12/2/2019 Lecture 28 - Basis of $\mathbb{R}^n$ & linearly dependent &

Recall that $B$ basis of $V \iff \begin{cases} B \text{ linearly independent} \\ \text{span}(B) = V \end{cases}$

$$\boxed{\forall u \in V: \exists! \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}: u = \lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_n u_n \quad (\text{with basis } B = \{u_1, u_2, \dots, u_n\})}$$

① $\left.\begin{array}{l} B \text{ basis of } V \\ |A| > |B| \end{array}\right\} \Rightarrow A$ linearly dependent

② $\left.\begin{array}{l} B_1 \text{ basis of } V \\ B_2 \text{ basis of } V \end{array}\right\} \Rightarrow |B_1| = |B_2|$

- $\dim(V)$ = size of any basis $B$ of $V$
- $\dim(\mathbb{R}^n) = n$

  $\dim(M_{nm}(\mathbb{R})) = nm$

  $\dim(M_n(\mathbb{R})) = n^2$

<u>theorem</u>: given vector space $(V, +, \cdot)$ and $A \subseteq V$, then

$\begin{cases} A \text{ linearly independent} \\ |A| = \dim(V) \end{cases} \iff A$ basis of $V$

<u>Proof</u> ($\Leftarrow$) obvious, by definition

($\Rightarrow$) Assume that $A = \{x_1, x_2, \dots, x_n\}$ linearly independent and $\dim(V) = n$. Sufficient to show that $\text{span}(A) = V$

Since $\begin{cases} A \subseteq V \\ V \text{ vector space} \end{cases} \Rightarrow \text{span}(A) \subseteq V$ Eq(1)

Let $u \in V$ be given. Want to show that $u \in \text{span}(A)$

12/2/2019 Basis in $\mathbb{R}^n$ linear independence examples

Proof of theorem: given vector space $(V, +, \cdot)$ and $A \subseteq V$

$$\left\{\begin{array}{l} A \text{ linearly independent} \\ |A| = \dim(V) \end{array}\right. \Leftrightarrow A \text{ basis of } V$$

Proof continued to show for $u \in V$, $u \in \text{span}(A)$

we distinguish between the following cases

Case 1 Assume that $u \in A \Rightarrow \exists a \in [n] : u = x_a \in A \subseteq \text{span}(A)$

$$\Rightarrow \underline{u \in \text{span}(A)}$$

Case 2 Assume that $u \notin A$. then

$$|A \cup \{u\}| = |\{x_1, x_2, \ldots, x_n, u\}| = n+1 > n = \dim(V)$$

$\Rightarrow |A \cup \{u\}| > \dim(V) \Rightarrow \left.\begin{array}{l} A \cup \{u\} \text{ linearly dependent} \\ A \text{ linearly independent} \end{array}\right\} \Rightarrow$

$\Rightarrow u \in \text{span}(A)$, [we show that for all $u \in V$, $u \in \text{span}(A)$]

we conclude that

$$(\forall u \in V : u \in \text{span}(A)) \Rightarrow V \subseteq \text{span}(A) \quad \text{Eq (2)}$$

From Eq(1) and Eq(2) $\left.\begin{array}{l} \text{span}(A) = V \\ A \text{ linearly independent} \end{array}\right\} \Rightarrow A$ basis of $V$

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Examples Show that $B = \{(a, a+1), (a+1, a+2)\}$ is basis of $\mathbb{R}^2$ for all $a \in \mathbb{R}$

Solution Let $a \in \mathbb{R}$ be given

Define $x = (a, a+1)$ and $y = (a+1, a+2)$, then

$$\det([x\,y]) = \begin{vmatrix} a & a+1 \\ a+1 & a+2 \end{vmatrix} = a(a+2) - (a+1)^2 = a^2 + 2a - (a^2 + 2a + 1)$$

$= a^2 + 2a - a^2 - 2a - 1 = -1 \neq 0 \Rightarrow \left.\begin{array}{l} B = \{x, y\} \text{ linearly independent} \\ |B| = 2 = \dim(\mathbb{R}^2) \end{array}\right\} \Rightarrow$

$\Rightarrow B$ basis of $\mathbb{R}^2$

12/2/2019 Basis in $\mathbb{R}^n$ examples

Example $B = \{(3a-1, a), (3a, a+1)\}$ Find all $a \in \mathbb{R}$ such that $B$ basis of $\mathbb{R}^2$

Solution define $x = (3a-1, a)$ and $y = (3a, a+1)$, then

$$\det([x \ y]) = \begin{vmatrix} 3a-1 & 3a \\ a & a+1 \end{vmatrix} = (3a-1)(a+1) - (3a)(a) =$$

$$= 3a^2 + 3a - a - 1 - 3a^2 = 2a - 1 \neq 0$$

Since $|B| = 2 = \dim(\mathbb{R}^2)$ it follows that $B$ basis of $\mathbb{R}^2 \Leftrightarrow$

$\Leftrightarrow B$ linearly independent $\Leftrightarrow \det([x \ y]) \neq 0 \Leftrightarrow 2a-1 \neq$

$\Leftrightarrow a \neq 1/2 \Leftrightarrow a \in \mathbb{R} - \{1/2\}$

We conclude that $B$ basis of $\mathbb{R}^2 \Leftrightarrow a \in \mathbb{R} - \{1/2\}$

Example Let $B = \{x, y\}$ be a basis of $\mathbb{R}^2$, define $u = x+y$ and $v = 2x$
Show that $B_2 = \{u, v\}$ basis of $\mathbb{R}^2$

Solution since

$B = \{x, y\}$ basis of $\mathbb{R}^2 \Rightarrow \{x, y\}$ linearly independ

$\Rightarrow \forall a, b \in \mathbb{R} : (ax + by = \mathbf{0} \Rightarrow a = b = 0)$ Eq(1)

sufficient to show that $\forall a, b \in \mathbb{R} : (au + bv = \mathbf{0} \Rightarrow a = b = 0)$

Let $a, b \in \mathbb{R}$ be given and assume that $au + bv = \mathbf{0}$, then

$au + bv = \mathbf{0} \Rightarrow a(x+y) + b(2x-y) = \mathbf{0} \Rightarrow ax + ay + b2x + (-by) = \mathbf{0}$

$\Rightarrow$ rearrange $\Rightarrow (a+2b)x + (a-b)y = \mathbf{0} \Rightarrow$ Eq(1) $\begin{cases} a+2b=0 \\ a-b=0 \end{cases} \Rightarrow$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ Eq(2) $\Rightarrow$ has obvious solution $a = b = 0$

by Cramer rule, since $\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = ((1)(-1) - (1)(2)) = -1 - 2 = -$

$\Rightarrow$ Eq(2) has a unique solution $\Rightarrow a = b = 0$

We conclude that $\forall a, b \in \mathbb{R} : (au + bv = \mathbf{0} \Rightarrow a = b = 0) \Rightarrow$

$\left.\begin{matrix} \Rightarrow \{u, v\} \text{ linearly independent} \\ |\{u, v\}| = 2 = \dim(\mathbb{R}^2) \end{matrix}\right\} \Rightarrow \{u, v\}$ basis of $\mathbb{R}$

12/2/2019 $\boxed{\text{Homework S1 - S9 Basis}}$

**Problem** Let $(V, +, \cdot)$ be vector space and let $B \subseteq V$

Want $\dim(\text{span}(B))$?

$$\left.\begin{array}{r}\text{If } B \text{ linearly independent} \\ \text{span}(B) = \text{span}(B)\end{array}\right\} \Rightarrow B \text{ basis of } \text{span}(B)$$

$\Rightarrow$

$\Rightarrow$ $\Rightarrow$ $\underline{\dim(\text{span}(B)) = |B|}$

<u>What if $B$ linearly dependent?</u>

**theorem**
$$\left.\begin{array}{l}\{x_1, x_2, \ldots, x_p\} \text{ linearly independent} \\ \forall u \in \{x_{p+1}, x_{p+2}, \ldots, x_n\} : \{x_1, \ldots, x_p, u\} \text{ linearly dependent}\end{array}\right\} \Rightarrow$$

$\Rightarrow \text{span}\{x_1, x_2, \ldots, x_n\} = \text{span}\{x_1, x_2, \ldots, x_p\}$

then it follows that

$$\dim(\text{span}\{x_1, \ldots, x_n\}) = p$$

<u>Project</u> - how to computerize this

Basis can write belonging condition for space

12/4/2019 Lecture 29, Dimension of span(B)

send project as PDF via email, pages vary - figures, pictures

due weds Dec 11 23:59

- returned exam 3

Given $(V, +, \cdot)$ vector space: Let $B \subseteq V$,

$$\boxed{B \text{ linearly independent} \Rightarrow \dim(\text{span}(B)) = |B|}$$

Theorem: For $p < n$ and:

$$\begin{cases} \{x_1, x_2, \ldots, x_p\} \text{ linearly independent} \\ \forall u \in \{x_{p+1}, \ldots, x_n\} : \{x_1, \ldots, x_p, u\} \text{ linearly depende} \end{cases}$$

$$\Rightarrow \text{span}\{x_1, \ldots, x_n\} = \text{span}\{x_1, \ldots, x_p\}$$

Proof since: $\{x_1, \ldots, x_p\} \subseteq \{x_1, \ldots, x_n\} \Rightarrow$

$$\Rightarrow \text{span}\{x_1, \ldots, x_p\} \subseteq \text{span}\{x_1, \ldots, x_n\} \quad (1)$$

- Preliminary argument - let $a \in \mathbb{N}$ with $1 \leq a \leq n-p$ and consider $x_{p+a}$

It follows that $\begin{cases} \{x_1, \ldots, x_p\} \text{ linearly independent} \\ \{x_1, \ldots, x_p, x_{p+a}\} \text{ linearly dependent} \end{cases} \Rightarrow$

$$\Rightarrow x_{p+a} \in \text{span}\{x_1, \ldots, x_p\} \Rightarrow$$

$$\Rightarrow \exists \mu_{a1}, \mu_{a2}, \ldots, \mu_{ap} \in \mathbb{R} : x_{p+a} = \sum_{b=1}^{p} \mu_{ab} x_b \ldots$$

cont.

12/4/2019 Dimension of span(B)

**Prop:** Let $B = \{x_1, \ldots, x_p\} \subseteq V$ and assume that $B$ is basis of $V$ then...

$$u \in V \iff \{x_1, \ldots, x_p, u\} \text{ linearly dependent}$$

**Proof:** ($\Rightarrow$): Assume that $u \in V$, then

$u \in V \Rightarrow u \in \text{span}(B)$ ← [via $\text{span}(B) = V$]

$\Rightarrow u \in \text{span}(\{x_1, \ldots, x_p\})$

$\Rightarrow \{x_1, \ldots, x_p, u\}$ linearly dependent

($\Leftarrow$): Assume that $\{x_1, \ldots, x_p, u\}$ linearly dependent, since $\{x_1, \ldots, x_p\}$ basis of $V$ $\Rightarrow$

$$\Rightarrow \begin{cases} \{x_1, \ldots, x_p\} \text{ linearly independent} \\ \{x_1, \ldots, x_p, u\} \text{ linearly dependent} \end{cases} \Rightarrow$$

$\Rightarrow u \in \text{span}(\{x_1, \ldots, x_p\}) \Rightarrow u \in \text{span}(B)$

$\Rightarrow u \in V$ (via $V = \text{span}(B)$)

**Example** $V = \text{span}\{x_1, x_2, x_3, x_4\}$ $x_1 = (1,2,0,3)$ $x_2 = (2,0,3,1)$ $x_3 = (-1,2,-3,2)$ $x_4 = (3,-2,6,-1)$

**Find $\dim(V)$ condition $(a,b,c,d) \in V$**

**Solution** ▸ check $\{x_1, x_2, x_3, x_4\}$ for linear dependence/independence

$$\det[x_1 \ x_2 \ x_3 \ x_4] = \begin{vmatrix} 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 1 \\ -1 & 2 & -3 & 2 \\ 3 & -2 & 6 & -1 \end{vmatrix} = \ldots = 0 \Rightarrow$$

transpose either works

$\Rightarrow \{x_1, x_2, x_3, x_4\}$ dependent

▸ now check $\{x_1, x_2, x_3\}$

↓ cont.