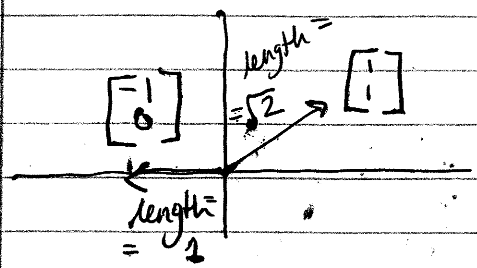
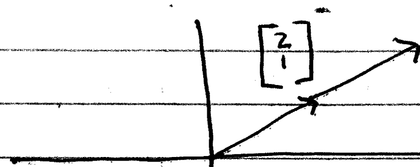
10/21/2019 Lecture 16 , Eigenvalues & Eigenvectors

Say $A = \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix}$ $\quad A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$



What about $A \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = ?$

$$A \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow$ eigenvalue $= 2$, eigenvector $= \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

eigenvector, $\boxed{\text{characteristic}} = \boxed{\text{eigen}}$ in German
eigenvalue vector/value

What about $A \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = ? = \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ eigen value 1, eigen vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

▶ Definition Let $A \in M_n(\mathbb{R})$
we say at $\lambda \in \mathbb{C}$ is an eigenvalue of A
with eigenvector $x \in M_{n,1}(\mathbb{C})$ if and only if

$$Ax = \lambda x \quad \text{with } x \neq 0$$

10/21/2019 Eigen Values & Vectors example

Example Let $A = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix}$ a) is $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ an eigenvector of A?

a) Solution $\begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -4 \end{bmatrix} \overset{?}{=} \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$? NO it is not an eigenvector

NO because you cannot factor out a λ so that definition is true

b) is $\bar{x} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ an eigenvector of A?

Solution $\begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

Yes, eigenvalue is $0 = \lambda$

c) is $\bar{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ an eigenvector of A?

Solution $\begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Yes, eigenvalue is $3 = \lambda$

d) is $\lambda = 2$ an eigenvalue of A? $A\bar{x} = 2\bar{x}$, $\bar{x} = ???$

I identity matrix

Solution $A\bar{x} = \lambda\bar{x} \Rightarrow A\bar{x} = \lambda I \bar{x} \Rightarrow A\bar{x} - \lambda I \bar{x} = 0$

$\Rightarrow (A - \lambda I)\bar{x} = 0$

10/21/2019 Eigen Values & Vectors example

Example Let $A = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix}$ d) is $\lambda = 2$ an eigenvalue of A?

d) Solution $(A - \lambda I)\bar{x} = 0$, $\bar{x} \neq \bar{0}$ per definition of eigenvector

$(A - \lambda I) = 0$ nonsingular $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$m\bar{x} = \bar{0}$ $m^{-1}m\bar{x} = m^{-1}\bar{0} \Rightarrow \bar{x} = \bar{0}$

Conclusion

$A - \lambda I$ must be singular to have a nonzero solution to $A\bar{x} = \lambda\bar{x}$, $\Rightarrow (A - \lambda I)\bar{x} = \bar{0}$

SO $\det(A - \lambda I) = 0$

d) Solution continued $\det(A - 2I) = 0$, $\lambda = 2$

$$A - 2I = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-2 & 0 & 0 \\ -4 & 6-2 & 2 \\ 16 & -15 & -5-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 4 & 2 \\ 16 & -15 & -7 \end{bmatrix} =$$

$$= 1 \det\begin{vmatrix} 4 & 2 \\ -15 & -7 \end{vmatrix} = 1\left(4(-7) - (-15)(2)\right) = -28 + 30 = 2$$

$\lambda = 2$ is not an eigenvalue of A because $\det(A - 2I) \neq 0$

SO why not solve for eigenvalues using $\det(A - \lambda I) = 0$?

10/21/2019 Eigenvalues & Eigenvectors

Example Find Eigenvalues & eigenvectors of $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$

Solution $A - \lambda I = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$

$= \begin{bmatrix} 4-\lambda & -2 & 1 \\ 2 & 0-\lambda & 1 \\ 2 & -2 & 3-\lambda \end{bmatrix}$ so $\det(A-\lambda I) = \det \begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 2 & -2 & 3-\lambda \end{vmatrix}$

$= (4-\lambda)\det \begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} - (-2)\det \begin{vmatrix} 2 & 1 \\ 2 & 3-\lambda \end{vmatrix} +$

$+ (1)\det \begin{vmatrix} 2 & -\lambda \\ 2 & -2 \end{vmatrix} =$

$= (4-\lambda)\left[(-\lambda)(3-\lambda) - (1)(-2)\right] + 2\left[2(3-\lambda) - (2)(1)\right] + \left[2(-2) - (-\lambda)(2)\right] =$

$= (4-\lambda)\left[-3\lambda + \lambda^2 + 2\right] + 2\left[6 - 2\lambda - 2\right] + \left[-4 + 2\lambda\right] =$

$= \left[4\lambda^2 - 12\lambda + 8 - \lambda^3 + 3\lambda^2 - 2\lambda\right] + \left[8 - 4\lambda\right] - \left[4 + 2\lambda\right] =$

$= -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0$ Factor or $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

Rational Root Theorem $\lambda = \dfrac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1}$

$= \dfrac{\pm \text{Factors of } |12|}{\pm \text{Factors of } |1|}$

10/21/2019 Eigenvalues & Eigenvectors

Example Find Eigenvalues & eigenvectors of $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$

Solution

$$\det(A - \lambda I) = \det \begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 2 & -2 & 3-\lambda \end{vmatrix} =$$

$= \lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$, check λ based on rational root thereom

$\lambda = 2$ is a root, and an eigenvalue

use division to find other roots

$$\frac{\lambda^3 - 7\lambda^2 + 16\lambda - 12}{\lambda - 2} = \lambda - 2 \overline{\big) \lambda^3 - 7\lambda^2 + 16\lambda - 12}$$

$$\begin{array}{r} \lambda^2 - 5\lambda + 6 \quad R\ 0 \\ \lambda - 2 \overline{\big) \lambda^3 - 7\lambda^2 + 16\lambda - 12} \\ -\ \underline{\lambda^3 - 2\lambda^2 \qquad\qquad} \\ -5\lambda^2 + 16\lambda - 12 \\ -\ \underline{-5\lambda^2 + 10\lambda \qquad} \\ 6\lambda - 12 \\ \underline{6\lambda - 12} \\ 0 \end{array}$$

So

$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0 \Rightarrow$

$\Rightarrow (\lambda - 2)(\lambda^2 - 5\lambda + 6)$

$\Rightarrow (\lambda - 2)(\lambda - 2)(\lambda - 3)$

$\Rightarrow \lambda = 2, 2, 3$ = eigenvalues

Now what are the eigenvectors? use $\underline{A - \lambda I}$

$$A - 2I = \begin{bmatrix} 4-2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 3-2 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \approx$$

10/21/2019 Eigenvalues & Eigenvectors

Example Find Eigenvectors of $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$ if λ eigenvalues = 2, 3

Solution $A - 2I = \begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ -1R_1 + R_2 \\ -1R_1 + R_3 \end{matrix} \sim$

$\sim \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ so $x_1 = 1 \quad x_2 = -1 \quad x_3 = 1/2$

so $1x_1 - x_2 + 1/2 x_3 = 0$ say $x_2 = s$ / $x_3 = t$

so $x_1 - s + 1/2 t = 0 \Rightarrow x_1 = s - 1/2 t \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s - 1/2 t \\ s \\ t \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$ Check by multiplying

$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ so $X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ so $X = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

both eigenvectors

Must check other eigenvalue $\lambda = 3$

10/21/2019 <u>Eigenvalues & Eigenvectors</u>

<u>Example</u> Find Eigenvectors of A if λ eigenvalue = 3

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

<u>Solution</u> $A - 3I = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$

so $-2 \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 2 & -2 & 0 \end{bmatrix} \sim -2 \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix} \sim 2 \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim$

$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ reduced row echelon form

$X_1 + 0X_2 - 1X_3 = 0$

$X_1 = X_3$

$X_2 - X_3 = 0$

$X_2 = X_3$

$X_3 = t$

$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = 1 \begin{bmatrix} t \\ t \\ t \end{bmatrix}$

$\lambda = 3$ has eigenvector $= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

10/23/2019 Lecture 17 Characteristic Polynomial

- Definition Let $A \in M_n(\mathbb{R})$ - characteristic polynomial

$$p(\lambda) = \det(A - \lambda I)$$

☆ how do you find all matrices with a specific characteristic polynomial? — project idea ☆

- Property For $A \in M_n(\mathbb{R})$ we have

$$p(\lambda) = \det(A - \lambda I)$$

$$= (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$$

- Proof $p(\lambda) = \det(A - \lambda I) = \sum_{\sigma \in S_n} \left[S(\sigma) \prod_{a=1}^{n} (A - \lambda I)_{a,\sigma(a)} \right] =$

$$= (+1) \prod_{a=1}^{n} (A - \lambda I)_{aa} + \sum_{\sigma \in S - \{\sigma_0\}} \left[S(\sigma) \prod_{a=1}^{n} (A - \lambda I)_{a,\sigma(a)} \right] = \bigstar'$$

$$= \prod_{a=1}^{n} (A_{aa} - \lambda) + g(\lambda) = (-\lambda)^n + h(\lambda) + g(\lambda) = (-1)^n \lambda^n + h(\lambda) + g(\lambda)$$

Since degree $d(h(\lambda)) \leq n-1$ because for some factors
* you choose A_{aa} instead of $-\lambda$, and,
Since degree $\deg(g(\lambda)) \leq n-1$ because
for $\sigma \in S_n - \{\sigma_0\}$ we have $\exists_a \in [n] : \sigma(a) \neq a$
and the corresponding $(A - \lambda I)_{a,\sigma(a)}$ is not λ dependent
the claim follows

$\bigstar'$: with σ_0 the permutation $\forall a \in [n]: \sigma_0(a) = a$

10/23/2019 <u>Trace of a Matrix</u>

▶ Recall fundamental theorem of algebra
For every polynomial...
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$
...there is a unique factorization...
$$f(x) = a_n (x - x_1)(x - x_2) \dots (x - x_n)$$
...with $x_1, x_2, \dots x_n \in \mathbb{C}$

▶ It follows that for $A \in M_n(\mathbb{R})$
$$\det(A - \lambda I) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$$
$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$$
for $\lambda = 0$:
$$\boxed{\det(A) = \lambda_1 \lambda_2 \lambda_3 \dots \lambda_n}$$
if some eigenvalue is zero, then $A \notin GL(n, \mathbb{R})$
If all eigenvalues are nonzero, then $A \in GL(n, \mathbb{R})$

▶ <u>Trace of a matrix</u> Let $A \in M_n(\mathbb{R})$ with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$
we define
$$tr(A) = \sum_{a=1}^{n} A_{aa} = A_{11} + A_{22} + \dots + A_{nn}$$

we can show that $\lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n = tr(A)$

▶ <u>Cauchy identities</u>
$$a^2 + b^2 = (a+b)^2 - 2ab$$
$$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$
$$\vdots$$

★ How to generalize? Maxima, binomial expansion
Project idea - show it is possible + results

10/23/2019 Application of Trace & Cauchy identity.

Example Let $A = \begin{bmatrix} a+3 & 1 \\ 2a & 2 \end{bmatrix}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ be eigenvalues of A

Find all $a \in \mathbb{R}$ such that $\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} = 1$

Solution since $\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} = \frac{\lambda_2^2 + \lambda_1^2}{\lambda_1^2 \lambda_2^2} = \frac{(\lambda_1+\lambda_2)^2 - 2\lambda_1\lambda_2}{(\lambda_1\lambda_2)^2}$

$= \frac{[tr(A)]^2 - 2\det(A)}{(\det(A))^2}$, and $\det(A) = \begin{vmatrix} a+3 & 1 \\ 2a & 2 \end{vmatrix} =$

$= 2(a+3) - 1(2a) = 2a+6-2a = 6$, and $tr(A) = A_{11} + A_{22} =$

$= (a+3)+2 = a+5$, it follows that $\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} = 1 \iff$

$\iff \frac{[tr(A)]^2 - 2\det(A)}{(\det(A))^2} = 1 \iff \frac{(a+5)^2 - 2\cdot 6}{6^2} = 1 \iff \frac{(a+5)^2 - 12}{36}$

$\iff (a+5)^2 - 12 = 36 \iff (a+5)^2 = 48 \iff a+5 = \sqrt{48} \iff a+5 = \pm 4\sqrt{3}$

$\iff a = -5+4\sqrt{3} \vee a = -5-4\sqrt{3} \iff a \in \{-5+4\sqrt{3}, -5-4\sqrt{3}\}$

We conclude that: $\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} = 1 \iff a \in \{-5+4\sqrt{3}, -5-4\sqrt{3}\}$

Homework 9-15

Homework

10/23/2019

Homework

(12) Let $A \in M_n(\mathbb{R})$ and $P \in GL(n, \mathbb{R})$

Define $B = P^{-1}AP$

Then A, B have the same eigenvalues

(prove they have same characteristic polynomials)

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} =$$

$$= (3-\lambda)(1-\lambda)(5-\lambda)$$

(13) Let $A, B \in M_n(\mathbb{R})$ and A non-singular

Then AB and BA have the same eigenvalues

(and the same characteristic polynomial)

Exam next weds

due Mon