11/25/2019 Lecture 26: Linear Independence

▶ Recall

Let V be vector space and $A = \{x_1, x_2, \dots, x_n\} \subseteq V$

A linearly dependent $\Leftrightarrow$

$$\Leftrightarrow \exists (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n : \begin{cases} \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n = \mathbf{0} \\ (\lambda_1, \lambda_2, \dots, \lambda_n) \neq \mathbf{0} \end{cases}$$

A linearly independent $\Leftrightarrow$

$$\Leftrightarrow \forall (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n : (\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n = \mathbf{0} \Rightarrow$$
$$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0)$$

@ Example Let $f, g \in \mathcal{F}(\mathbb{R})$ with $\forall x \in \mathbb{R} : (f(x) = \sin x \wedge g(x) = \cos x)$
Show that $\{f, g\}$ linearly independent

Solution

▶ Sufficient to show that $\forall a, b \in \mathbb{R} : (af + bg = \mathbf{0} \Rightarrow a = b = 0)$
Let $a, b \in \mathbb{R}$ be given. Assume $af + bg = \mathbf{0}$ .... then,

$af + bg = \mathbf{0} \Rightarrow \forall x \in \mathbb{R} : (af + bg)(x) = 0 \Rightarrow \forall x \in \mathbb{R} : af(x) + bg(x) = 0$

Eq(1) $\Rightarrow \forall x \in \mathbb{R} : a\sin(x) + b\cos(x) = 0$ — can prove by contradiction or pick 2 arbitrary x, system o

For $x = 0$, Eq(1) gives: $a\sin 0 + b\cos 0 = 0 \Rightarrow 0a + 1b = 0 \Rightarrow \mathbf{b = 0}$
For $x = \pi/2$, Eq(1) gives: $a\sin(\pi/2) + b\cos(\pi/2) = 0 \Rightarrow 1a + 0b = 0 \Rightarrow \mathbf{a = 0}$

We conclude that $\mathbf{a = b = 0}$. It follows that

$\forall a, b \in \mathbb{R} : (af + bg = \mathbf{0} \Rightarrow a = b = 0) \Rightarrow \{f, g\}$ linearly independent

11/25/2019 <u>Linear Independence Examples</u>

(b) <u>Example</u> Let $f, g \in F(\mathbb{R})$ with $\forall x \in \mathbb{R}: (f(x) = 2x \wedge g(x) = x^2)$
Show that $\{f, g\}$ linearly independent

<u>Solution</u>

▶ Sufficient to show that $\forall a, b \in \mathbb{R}: (af + bg = \mathbf{0} \Rightarrow a = b = 0)$
Let $a, b \in \mathbb{R}$ be given, assume that $af + bg = \mathbf{0}$, then

$af + bg = \mathbf{0} \Rightarrow \forall x \in \mathbb{R}: (af + bg)(x) = 0 \Rightarrow \forall x \in \mathbb{R}: af(x) + bg(x) = 0 \Rightarrow$

(1) $\Rightarrow \forall x \in \mathbb{R}: 2ax + bx^2 = 0$

For $x = 1$ Eq(1) gives $\underline{2a + b = 0}$, Eq(2) Eq(3)
For $x = 2$ Eq(1) gives $2a \cdot 2 + b \cdot 2^2 = 0 \Rightarrow 4a + 4b = 0 \Rightarrow \underline{a + b = 0}$
From Eq(2) and Eq(3):

$$\begin{cases} 2a + b = 0 \\ a + b = 0 \end{cases} \Rightarrow \begin{cases} a + (a+b) = 0 \\ a + b = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ a + b = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \end{cases} \Rightarrow a = b = 0$$

We conclude that $\forall a, b \in \mathbb{R}: (af + bg = \mathbf{0} \Rightarrow a = b = 0) \Rightarrow$
$\Rightarrow \{f, g\}$ linearly independent

▶ In general $(\forall x \in \mathbb{R}: a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0) \Leftrightarrow$
$\Leftrightarrow \forall k \in [n]: a_k = 0$

Can be proved for theoretical project
$n \times n$ system, calculate determinant
or proof by induction

11/25/2019 <u>Linear Dependence Examples</u>

Recall A dependent $\Leftrightarrow \exists x \in A : x \in \text{span}(A - \{x\})$

© <u>Example</u> Define $f, g, h \in F(\mathbb{R})$ with
$\forall x \in \mathbb{R} : (f(x) = \cos x \wedge g(x) = \cos x \cos(2x) \wedge h(x) = \sin x \sin(2x$

<u>Remember:</u> $\cos(a \pm b) = \cos a \cos b \pm \sin a \sin b$

Show that $\{f, g, h\}$ linearly dependent

<u>Solution</u> $f(x) = \cos x = \cos(2x - x) = \cos(2x)\cos x + \sin(2x)\sin x =$
$= g(x) + h(x) = (g+h)(x), \; \forall x \in \mathbb{R} \Rightarrow$

$\Rightarrow f = g + h \Rightarrow f \in \text{span}\{g, h\} \Rightarrow$ $\{f, g, h\}$ linearly dependent

Homework 34 - 40*, note <u>40</u> could be a project or <u>41</u>.

<u>Linear dependence / independence of geometric vectors in $\mathbb{R}^n$</u>

<u>Definition</u> Let $A = \{x_1, x_2, \dots, x_k\} \subseteq \mathbb{R}^n$. We define

$\text{Mat}(A) = \begin{bmatrix} x_1 & x_2 & \cdots & x_k \end{bmatrix} \in M_{nk}(\mathbb{R})$
such that
$\forall a \in [n] : \forall b \in [k] : [\text{Mat}(A)]_{ab} = (x_b)_a$

<u>Definition</u> Let $M \in M_{nk}(\mathbb{R})$ with $k \leq n$. We define $\text{Sub}(M)$ to be the set of all matrices $A \in M_k(\mathbb{R})$ obtained by deleting abitrary choice of $n-k$ rows.

11/25/2019 Mat(A) & sub(A) example / purpose

<u>Example</u> Let $x_1 = (2,5,3,1)$ and $x_2 = (3,1,4,7)$

Let $A = \{x_1, x_2\}$ $\text{Mat}(A) = [x_1 \;\; x_2] = \begin{bmatrix} 2 & 3 \\ 5 & 1 \\ 3 & 4 \\ 1 & 7 \end{bmatrix} \Rightarrow$

$\Rightarrow \text{Sub}(\text{Mat}(A)) = \left\{ \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}, \right.$

$\left. \begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 1 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix} \right\}$

▶ <u>Theorem</u> Let $A = \{x_1, x_2, \ldots, x_k\} \subseteq \mathbb{R}^n$ with $k \leq n$

a) A linearly independent $\Leftrightarrow \exists M \in \text{Sub}(\text{Mat}(A)) : \det(M) \neq 0$

b) A linearly dependent $\Leftrightarrow \forall M \in \text{Sub}(\text{Mat}(A)) : \det(M) = 0$

For $k = n$, then $\text{Sub}(\text{Mat}(A)) = \{\text{Mat}(A)\}$, and therefore

A linearly independent $\Leftrightarrow \det(\text{Mat}(A)) \neq 0$

A linearly dependent $\Leftrightarrow \det(\text{Mat}(A)) = 0$

<u>Example</u> Let $x = (3,9,1)$ and $y = (a, 2a-1, 1-3a)$

Find all values of a such that x, y linearly dependent

<u>Solution</u> Define $M = [x \;\; y] = \begin{bmatrix} 3 & a \\ 9 & 2a-1 \\ 1 & 1-3a \end{bmatrix} \Rightarrow \text{Sub}(M) = \{$

$\Rightarrow \text{Sub}(M) = \left\{ \begin{bmatrix} 3 & a \\ 9 & 2a-1 \end{bmatrix}, \begin{bmatrix} 3 & a \\ 1 & 1-3a \end{bmatrix}, \begin{bmatrix} 9 & 2a-1 \\ 1 & 1-3a \end{bmatrix} \right\} = \{M_1, M_2, M_3\}$

Find det of each matrix M_1, M_2, M_3

11/25/2019 Application of Mat(A)/Sub(A) in Parametric Eqn

Example let $x = (3, 9, 1)$ $y = (a, 2a-1, 1-3a)$
Find all values of a such that x, y linearly dependent

Solution $\text{sub}(M) = \left\{ \begin{bmatrix} 3 & a \\ 9 & 2a-1 \end{bmatrix}, \begin{bmatrix} 3 & a \\ 1 & 1-3a \end{bmatrix}, \begin{bmatrix} 9 & 2a-1 \\ 1 & 1-3a \end{bmatrix} \right\}$

$\text{sub}(M) = \{ M_1 , M_2 , M_3 \}$

then $\det(M_1) = \begin{vmatrix} 3 & a \\ 9 & 2a-1 \end{vmatrix} = 3(2a-1) - 9a =$

$= 6a - 3 - 9a =$ $\boxed{= -3a - 3}$

$\det(M_2) = \begin{vmatrix} 3 & a \\ 1 & 1-3a \end{vmatrix} = 3(1-3a) - a \cdot 1 =$

$= 3 - 9a - a =$ $\boxed{= 3 - 10a}$

$\det(M_3) = \begin{vmatrix} 9 & 2a-1 \\ 1 & 1-3a \end{vmatrix} = 9(1-3a) - 1(2a-1) -$

$= 9 - 27a - 2a + 1 =$ $\boxed{= 10 - 29a}$

then $\{x, y\}$ linearly dependent $\Leftrightarrow$

$\Leftrightarrow \det(M_k) = 0, \forall k \in \{1, 2, 3\} \Leftrightarrow$

$\Leftrightarrow \det(M_1) = 0 \wedge \det(M_2) = 0 \wedge \det(M_3) = 0 \Leftrightarrow$

$\Leftrightarrow -3a - 3 = 0 \wedge 3 - 10a = 0 \wedge 10 - 29a = 0 \Leftrightarrow$

$\Leftrightarrow \left. a = -1 \wedge a = \frac{3}{10} \wedge a = \frac{10}{29} \right\}$ impossible, contradiction

we conclude that $\forall a \in \mathbb{R}$: $\{x, y\}$ linearly independent

11/27/2019 Lecture 27. Linear Independence in $\mathbb{R}$

Exam 3 given - due Mon Dec 2 in class
Q 5 linear independence - do not use Wronskian

Final Project - LaTeX optional - due weds Dec 11 23
email PDF

- Recall that $A = \{x_1, x_2, \ldots, x_k\} \subseteq \mathbb{R}^n$ then

A linearly independent $\Leftrightarrow \exists M \in Sub(Mat(A)) : det(M)$

A linearly dependent $\Leftrightarrow \forall M \in Sub(Mat(A)) : det(M) = C$

- When $n = K$... A linearly independent $\Leftrightarrow det(Mat(A)) \neq 0$
A linearly dependent $\Leftrightarrow det(Mat(A)) = 0$

Example: Let $x = (1,2,1)$ and $y = (1,1,0)$ and $z = (a, 2a+3,$
Find all $a \in \mathbb{R}$ such that $\{x, y, z\}$ linearly independe

Solution Since $det([x \; y \; z]) = \begin{vmatrix} 1 & 1 & a \\ 2 & 1 & 2a+3 \\ 1 & 0 & 2 \end{vmatrix}$ (-1) second col will be 1, 0,

$$= \begin{vmatrix} 1 & 1 & a \\ 2+1(-1) & 0 & 2a+3-a \\ 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & a \\ 1 & 0 & a+3 \\ 1 & 0 & 2 \end{vmatrix}$$ make minor

$$= (-1)(1)\begin{vmatrix} 1 & a+3 \\ 1 & 2 \end{vmatrix} = -[1 \cdot 2 - 1 \cdot (a+3)] = -(2-a-3)$$

$$= -(-1-a) = a+1$$

11/27/2019 Linear Independence Examples, + Basis

(d) Example Let $x=(1,2,1)$ and $y=(1,1,0)$ and $z=(a, 2a+3, 2)$
Find all $a \in \mathbb{R}$ such that $\{x,y,z\}$ linearly independent

Solution Since $\det([x\ y\ z]) = a+1$ it follows that

$\{x,y,z\}$ linearly independent $\Leftrightarrow \det([x\ y\ z]) \neq 0 \Leftrightarrow$

$\Leftrightarrow a+1 \neq 0 \Leftrightarrow a \neq -1 \Leftrightarrow \underline{a \in \mathbb{R} - \{-1\}}$

Homework 43-50 ↑↑↑ everything covered up to now is on Exam 3

▶ Basis of a vector space

Definition: Let $(V,+,\cdot)$ be a vector space and let $B = \{x_1, x_2, \ldots, x_n\} \subseteq V$

B is basis of V $\Leftrightarrow \begin{cases} B \text{ linearly independent} \\ \text{span}(B) = V \end{cases}$

notation $|B| = |\{x_1, x_2, \ldots, x_n\}| = n$ (cardinality)

theorem Assume that $B = \{x_1, \ldots, x_n\}$ basis of vector space V then

$$\forall u \in V : \exists! (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : u = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n$$

For every vector in space, there will exist unique #s such that you can use those numbers to recreate $\vec{u}$ (coordinate system)

11/27/2019 **Basis of a Vector Space**

Theorem Assume $B = \{x_1, \ldots, x_n\}$ basis of vector space V then

$$\forall u \in V : \exists! (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : u = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x$$

Proof Since B basis of $V \Rightarrow \begin{cases} B \text{ linearly independent} & \text{Eq(1)} \\ \text{span}(B) = V & \text{Eq(2)} \end{cases}$

Let $u \in V$ be given, then...

$$\begin{aligned} u \in V &\Rightarrow u \in \text{span}(B) \quad [\text{from Eq(2)}] \\ &\Rightarrow u \in \{\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \mid (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n\} \\ &\Rightarrow \exists (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : u = \lambda_1 x_1 + \ldots + \lambda_n x_n \end{aligned}$$

To show uniqueness, assume that for $(\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n$ we have

$$u = \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 + \ldots + \mu_n x_n$$

Since $\sum_{a=1}^{n} (\lambda_a - \mu_a) x_a = \sum_{a=1}^{n} \lambda_a x_a - \sum_{a=1}^{n} \mu_a x_a = u - u = 0 \Rightarrow$

$\Rightarrow \forall a \in [n] : \lambda_a - \mu_a = 0 \quad [\text{from Eq(1)}] \Rightarrow \forall a \in [n] : \lambda_a = \mu_a$

This establishes uniqueness of $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, this concludes the argument

11/27/2019 **Dimension of Vector space**

theorem Let $(V, +, \cdot)$ be vector space, let $B \subseteq V$ and $A \subseteq V$ then

$$\begin{cases} B \text{ basis of } V \\ |A| > |B| \end{cases} \Rightarrow A \text{ linearly dependent}$$

Proof Define $A = \{x_1, x_2, \dots, x_n\}$ and $B = \{y_1, y_2, \dots, y_n\}$ with $|A| > |B| \Rightarrow n > m$

Since B basis of $V \Rightarrow \begin{cases} B \text{ linearly independent} & \text{Eq(1)} \\ \text{span}(B) = V & \text{Eq(2)} \end{cases}$

▶ Sufficient to show that $\exists (\lambda_1, \dots, \lambda_n) : \begin{cases} \lambda_1 x_1 + \dots + \lambda_n x_n = 0 & \text{Eq(3)} \\ (\lambda_1, \dots, \lambda_n) \neq 0 \end{cases}$

Since $(\forall a \in [n] : x_a \in V) \Rightarrow$

$\Rightarrow (\forall a \in [n] : x_a \in \text{span}\{y_1, y_2, \dots, y_m\}) \Rightarrow$

$\Rightarrow \forall a \in [n] : \exists (M_{a1}, M_{a2}, \dots, M_{am}) \in \mathbb{R}^m : x_a = \sum_{b=1}^{m} M_{ab} y_b$

it follows that $\lambda_1 x_1 + \dots + \lambda_n x_n = \sum_{a=1}^{n} \lambda_a x_a =$

$$= \sum_{a=1}^{n} \lambda_a \left[\sum_{b=1}^{m} M_{ab} y_b \right] = \sum_{a=1}^{n} \sum_{b=1}^{m} \lambda_a M_{ab} y_b = \sum_{b=1}^{m} \left[\sum_{a=1}^{n} \lambda_a M_{ab} \right] y_b =$$

$$= 0 \Leftrightarrow \forall b \in [m] : \sum_{a=1}^{n} \lambda_a M_{ab} = 0 \quad [\text{Eq (1)}]$$

nned) Since this is an underdetermined system it either has no solutions $(\lambda_1, \dots, \lambda_n)$ or has infinite solutions

11/27/2019 Proofs of Basis

<u>theorem</u> Let $(V, +, \cdot)$ be vector space, let $B \subseteq V$ and $A \subseteq V$ then

$$\begin{cases} B \text{ basis of } V \\ |A| > |B| \end{cases} \Rightarrow A \text{ linearly dependent}$$

↑ (continued)

<u>Proof</u> Since $(\lambda_1, \dots, \lambda_n) = 0$ is an obvious solution, then there exist additional solutions $(\lambda_1, \dots, \lambda_n) \neq 0$ such that $\lambda_1 X_1 + \dots + \lambda_n X_n = 0$

From Eq(3) $\Rightarrow \{X_1, \dots, X_n\}$ linearly dependent

<u>theorem</u> $\begin{cases} B_1 \text{ basis of } V \\ B_2 \text{ basis of } V \end{cases} \Rightarrow |B_1| = |B_2|$

<u>Solution</u> Assume that $B_1 B_2$ basis of V, to show a contradict assume that $|B_1| > |B_2|$ with no loss generality

It follows that $\begin{cases} B_2 \text{ basis of } V \\ |B_1| > |B_2| \end{cases} \Rightarrow B_1$ linearly dependent $\Rightarrow B_1$ not a basis of V

↗ contradiction!

We conclude that $|B_1| = |B_2|$

<u>notation</u> $\dim(V) = |B|$ for any basis B of V

for R^n: $e_1 = (1, 0, \dots, 0)$
$e_2 = (0, 1, \dots, 0)$
$\vdots$
$e_n = (0, 0, \dots, 1)$

↓ continued

11/27/2019 Linear Dependence in basis of $\mathbb{R}^n$

<u>notation</u> $\dim(V) = |B|$ for any basis B of V

For $\mathbb{R}^n$:

$$e_1 = (1, 0, \dots, 0)$$
$$e_2 = (0, 1, \dots, 0)$$
$$\vdots$$
$$e_n = (0, 0, \dots, 1)$$

$B = \{e_1, \dots, e_n\}$ basis of $\mathbb{R}^n \Rightarrow \dim(\mathbb{R}^n) = n$

<u>corollary</u> $|A| > \dim(V) \Rightarrow A$ linearly dependent