11/11/2019 Lecture 22, Requirements for Vector Space

- Recall $(V, +, \cdot)$ real vector space $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} (V, +) \text{ group} \\ \forall \lambda, \mu \in \mathbb{R} : \forall u \in V : (\lambda + \mu) u = \lambda u + \mu u \\ \forall \lambda \in \mathbb{R} : \forall u, v \in V : \lambda (u+v) = \lambda u + \lambda v \\ \forall \lambda, \mu \in \mathbb{R} : \forall u \in V : \lambda(\mu u) = (\lambda \mu) u = \mu (\lambda u) \\ \forall u \in V : \mathbb{1} u = u \end{cases}$$

- Prop $(V, +, \cdot)$ real vector space $\Rightarrow$

$\Rightarrow (V, +)$ abelian group

- Proof Assume that $(V, +, \cdot)$ real vector space
Then $(V, +)$ group
Let $x, y \in V$ be given, then:

$(1+1)(x+y) = 1(x+y) + 1(x+y) =$

$= 1x + 1y + 1x + 1y = \underline{x+y+x+y}$ and

$(1+1)(x+y) = (1+1)x + (1+1)y =$

$= 1x + 1x + 1y + 1y = \underline{x+x+y+y}$ and therefore

$x+y+x+y = x+x+y+y \Rightarrow y+x+y = x+y+y$ [cancel x $(V, +)$

cancel y
$\Rightarrow y+x = x+y$ [$(V, +)$ group] $\Rightarrow \underline{y+x = x+y}$

We conclude that $(\forall x, y \in V : x+y = y+x) \rightarrow$ "+" commutative $\left.\begin{matrix} \\ (V,+) \text{ group} \end{matrix}\right\} \Rightarrow$

$\Rightarrow (V, +)$ abelian group

11/11/2019 Zero vector/Neutral Element,

▸ Notation Let $\mathbf{0}$ be the neutral element of $(V, +)$
such that $\forall x \in V : x + \mathbf{0} = \mathbf{0} + x = x$
we can show that

1) $\forall \lambda \in \mathbb{R} : \lambda \mathbf{0} = \mathbf{0}$

Proof. Let $\lambda \in \mathbb{R}$ be given. Choose some $x \in V$, then

$$\lambda x + \lambda \mathbf{0} = \lambda(x + \mathbf{0}) = \lambda x = \lambda x + \mathbf{0} \Rightarrow \lambda \mathbf{0} = \mathbf{0}$$

2) $\forall x \in V : 0x = \mathbf{0}$

Proof. Let $x \in V$ be given. Choose some $\lambda \in \mathbb{R}$, then

$$0x + \lambda x = (0 + \lambda)x = \lambda x = \mathbf{0} + \lambda x \Rightarrow 0x = \mathbf{0}$$

3) $\forall \lambda \in \mathbb{R} : \forall x \in V : (\lambda x = \mathbf{0} \Rightarrow (\lambda = 0 \vee x = \mathbf{0}))$

Proof. Let $\lambda \in \mathbb{R}$ and $x \in V$ be given. Assume $\lambda x = \mathbf{0}$
We distinguish between the following cases

Case 1 Assume that $\lambda = 0$. Then
$\lambda = 0 \Rightarrow \lambda = 0 \vee x = \mathbf{0}$

Case 2 Assume that $\lambda \neq 0$. Then $1/\lambda = \lambda^{-1}$ is defined
It follows that

$$x = 1x = (\lambda^{-1}\lambda)x = \lambda^{-1}(\lambda x) =$$

use assume $\lambda x = \mathbf{0}$

$$= \lambda^{-1}\mathbf{0} = \mathbf{0} \Rightarrow x = \mathbf{0} \Rightarrow \lambda = 0 \vee x = \mathbf{0}$$

In both cases we show that $\lambda = 0 \vee x = \mathbf{0}$

11/11/2019 · Examples of Vector Spaces

another zero vector property

(4) $\forall \lambda \in \mathbb{R} : \forall x \in \mathbb{R} : (-\lambda)x = \lambda(-x) = -\lambda x$

Examples of Vector Spaces

(1) The space $(\mathbb{R}^n, +, \cdot)$

Let $x, y \in \mathbb{R}^n$ with $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$

Then:

$$\forall x, y \in V : x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$
$$\forall \lambda \in \mathbb{R} : \forall x \in V : \lambda x = (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$$

(2) Function space $(F(A), +, \cdot)$

Let $A \subseteq \mathbb{R}$ then $F(A)$ is the set of all functions $f : A \to \mathbb{R}$

We define

$$\begin{cases} \forall f, g \in F(A) : \forall x \in A : (f+g)(x) = f(x) + g(x) \\ \forall f \in F(A) : \forall \lambda \in \mathbb{R} : \forall x \in A : (\lambda f)(x) = \lambda f(x) \end{cases}$$

Note that

$$\forall f, g \in F(A) : (f = g \Leftrightarrow \forall x \in A : f(x) = g(x))$$

For $A = [n] = \{1, 2, 3, 4, \dots, n\}$

(3) Matrix space $(M_{nm}(\mathbb{R}), +, \cdot)$

$+$: addition matrix + matrix

$\cdot$: multiplication number $\cdot$ matrix

Example Show that "+" is associative on $F(A)$

Solution sufficient to show that

$$\forall f, g, h \in F(A) : \forall x \in A : [(f+g)+h](x) = [f+(g+h)](x)$$

|2019 Vector Space Examples, & Subspaces

ple Show that "+" is associative on $F(A)$

ion Let $f, g, h \in F(A)$ and let $x \in A$ be given. Then

$$[(f+g)+h](x) = (f+g)(x) + h(x) = [f(x)+g(x)]+h(x)] = \text{arithmetic associative property}$$

$$= f(x) + [g(x)+h(x)] = \text{definition} = f(x) + (g+h)(x) =$$

$$= [f+(g+h)](x)$$ we conclude that

$$\forall f, g, h \in F(A) : \underbrace{\forall x \in A : [(f+g)+h](x) = [f+(g+h)](x)}_{} \Rightarrow$$

$$\Rightarrow \forall f, g, h \in F(A) : (f+g)+h = f+(g+h) \Rightarrow \text{"+" associative on } F(A)$$

☆ Project Idea, prove $F(A)$ is a vector space

$\boxed{HW}$ 17, 18

▶ Vector Subspaces - definition - let $(V, +, \cdot)$ be real vector space

let V_0 be a set

$$V_0 \text{ subspace of } V \Leftrightarrow \begin{cases} V_0 \neq \emptyset \land V_0 \subseteq V \\ (V_0, +, \cdot) \text{ real vector space} \end{cases}$$

Let $(V, +, \cdot)$ be vector space, and $V_0 \neq \emptyset$, and $V_0 \subseteq V$

① V_0 subspace of $V \Leftrightarrow \forall \lambda \in \mathbb{R} : \forall x, y \in V_0 : (\lambda x \in V_0 \land x+y \in V_0)$

② V_0 subspace of $V \Leftrightarrow \forall \lambda, \mu \in \mathbb{R} : \forall x, y \in V_0 : (\lambda x + \mu y) \in V_0$

③ V_0 subspace of $V \Rightarrow \mathbf{0} \in V_0$

↳ $\mathbf{0} \notin V_0 \Rightarrow V_0$ not subspace of V

④ V_1 subspace of V, V_2 subspace of V $\Rightarrow V_1 \cap V_2$ subspace of V

11/13/2019 Lecture 23, Subspaces V_0

▶ Recall that

Given $(V, +, \cdot)$ vector space

$$V_0 \text{ subspace of } V \Leftrightarrow \begin{cases} V_0 \neq \emptyset \wedge V_0 \subseteq V \\ (V_0, +, \cdot) \text{ vector space} \end{cases}$$

① Given $V_0 \subseteq V$ and $V_0 \neq \emptyset$

V_0 subspace of $V \Leftrightarrow \forall u, v \in V_0 : \forall \lambda \in \mathbb{R} : (\lambda u \in V_0 \wedge u$

② Given $V_0 \subseteq V$ and $V_0 \neq \emptyset$ then

V_0 subspace of $V \Leftrightarrow \forall x, y \in V_0 : \forall \lambda, \mu \in \mathbb{R} : \lambda x + \mu y \in V_0$

Proof Assume that $V_0 \subseteq V$ and $V_0 \neq \emptyset$, then...

($\Rightarrow$): Assume that V_0 subspace of V.

Let $x, y \in V_0$ and let $\lambda, \mu \in \mathbb{R}$ be given. Then...

$$\left.\begin{aligned} \lambda \in \mathbb{R} \wedge x \in V_0 &\overset{(1)}{\Rightarrow} \lambda x \in V_0 \\ \mu \in \mathbb{R} \wedge y \in V_0 &\overset{(1)}{\Rightarrow} \mu y \in V_0 \end{aligned}\right\} \overset{(1)}{\Rightarrow} \lambda x + \mu y \in V_0$$

It follows that $\forall x, y \in V_0 : \forall \lambda, \mu \in \mathbb{R} : \lambda x + \mu y \in V_0$

($\Leftarrow$): Assume that $\forall x, y \in V_0 : \forall \lambda, \mu \in \mathbb{R} : \lambda x + \mu y \in V_0$

▶ we will show that $\forall x, y \in V_0 : \forall \lambda \in \mathbb{R} : (\underline{\lambda x \in V_0} \wedge \underline{x + y \in V_0})$

Let $\underline{x, y \in V_0}$ and $\underline{\lambda \in \mathbb{R}}$ be given. Then...

$$\lambda x = \lambda x + 0y \wedge \lambda, 0 \in \mathbb{R} \wedge x, y \in V_0 \Rightarrow \underline{\lambda x \in V_0}$$

$$x + y = 1x + 1y \wedge 1 \in \mathbb{R} \wedge x, y \in V_0 \Rightarrow \underline{x + y \in V_0}$$

we conclude that

$\forall x, y \in V_0 : \forall \lambda \in \mathbb{R} : (\lambda x \in V_0 \wedge x + y \in V_0) \overset{(1)}{\Rightarrow} V_0$ subspace of

11/13/2019 <u>Subspace Properties</u>

(3) V_0 subspace of $V \Rightarrow \mathbf{0} \in V_0$

<u>Proof</u>: Assume that V_0 subspace of V. then...
V_0 subspace of $V \Rightarrow V_0 \neq \emptyset$
therefore choose some $x \in V_0$

Since $\begin{cases} V_0 \text{ subspace of } V \\ x \in V_0 \end{cases} \Rightarrow \forall \lambda \in \mathbb{R}: \lambda x \in V_0 \Rightarrow$

$\Rightarrow 0x \in V_0$ [when $\lambda = 0$) $\Rightarrow \mathbf{0} \in V_0$

<u>Contrapositive</u> $\mathbf{0} \notin V_0 \Rightarrow V_0$ not subspace of V

(4) $\begin{cases} V_1 \text{ subspace of } V \\ V_2 \text{ subspace of } V \end{cases} \Rightarrow V_1 \cap V_2$ subspace of V

<u>Proof</u>: $V_1 \cap V_2 \subseteq V$... since $\begin{cases} V_1 \text{ subspace of } V \\ V_2 \text{ subspace of } V \end{cases} \overset{(3)}{\Rightarrow} \begin{cases} \mathbf{0} \in V_1 \\ \mathbf{0} \in V_2 \end{cases} \Rightarrow$

$\Rightarrow \mathbf{0} \in V_1 \cap V_2 \Rightarrow V_1 \cap V_2 \neq \emptyset$
Let $x, y \in V_1 \cap V_2$ and $\lambda, \mu \in \mathbb{R}$ be given.
$\begin{cases} \lambda, \mu \in \mathbb{R} \\ x, y \in V_1 \cap V_2 \end{cases} \Rightarrow \begin{cases} \lambda, \mu \in \mathbb{R} \\ x, y \in V_1 \end{cases} \wedge \begin{cases} \lambda, \mu \in \mathbb{R} \\ x, y \in V_2 \end{cases} \Rightarrow$

$\Rightarrow \lambda x + \mu y \in V_1 \wedge \lambda x + \mu y \in V_2$ [V_1, V_2 subspaces of V] $\Rightarrow$

$\Rightarrow \lambda x + \mu y \in V_1 \cap V_2$ we conclude that...

$\begin{cases} \forall x, y \in V_1 \cap V_2 : \forall \lambda, \mu \in \mathbb{R} : \lambda x + \mu y \in V_1 \cap V_2 \\ V_1 \cap V_2 \neq \emptyset \end{cases} \Rightarrow V_1 \cap V_2$ subspace of V

11/13/2019 <u>Vector Space Examples</u>

ⓐ <u>Example</u> Show that $V = \{(a,b) \in \mathbb{R}^2 \mid 2a+3b=0\}$ is subspace of $\mathbb{R}^2$

<u>Solution</u> Let $\lambda, \mu \in \mathbb{R}$ and $x, y \in V$ be given. Since...

$$x \in V \Rightarrow \exists a_1, b_1 \in \mathbb{R} : x = (a_1, b_1) \land 2a_1 + 3b_1 = 0$$
$$y \in V \Rightarrow \exists a_2, b_2 \in \mathbb{R} : y = (a_2, b_2) \land 2a_2 + 3b_2 = 0$$

▸ we will show that $\lambda x + \mu y = (c_1, c_2)$ such that $2c_1 + 3c_2 = 0$

Since
$$\begin{aligned}\lambda x + \mu y &= \lambda(a_1, b_1) + \mu(a_2, b_2)\\ &= (\lambda a_1, \lambda b_1) + (\mu a_2, \mu b_2)\\ &= (\lambda a_1 + \mu a_2, \lambda b_1 + \mu b_2) = (c_1, c_2) \text{ with}\end{aligned}$$
$$c_1 = \lambda a_1 + \mu a_2$$
$$c_2 = \lambda b_1 + \mu b_2$$

and
$$\begin{aligned}2c_1 + 3c_2 &= 2(\lambda a_1 + \mu a_2) + 3(\lambda b_1 + \mu b_2)\\ &= \underline{2\lambda a_1} + \underline{\underline{2\mu a_2}} + \underline{3\lambda b_1} + \underline{\underline{3\mu b_2}}\\ &= \lambda(2a_1 + 3b_1) + \mu(2a_2 + 3b_2) = 0\lambda + 0\mu =\end{aligned}$$

it follows that $\lambda x + \mu y \in V$

we conclude that $\forall \lambda, \mu \in \mathbb{R} : \forall x, y \in V : \lambda x + \mu y \in V \Rightarrow$

$\Rightarrow V$ subspace of $\mathbb{R}^2$

11/13/2019 Vector Space Examples

(b) Example show that $V = \{ f \in F(\mathbb{R}) \mid f \text{ continuous on } \mathbb{R} \}$ is subspace of $F(\mathbb{R})$

Solution Let $\lambda, \mu \in \mathbb{R}$ and $f, g \in V$ be given

▸ we will show that $\lambda f + \mu g \in V$

Since, $f \in V \Rightarrow f$ continuous on $\mathbb{R} \Rightarrow \forall x_0 \in \mathbb{R}: f$ continuous on $x_0 \Rightarrow$
$\Rightarrow \forall x_0 \in \mathbb{R}: \lim_{x \to x_0} f(x) = f(x_0)$

and $g \in V \Rightarrow \ldots \Rightarrow \forall x_0 \in \mathbb{R}: \lim_{x \to x_0} g(x) = g(x_0)$

▸ we will show that $\forall x_0 \in \mathbb{R}: \lim_{x \to x_0} (\lambda f + \mu g)(x) = (\lambda f + \mu g)(x_0)$

Since $\forall x_0 \in \mathbb{R}: \lim_{x \to x_0} (\lambda f + \mu g)(x) = \lim_{x \to x_0} [(\lambda f)(x) + (\mu g)(x)] =$

$= \lim_{x \to x_0} [\lambda f(x) + \mu g(x)] = \lambda \lim_{x \to x_0} f(x) + \mu \lim_{x \to x_0} g(x) = \lambda f(x_0) + \mu g(x_0) =$

$= (\lambda f)(x_0) + (\mu g)(x_0) = (\lambda f + \mu g)(x_0) \Rightarrow \forall x_0 \in \mathbb{R}: \lambda f + \mu g$ continuous on x_0 $\Rightarrow$

$\Rightarrow \lambda f + \mu g$ continuous on $\mathbb{R} \Rightarrow \lambda f + \mu g \in V$

we conclude that $\forall \lambda, \mu \in \mathbb{R}: \forall f, g \in V: \lambda f + \mu g \in V \Rightarrow$

$\Rightarrow V$ subspace of $F(\mathbb{R})$

Homework: 19–25