11/4/2019 Lecture 20, vector spaces Example continue

(a) Let $A = \mathbb{R} - \{\lambda\}$ with $\lambda \in \mathbb{R}$ and
define $\forall x, y \in A: x * y = xy - \lambda(x+y) + \lambda(\lambda+1)$

we showed (1) $*$ closed on A (2) $*$ commutative

Show that (3) $*$ has a unit element on A

Solution [... (1) ... (2) ...]

▶ because $*$ commutative, sufficient to show one way
$\forall x \in A: x * e = e * x = x$ $x * e$ or $e * x$

Let $x \in A$ be given. Then $x \in A \Rightarrow x \in \mathbb{R} - \{\lambda\} \Rightarrow x \neq \lambda$

Solve - note. $x - \lambda \neq 0$ (1)

$x * e = x \Leftrightarrow x * e - x = 0 \Leftrightarrow$ plug in original eqn $\Leftrightarrow$

$\Leftrightarrow xe - \lambda(x+e) + \lambda(\lambda+1) - x = 0 \Leftrightarrow$ simplify $\Leftrightarrow$

$\Leftrightarrow xe - \lambda x - \lambda e + \lambda^2 + \lambda - x = 0 \Leftrightarrow$ factor by grouping

$\Leftrightarrow (x - \lambda)e = \lambda x - \lambda^2 - \lambda + x \Leftrightarrow$ factor by grouping $\Leftrightarrow$

$\Leftrightarrow (x - \lambda)e = (\lambda+1)x - \lambda(\lambda+1) \Leftrightarrow (x-\lambda)e = (x-\lambda)(\lambda+1)$

$\Leftrightarrow e = \lambda + 1$ [via $x - \lambda \neq 0$ (1)]

It follows that for $e = \lambda + 1$ then

$\begin{cases} \forall x \in A: x * e = x \\ * \text{ commutative} \end{cases} \Rightarrow$

$\Rightarrow \forall x \in A: x * e = e * x = x \Rightarrow$

$\Rightarrow *$ has unit element $e = \lambda + 1$ on A

11/4/2019 Vector Spaces, examples

b) Define $\forall x, y \in \mathbb{R}: x * y = xy + 2ax + by$

Find all $a, b \in \mathbb{R}$ such that "$*$" associative on $\mathbb{R}$

Solution,

Let $x, y, z \in \mathbb{R}$ be given. We note that

$$x*(y*z) = x*(yz + 2ay + bz)$$
$$= x(yz + 2ay + bz) + 2ax + b(yz + 2ay + bz)$$
$$= \underline{xyz} + \underline{2axy} + bxz + 2ax + \underline{byz} + \underline{2aby} + b^2 z$$
$$(x*y)*z = (xy + 2ax + by)*z$$
$$= (xy + 2ax + by)z + 2a(xy + 2ax + by) + bz$$
$$= \underline{xyz} + 2axz + \underline{byz} + \underline{2axy} + 4a^2 x + \underline{2aby} + bz$$

we want these to be the same for $*$ to be associative

4 terms in common, 3 terms different

consider x, y, z as polynomial variables, a & b constant

Therefore $x*(y*z) - (x*y)*z =$

$$= bxz + 2ax + b^2 z - (2axz + 4a^2 x + bz)$$ for associative

$$= (b-2a)xz + (2a - 4a^2)x + (b^2 - b)z$$ coefficients should be zero

"$*$" associative on $\mathbb{R} \Leftrightarrow \forall x, y, z \in \mathbb{R}: x*(y*z) = (x*y)*z \Leftrightarrow$

$\Leftrightarrow \forall x, y, z \in \mathbb{R}: x*(y*z) - (x*y)*z = 0 \Leftrightarrow$

$\Leftrightarrow \forall x, y, z \in \mathbb{R}: (b-2a)xz + (2a-4a^2)x + (b^2-b)z = 0 \Leftrightarrow$ logical and distributes to logical or

$\Leftrightarrow b - 2a = 0 \land 2a - 4a^2 = 0 \land b^2 - b = 0 \Leftrightarrow$ $\boxed{\begin{array}{l} \text{recall } p \land (q \lor r) \Leftrightarrow (p \land q) \lor (p \land r) \\ p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r) \end{array}}$

$\Leftrightarrow b = 2a \land 2a(1-2a) = 0 \land b(b-1) = 0 \Leftrightarrow$

$$\Leftrightarrow \begin{cases} b = 2a \\ 2a(1-2a) = 0 \\ b = 0 \end{cases} \lor \begin{cases} b = 2a \\ 2a(1-2a) = 0 \\ b = 1 \end{cases} \Leftrightarrow$$

Find consistent systems

$$\Leftrightarrow \begin{cases} b = 2a \\ a = 0 \\ b = 0 \end{cases} \lor \begin{cases} b = 2a \\ a = 1/2 \\ b = 0 \end{cases} \lor \begin{cases} b = 2a \\ a = 0 \\ b = 1 \end{cases} \lor \begin{cases} b = 2a \\ a = 1/2 \\ b = 1 \end{cases} \Leftrightarrow$$

11/4/2019 Vector Spaces, examples + Groups

ⓑ Define $\forall x, y \in \mathbb{R}: x * y = xy + 2ax + by$

Find all $a, b \in \mathbb{R}$ such that "$*$" associative on $\mathbb{R}$

Solution:

$$\Leftrightarrow \begin{cases} b = 2a \\ a = 0 \\ b = 0 \end{cases} \vee \begin{cases} b = 2a \\ a = 1/2 \\ b = 0 \end{cases} \vee \begin{cases} b = 2a \\ a = 0 \\ b = 1 \end{cases} \vee \begin{cases} b = 2a \\ a = 1/2 \\ b = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} b = 0 \\ a = 0 \\ b = 0 \end{cases} \vee \underbrace{\begin{cases} b = 1 \\ a = 1/2 \\ b = 0 \end{cases}}_{\text{False}} \vee \underbrace{\begin{cases} b = 0 \\ a = 0 \\ b = 1 \end{cases}}_{\text{False}} \vee \begin{cases} b = 1 \\ a = 1/2 \\ b = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} a = 0 \\ b = 0 \end{cases} \vee \begin{cases} a = 1/2 \\ b = 1 \end{cases} \Leftrightarrow (a, b) \in \{(0,0), (1/2$$

$\boxed{\text{Homework: } 1 - 8}$

Groups

Definition Let G be a set and let "$*$" be an internal operation on G

ⓐ $(G, *)$ group $\Leftrightarrow \begin{cases} * \text{ closed on } G \\ \forall a, b, c \in G: (a*b)*c = a*(b*c) \\ \exists e \in G: \forall a \in G: a*e = e*a = a \\ \forall a \in G: \exists a' \in G: a*a' = a'*a = e \end{cases}$

ⓑ $(G, *)$ abelian group $\Leftrightarrow \begin{cases} (G, *) \text{ group} \\ \forall a, b \in G: a*b = b*a \end{cases}$

Abel

11/4/2019 What are Abelian Groups? Examples

$(\mathbb{R}, +)$ abelian group

$(\mathbb{R} - \{0\}, \cdot)$ abelian group vs $(GL(n,\mathbb{R}), \cdot)$ group

$(M_{nm}(\mathbb{R}), +)$ abelian group

Theorem Let G be a set and $*$ internal operation on G

$$(G, *) \text{ group} \iff \begin{cases} * \text{ closed on } G \\ * \text{ associative on } G \\ \exists e \in G : \forall a \in G : a * e = a \\ \forall a \in G : \exists a' \in G : a * a' = e \end{cases}$$

Theorem Let $(G, *)$ be a group, then

$\forall a, b \in G : (a * b)' = b' * a'$

$\forall a \in G : (a')' = a$

note on $(GL(n,\mathbb{R}), \cdot)$

$\forall A, B \in GL(n,\mathbb{R}) : (AB)^{-1} = B^{-1}A^{-1}$

$\forall A \in GL(n,\mathbb{R}) : (A^{-1})^{-1} = A$

Theorem Let $(G, *)$ be a group, then

$\forall a, b, c \in G : (c * a = c * b \iff a = b)$

$\forall a, b, c \in G : (a * c = b * c \iff a = b)$

any group not just abelian

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Example-

Show that $(\mathbb{R} - \{1/3\}, *)$ with $\forall a,b \in \mathbb{R} - \{1,3\}$: $a * b = a + b - 3ab$ is an abelian group

Solution-

- Closure

▶ want to show that $\forall a,b \in \mathbb{R} - \{1/3\}$: $a * b \in \mathbb{R} - \{1/3\}$

let $a,b \in \mathbb{R} - \{1,3\}$ be given.

$$a * b - 1/3 = \underline{a} + b - \underline{3ab} - 1/3$$

$$= a(1-3b) + (b - 1/3)$$

$$= -3a(b - 1/3) + (b - 1/3)$$

$$= (b - 1/3)(1 - 3a)$$

$$= -3(a - 1/3)(b - 1/3)$$

and therfore:

$$a,b \in \mathbb{R} - \{1/3\} \Rightarrow a \neq 1/3 \wedge b \neq 1/3$$

$$\Rightarrow a - 1/3 \neq 0 \wedge b - 1/3 \neq 0$$

$$\Rightarrow -3(a - 1/3)(b - 1/3) \neq 0$$

$$\Rightarrow a * b - 1/3 \neq 0$$

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$\Rightarrow a * b \neq 1/3$

$\Rightarrow a * b \in \mathbb{R} - \{1/3\}$

· Commutative-

Let $a * b \in \mathbb{R} - \{1/3\}$ be given. Then

$$\begin{aligned} a * b &= a + b - 3ab \\ &= b + a - 3ba \\ &= b * a \end{aligned}$$

therefore:

$\forall a, b \in \mathbb{R} - \{1/3\} : a * b = b * a$

$\Rightarrow$ " $*$ " commutative $\mathbb{R} - \{1/3\}$

· Associative-

Let $a, b, c \in \mathbb{R} - \{1/3\}$ be given. Then

$$\begin{aligned} a * (b * c) &= a * (b + c - 3bc) \\ &= a + (b + c - 3bc) - 3a(b + c - 3bc) \\ &= a + b + c - 3bc - 3ab - 3ac + 9abc \\ &= a + b + c - 3(ab + bc + ca) + 9abc \end{aligned}$$

and

$(a * b) * c = (a + b - 3ab) * c$

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$= (a+b-3ab)+c-3(a+b-3ab)c$

$= a+b-3ab+c-3ac-3bc+9abc$

$= a+b+c-3(ab+bc+ca)+9abc$

therefore: $a*(b*c)=(a*b)*c$

We conclude:

$\forall a,b,c \in \mathbb{R}-\{1/3\}: a*(b*c)=(a*b)*c$

$\Rightarrow$ $*$ associative on $\mathbb{R}-\{1/3\}$

- Unit element: Let $x \in \mathbb{R}-\{1/3\}$ be given. Solve with respect to e the equation.

$\underline{x * e = x} \Leftrightarrow x+e-3xe=x$

$\Leftrightarrow e-3xe=0$

$\Leftrightarrow (1-3x)e=0$

$\Leftrightarrow 1-3x=0 \vee e=0$

$\Leftrightarrow x=1/3 \vee e=0$

$\Leftrightarrow \underline{e=0}$

We conclude that:
$\forall x \in \mathbb{R}-\{1/3\}: x*0=0*x=x$
$\Rightarrow$ $*$ has $e=0$ as a unit element.

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- Symmetric element:

▶ we need to show that

$\forall x \in \mathbb{R} - \{1/3\} : \exists y \in \mathbb{R} - \{1/3\} : x * y = y * x = 0$

Let $x \in \mathbb{R} - \{1/3\}$ be given. Then

$x * y = 0 \iff x + y - 3xy = 0$

$\iff y - 3xy = -x$

$\iff (1 - 3x)y = -x \qquad (1)$

Since
$x \in \mathbb{R} - \{1/3\} \Rightarrow x \neq 1/3 \Rightarrow 3x \neq 1$

$\Rightarrow 1 - 3x \neq 0 \qquad (2)$

From eq (2)

Eq (1) $\iff y = \dfrac{-x}{1-3x}$

we conclude that

$\forall x \in \mathbb{R} - \{1/3\} : \exists y \in \mathbb{R} - \{1/3\}$
$: x * y = y * x = 0$

From all of the above
$(\mathbb{R} - \{1/3\}, *)$
abelian group

HW : 9-16

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Vector Spaces

Def. An external operation on A with coefficients from G is a mapping

$$f : G \times A \to A.$$

Def. Given a set $\mathbb{V}$, an internal operation $+$ on $\mathbb{V}$, an external operation $\cdot$ on $\mathbb{V}$ with coefficients from $\mathbb{R}$, we say that $(\mathbb{V}, +, \cdot)$ is real vector space if and only if.

(a) $(\mathbb{V}, +)$ group

(b) $\forall \lambda \in \mathbb{R} : \forall x, y \in \mathbb{V} : \lambda(x+y) = \lambda x + \lambda y$

(c) $\forall \lambda, \mu \in \mathbb{R} : \forall x \in \mathbb{V} : (\lambda + \mu) x = \lambda x + \mu x$

(d) $\forall \lambda, \mu \in \mathbb{R} : \forall x \in \mathbb{V} : \lambda(\mu x) = (\lambda\mu) x$

(e) $\forall x \in \mathbb{V} : 1 x = x$

Thm: $(\mathbb{V}, +, \cdot)$ real vector space $\Rightarrow$

$\Rightarrow (\mathbb{V}, +)$ abelian group.