11/18/2019 **Lecture 24**, subspace examples

(c) **Example** Let $A \in M_n(\mathbb{R})$ and let $V = \{X \in M_n(\mathbb{R}) \mid AX = X$

Show that V subspace of $M_n(\mathbb{R})$

Solution Let $\lambda, \mu \in \mathbb{R}$ be given, let $X, Y \in V$ be given

▶ Sufficient to show that $\lambda X + \mu Y \in V$

Since $X \in V \Rightarrow AX = XA$

$Y \in V \Rightarrow AY = YA$

It follows that

$$\begin{aligned} A(\lambda X + \mu Y) &= A(\lambda X) + A(\mu Y) \quad \text{distributive} \\ &= \lambda(AX) + \mu(AY) \\ &= \lambda(XA) + \mu(YA) \quad [\text{via } X, Y \in V] \\ &= (\lambda X)A + (\mu Y)A \\ &= (\lambda X + \mu Y) A \Rightarrow \end{aligned}$$

$\Rightarrow \lambda X + \mu Y \in V$

we conclude that

$\forall \lambda, \mu \in \mathbb{R} : \forall X, Y \in V : \lambda X + \mu Y \in V \Rightarrow V$ subspace of $M_n(\mathbb{R})$

(d) **Example** Show that $V = \{f \in \mathcal{F}(\mathbb{R}) \mid f \text{ even}\}$ is a subspace of F

Solution Let $\lambda, \mu \in \mathbb{R}$ and let $f, g \in V$ be given

Then: $f, g \in V \Rightarrow \begin{cases} f \text{ even} \\ g \text{ even} \end{cases} \Rightarrow \begin{cases} \forall x \in \mathbb{R} : f(-x) = f(x) \\ \forall x \in \mathbb{R} : g(-x) = g(x) \end{cases}$ Eq (1

▶ Sufficient to show that

$\forall x \in \mathbb{R} : (\lambda f + \mu g)(-x) = (\lambda f + \mu g)(x)$

11/18/2019 <u>Subspace Examples.</u>

ⓓ <u>Example</u> Show that $V = \{ f \in F(\mathbb{R}) \mid f \text{ even} \}$ is a subspace of $F(\mathbb{R})$

<u>on</u> since

$$\forall x \in \mathbb{R}: (\lambda f + \mu g)(-x) = (\lambda f)(-x) + (\mu g)(-x)$$
$$= \lambda f(-x) + \mu g(-x)$$
$$= \lambda f(x) + \mu g(x) \quad [\text{via Eq } (1)]$$
$$= (\lambda f)(x) + (\mu g)(x)$$
$$= (\lambda f + \mu g)(x) \Rightarrow$$

$\Rightarrow \lambda f + \mu g$ even $\Rightarrow$ <u>$\lambda f + \mu g \in V$</u>

we conclude that:

$$\forall \lambda, \mu \in \mathbb{R}: \forall f, g \in V: \lambda f + \mu g \in V \Rightarrow V \text{ subspace of } F(\mathbb{R})$$

▶ <u>Subspaces spanned by vectors.</u>

Let $(V, +, \cdot)$ be a vector space and $x_1, x_2, \ldots, x_n \in V$

<u>tion</u> we define for $A = \{x_1, x_2, \ldots, x_n\}$

$$\text{span}(A) = \left\{ \sum_{k=1}^{n} \lambda_k x_k \;\middle|\; \forall a \in [n]: \lambda_a \in \mathbb{R} \right\}$$

$$= \text{span}\{x_1, x_2, \ldots, x_n\}$$

<u>ark</u> Belonging condition for span(A) is

$$x \in \text{span}(A) \Leftrightarrow \exists \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}: x = \sum_{a=1}^{n} \lambda_a x_a$$

11/18/2019 **Subspaces spanned by vectors.**

Theorem:

$$\begin{cases} A = \{x_1, x_2, \ldots, x_n\} \subseteq V \\ (V, +, \cdot) \text{ vector space} \end{cases} \Rightarrow \text{span}(A) \text{ subspace of } V$$

Property: Let $A \subseteq V$ and $B \subseteq V$ with $(V,+,\cdot)$ vector space

ⓐ $A \subseteq \text{span}(A)$

ⓑ $A \subseteq B \Rightarrow \text{span}(A) \subseteq \text{span}(B)$

Proof of the theorem Let $A = \{x_1, x_2, \ldots, x_n\}$

Let $\lambda, \mu \in \mathbb{R}$ and $x, y \in \text{span}(A)$ be given

▶ Sufficient to show that $\lambda x + \mu y \in \text{span}(A)$

Since

$$x \in \text{span}(A) \Rightarrow \exists p_1, p_2, \ldots, p_n \in \mathbb{R} : x = \sum_{a=1}^{n} p_a x_a \quad (1)$$

$$y \in \text{span}(A) \Rightarrow \exists q_1, q_2, \ldots, q_n \in \mathbb{R} : y = \sum_{a=1}^{n} q_a x_a \quad (2)$$

It follows that $\lambda x + \mu y = \lambda \sum_{a=1}^{n} p_a x_a + \mu \sum_{a=1}^{n} q_a x_a$ [via Eq(1)/Eq(2)]

$$= \sum_{a=1}^{n} (\lambda p_a x_a) + \sum_{a=1}^{n} (\mu q_a x_a) = \sum_{a=1}^{n} (\lambda p_a x_a + \mu q_a x_a) =$$

$$= \sum_{a=1}^{n} (\lambda p_a + \mu q_a) x_a \Rightarrow \exists \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R} : \lambda x + \mu y = \sum_{a=1}^{n} \lambda_a x_a =$$

$\Rightarrow \lambda x + \mu y \in \text{span}(A)$, we conclude that..

$$\forall \lambda, \mu \in \mathbb{R} : \forall x, y \in \text{span}(A) : \lambda x + \mu y \in \text{span}(A) \Rightarrow \text{span}(A) \text{ subspace}$$

Homework: 19-28 & 29-31

next Weds Exam 3

11/18/2019. Span examples.

ⓐ Example: Let $x_1 = (1,3,0)$ and let $x_2 = (0,2,-1)$
Define $V = \text{span}\{x_1, x_2\}$

Solution: Since for $a, b \in \mathbb{R}$: $ax_1 + bx_2 = a(1,3,0) + b(0,2,-1)$
$= (a, 3a, 0) + (0, 2b, -b) = (a, 3a+2b, -b)$
It follows that
$$V = \text{span}\{x_1, x_2\} = \{ax_1 + bx_2 \mid a, b \in \mathbb{R}\}$$
$$= \{(a, 3a+2b, -b) \mid a, b \in \mathbb{R}\}$$

ⓑ Example: Show that $V = \{(a+b, 2b, b-3a) \mid a, b \in \mathbb{R}\}$ is a vector space

Solution: since $(a+b, 2b, b-3a) = (a, 0, -3a) + (b, 2b, b) =$
$= a(1,0,-3) + b(1,2,1) = ax_1 + bx_2$
with $\begin{cases} x_1 = (1,0,-3) \\ x_2 = (1,2,1) \end{cases}$ It follows that

$V = \{(a+b, 2b, b-3a) \mid a, b \in \mathbb{R}\} = \{ax_1 + bx_2 \mid a, b \in \mathbb{R}\} = \text{span}\{x_1, x_2\} \Rightarrow$

$\Rightarrow V$ subspace of $\mathbb{R}^3 \Rightarrow (V, +, \cdot)$ vector space

ⓒ Define the subspace $V = \text{span}\{f, g\}$ with $\begin{cases} \forall x \in \mathbb{R}: f(x) = \sin(x) \\ \forall x \in \mathbb{R}: g(x) = \cos(x) \end{cases}$

Solution Let $a, b \in \mathbb{R}$, then
$\forall x \in \mathbb{R}: (af + bg)(x) = (af)(x) + (bg)(x) = af(x) + bg(x) =$
$= a\sin(x) + b\cos(x)$ and therefore

*!

$V = \text{span}\{f, g\} = \{af + bg \mid a, b \in \mathbb{R}\} = \{h \in F(\mathbb{R}) \mid \exists a, b \in \mathbb{R} : h = af + bg\} =$

$= \{h \in F(\mathbb{R}) \mid \exists a, b \in \mathbb{R} : \forall x \in \mathbb{R} : h(x) = (af + bg)(x)\} =$

$\updownarrow$

$= \{h \in F(\mathbb{R}) \mid \exists a, b \in \mathbb{R} : \forall x \in \mathbb{R} : h(x) = a\sin x + b\cos x\}$

11/20/2019 __Lecture 25__, Subspace Examples

__Example__ Show that $V = \{ f \in F(\mathbb{R}) \mid \exists a, b \in \mathbb{R} : \forall x \in \mathbb{R} :$
$f(x) = (ax+b)\sin x + (ax^2+bx+b)\cos \}$
is subspace of $F(\mathbb{R})$

__Solution__ since $f \in V \Leftrightarrow \exists a, b \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) =$

$$(ax+b)\sin x + (ax^2+bx+b)\cos x =$$
$$= ax\sin x + b\sin x + ax^2\cos x + bx\cos x + b\cos x =$$
$$= a(x\sin x + x^2\cos x) + b(\sin x + x\cos x + \cos x)$$
$$= a\,g_1(x) + b\,g_2(x) = (ag_1)(x) + (bg_2)(x) = (ag_1 + bg_2)(x)$$

with $g_1(x) = x\sin x + x^2\cos x, \ \forall x \in \mathbb{R}$
$g_2(x) = \sin x + x\cos x + \cos x, \ \forall x \in \mathbb{R}$

it follows that

$$f \in V \Leftrightarrow \exists a, b \in \mathbb{R} : \underbrace{\forall x \in \mathbb{R} : f(x) = (ag_1 + bg_2)(x)}, \Leftrightarrow$$

$$\Leftrightarrow \exists a, b \in \mathbb{R} : f = ag_1 + bg_2 \Leftrightarrow f \in \operatorname{span}\{g_1, g_2\}$$

and we conclude that $V = \operatorname{span}\{g_1, g_2\} \Rightarrow V$ subspace $F(\mathbb{R})$

__Example__ Show that $V = \left\{ \begin{bmatrix} a & 2c & c \\ 2b+c & b & -2b \\ a+3c & 2a+b & 3a \end{bmatrix} \middle| \ a, b, c \in \right.$
is a vector space
with respect to matrix addition and scalar multiplical

__Solution__ since $M(a,b,c) = \begin{bmatrix} a & 2c & c \\ 2b+c & b & -2b \\ a+3c & 2a+b & 3a \end{bmatrix} \Rightarrow$

Build matrix based on each variable $\Rightarrow$
"decomposition"

11/20/2019 <u>Matrix span Subspace Examples.</u>

<u>Example</u> Show that $V = \left\{ \begin{bmatrix} a & 2c & c \\ 2b+c & b & -2b \\ a+3c & 2a+b & 3a \end{bmatrix} \middle| \ a,b,c \in \mathbb{R} \right\}$ is a vector space with respect to matrix addition & scalar multiplication

<u>Solution</u> Since $M(a,b,c) = \begin{bmatrix} a & 2c & c \\ 2b+c & b & -2b \\ a+3c & 2a+b & 3a \end{bmatrix} =$

$$= \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ a & 2a & 3a \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2b & b & -2b \\ 0 & b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2c & c \\ e & 0 & 0 \\ 3c & 0 & 0 \end{bmatrix} =$$

$$= a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & -2 \\ 0 & 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

$= aM_1 + bM_2 + cM_3$ with

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} \wedge M_2 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & -2 \\ 0 & 1 & 0 \end{bmatrix} \wedge M_3 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

it follows that $V = \{ M(a,b,c) \mid a,b,c \in \mathbb{R} \} =$

$= \{ aM_1 + bM_2 + cM_3 \mid a,b,c \in \mathbb{R} \} = \text{span}\{M_1, M_2, M_3\} \Rightarrow$

$\Rightarrow V$ subspace of $(M_3(\mathbb{R}), +, \cdot) \Rightarrow (V, +, \cdot)$ vector space

HW - [redacted] 29-33

11/20/2019 <u>Linear Independence</u>

<u>Definition</u> Let $(V, +, \cdot)$ be a vector space and let $A = \{x_1, x_2, \dots, x_n\} \subseteq V$

ⓐ A linearly dependent $\Leftrightarrow \exists x \in A : x \in \text{span}(A - \{x\})$

▶ Recall $\overline{\exists x \in A : p(x)} \Leftrightarrow \forall x \in A : \overline{p(x)}$

A linearly dependent $\Leftrightarrow$ A <u>not</u> linearly dependent

$\Leftrightarrow \forall x \in A : x \notin \text{span}(A - \{x\})$

▶ Characterization of linear dependence / independence

A linearly dependent $\Leftrightarrow \exists (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n : \begin{cases} (\lambda_1, \lambda_2, \dots, \lambda_n) \\ \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n \end{cases}$

negate for independence

▶ Recall $\overline{p \Rightarrow q} \Leftrightarrow (p \wedge \bar{q})$ ▶ $\overline{(p \wedge \bar{q})} \Leftrightarrow (p \Rightarrow q)$

therefore A linearly independent $\Leftrightarrow$

$\Leftrightarrow \forall (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n : (\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n = \mathbf{0} \Rightarrow$
$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0)$

<u>Remark</u> $\begin{cases} A = \{u\} \\ u \neq \mathbf{0} \end{cases} \Rightarrow A$ linearly independent

cannot
the

<u>Proof</u> Let $\lambda \in \mathbb{R}$ be given, then $\lambda u = \mathbf{0} \Rightarrow \lambda = 0 \vee u = \mathbf{0} \Rightarrow$

$\Rightarrow \lambda = 0$ we conclude that $\forall \lambda \in \mathbb{R} : (\lambda u = \mathbf{0} \Rightarrow \lambda = 0) \Rightarrow$
$\Rightarrow \{u\}$ linearly independent

11/20/2019 Linear Dependence & Independence

Remark $A = \{\mathbf{0}\} \Rightarrow A$ linearly dependent

Proof For $\lambda = 1$: $\begin{cases} 1\mathbf{0} = \mathbf{0} \\ 1 \neq 0 \end{cases} \Rightarrow \exists \lambda \in \mathbb{R}: \begin{cases} \lambda\mathbf{0} = \mathbf{0} \\ \lambda \neq 0 \end{cases} \Rightarrow$

$\Rightarrow A = \{\mathbf{0}\}$ linearly dependent

▶ Properties of Linear dependence & independence

① $\begin{cases} B \text{ linearly dependent} \\ B \subset A \end{cases} \Rightarrow A$ linearly dependent

② $\begin{cases} A \text{ linearly independent} \\ B \subset A \end{cases} \Rightarrow B$ linearly independent

③ $\begin{cases} A \text{ linearly independent} \\ A \cup \{u\} \text{ linearly dependent} \end{cases} \Rightarrow u \in \operatorname{span}(A)$

Example Let $x, y \in \mathbb{R}^3$ with $x = (3,1,2)$ and $y = (1,0,3)$
Show that $\{x, y\}$ linearly independent

Solution ▶ Sufficient to show $\forall a, b \in \mathbb{R}: (ax + by = \mathbf{0} \Rightarrow a = b = 0)$
Let $a, b \in \mathbb{R}$ be given, assume that $ax + by = \mathbf{0}$, Then

$$ax + by = \mathbf{0} \Rightarrow a(3,1,2) + b(1,0,3) = (0,0,0) \Rightarrow (3a+b,\ a,\ 2a+3b) = (0,0,0)$$

$$\Rightarrow \begin{cases} 3a+b=0 \\ a=0 \\ 2a+3b=0 \end{cases} \Rightarrow \begin{cases} 3a+b=0 \\ a=0 \end{cases} \Rightarrow \begin{cases} 0+b=0 \\ a=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \end{cases} \Rightarrow a=b=0$$

We conclude that $\forall a, b \in \mathbb{R}: (ax + by = \mathbf{0} \Rightarrow a = b = 0) \Rightarrow$
$\Rightarrow \{x, y\}$ linearly independent