \neq 0$

$\Rightarrow$  Eq(2) has a unique solution  $\Rightarrow a=b=0$

we conclude that  $\forall a, b \in \mathbb{R}: (au + bv = \mathbf{0} \Rightarrow a = b = 0) \Rightarrow$

$\Rightarrow \{u, v\}$  linearly independent  $\Rightarrow \{u, v\}$  basis of  $\mathbb{R}^2$   
 $| \{u, v\} | = 2 = \dim(\mathbb{R}^2)$

12/2/2019 Homework 51-59 Basis

Problem Let  $(V, +, \cdot)$  be vector space and let  $B \subseteq V$

Want  $\dim(\text{span}(B))$ ?

$\rightarrow$  If  $B$  linearly independent  $\} \Rightarrow B$  basis of  $\text{span}(B)$   
 $\text{span}(B) = \text{span}(B)$

$\Rightarrow \Rightarrow \underline{\dim(\text{span}(B)) = |B|}$

What if  $B$  linearly dependent?

Theorem  $\{x_1, x_2, \dots, x_p\}$  linearly independent  $\} \Rightarrow$

$\forall u \in \{x_{p+1}, x_{p+2}, \dots, x_n\} : \{x_1, \dots, x_p, u\}$  linearly dependent

$\Rightarrow \text{span}\{x_1, x_2, \dots, x_n\} = \text{span}\{x_1, x_2, \dots, x_p\}$

then it follows that

$$\dim(\text{span}\{x_1, \dots, x_n\}) = p$$

Project - how to computerize this

Basis can write belonging condition for space