10/14/2019 <u>Lecture 14</u> $n \times n$ system of Equations

▶ <u>$n \times n$ system of equations</u>

Given $\begin{cases} A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n = b_1 \\ A_{21}x_1 + A_{22}x_2 + \dots + A_{2n}x_n = b_2 \\ \vdots \\ A_{n1}x_1 + A_{n2}x_2 + \dots + A_{nn}x_n = b_n \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \Leftrightarrow$$

$\Leftrightarrow$ (1) $Ax = b$ with $A \in M_n(\mathbb{R})$ and $x, b \in M_{n1}(\mathbb{R})$

Eq(1) homogerous $\Leftrightarrow b = \mathbf{0}$ / Eq(1) inhomogeneous $\Leftrightarrow b \neq \mathbf{0}$

▶ <u>Cramer's rule</u> Write $A = [A_1\ A_2 \dots A_n]$

$A_1, A_2, \dots, A_n \in M_{n1}(\mathbb{R})$, columns of A

Define $D = \det(A) = \det([A_1\ A_2 \dots A_n])$

$D_1 = \det([b\ A_2 \dots A_n])$

$D_2 = \det([A_1\ b \dots A_n])$

$\vdots$

$D_n = \det([A_1\ A_2 \dots b])$

Then

a) if $D \neq 0$ then eq(1) has a unique solution $(x_1, x_2, \dots, x_n)$ with $x_K = \frac{D_K}{D}$, $\forall K \in [n]$

b) $\begin{cases} D = 0 \\ \exists K \in [n] : D_K \neq 0 \end{cases} \Rightarrow$

$\Rightarrow$ Eq(1) has no solution

<u>Remark</u>

If we assume $\begin{cases} D = 0 \\ \forall K \in [n] : D_K = 0 \end{cases}$ argument is inconclusive

10/14/2019 Solving Linear systems - Cramers Rule

Example Solve $\begin{cases} \lambda x + (\lambda-2)y = \lambda+1 \\ (\lambda+1)x - (\lambda-2)y = \lambda \end{cases}$ (1)

with respect to x, y

Solution Eq(1) $\Leftrightarrow \begin{bmatrix} \lambda & \lambda-2 \\ \lambda+1 & -(\lambda-2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda+1 \\ \lambda \end{bmatrix}$

$$D = \begin{vmatrix} \lambda & \lambda-2 \\ \lambda+1 & -(\lambda-2) \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda & 1 \\ \lambda+1 & -1 \end{vmatrix} =$$

$$= (\lambda-2)[\lambda(-1) - (1)(\lambda+1)] = (\lambda-2)[-\lambda-\lambda-1] = (\lambda-2)(-\lambda-1)$$

$$= -(\lambda-2)(2\lambda+1)$$

$$D_x = \begin{vmatrix} \lambda+1 & \lambda-2 \\ \lambda & -(\lambda-2) \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda+1 & 1 \\ \lambda & -1 \end{vmatrix} =$$

$$= (\lambda-2)((\lambda+1)(-1) - 1\cdot\lambda) = (\lambda-2)(-\lambda-1-\lambda) = -(\lambda-2)(2\lambda+1)$$

$$D_y = \begin{vmatrix} \lambda & \lambda+1 \\ \lambda+1 & \lambda \end{vmatrix} = \lambda^2 - (\lambda+1)^2 = [\lambda-(\lambda+1)][\lambda+(\lambda+1)] =$$

$$= (\lambda-\lambda-1)(2\lambda+1) = -(2\lambda+1)$$

distinguish between cases 1, 2, & 3

case 1 - $\lambda \in \mathbb{R} - \{-1/2, 2\}$

case 2 - $\lambda = 2$

case 3 - $\lambda = -1/2$

10/14/2019 **Example Cramer's Rule**

Solve with respect to x, y
$$\begin{cases} \lambda x + (\lambda - 2)y = \lambda + 1 \\ (\lambda+1)x - (\lambda-2)y = \lambda \end{cases}$$

Solution Since $D = 0 \Leftrightarrow -(\lambda-2)(2\lambda+1) = 0$
$\Leftrightarrow \lambda - 2 = 0 \lor 2\lambda + 1 = 0 \Leftrightarrow$
$\Leftrightarrow \lambda = 2 \lor \lambda = -1/2 \Leftrightarrow \lambda \in \{-1/2, 2\}$

we distinguish between 3 cases

Case 1 Assume that $\lambda \in \mathbb{R} - \{-1/2, 2\}$ then $D \neq 0$
therefore eq(1) has a unique solution (x, y) with
$$x = \frac{D_x}{D} = \frac{-(\lambda-2)(2\lambda+1)}{-(\lambda-2)(2\lambda+1)} = 1 \quad y = \frac{D_y}{D} = \frac{-(2\lambda+1)}{-(\lambda-2)(2\lambda+1)} = \frac{1}{\lambda-2}$$

Case 2 Assume that $\lambda = 2$ then $D = 0 \land D_y \neq 0 \Rightarrow$
$\Rightarrow$ Eq(1) has no solutions

Case 3 Assume that $\lambda = -1/2$
$$\text{Eq(1)} \Leftrightarrow \begin{cases} (-1/2)x + ((-1/2)-2)y = (-1/2)+1 \\ ((-1/2)+1)x - ((-1/2)-2)y = -1/2 \end{cases} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} (-1)x + (-1-4)y = -1+2 \\ (-1+2)x - (-1-4)y = -1 \end{cases} \Leftrightarrow \begin{cases} -x - 5y = 1 \\ x + 5y = -1 \end{cases}$$

Consi[stent] syste[m]
if in[consistent] no s[olutions]

$\Leftrightarrow x + 5y = -1 \Leftrightarrow x = -5y - 1 \Leftrightarrow (x, y) = (-5y-1, y) \Leftrightarrow$
$\Leftrightarrow (x, y) \in \{(-5t-1, t) \mid t \in \mathbb{R}\}$

we conclude that the solution set for eq(1) is given by
$$S \begin{cases} \{(1, \frac{1}{\lambda-2})\}, & \text{if } \lambda \in \mathbb{R} - \{-1/2, 2\} \\ \emptyset, & \text{if } \lambda = 2 \\ \{(-5t-1, t) \mid t \in \mathbb{R}\}, & \text{if } \lambda = -1/2 \end{cases}$$

10/14/2019 Cramer's Rule + Gaussian Elimination Introduction

Homework - 17, 18

general analysis of 2×2, 3×3 case of Cramers rule } project idea

what happens in inconclusive cases?
theoretical, hard to do for 3×3 case
use 2×2, assume z, go from there

Gaussian Elimination Overview

No inconclusive cases
can work with non square systems
not pratical for parametric systems

$n = \#$ equations
$m = \#$ unknown variables

Consider system $Ax = b$ with $A \in M_{nm}(\mathbb{R})$
with $x, b \in M_{m1}(\mathbb{R})$

we represent it with an augmented matrix

$$M = \left[\begin{array}{cccc|c} A_{11} & A_{12} & \dots & A_{1m} & b_1 \\ A_{21} & A_{22} & \dots & A_{2m} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nm} & b_m \end{array}\right]$$

notation

$S(M)$ = solution set of system represented by M

$$M_1 \sim M_2 \Leftrightarrow S(M_1) = S(M_2)$$

10/16/2019 Lecture 15, Gaussian Elimination

Properties of Gaussian Elimination

1) Transposition You can switch two rows, but NOT colu[mns]

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

2) Scalar multiplication You can multiply any r[ow] with $\lambda \in \mathbb{R} - \{0\}$

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] \cdot \lambda \sim \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ \lambda a_2 & \lambda b_2 & \lambda c_2 & \lambda d_2 \\ a_3 & b_2 & c_3 & d_3 \end{array}\right]$$

3) Linear Combination You can add a multiple of r[ow] into another row

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] \cdot \lambda \sim \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2+\lambda a_1 & b_2+\lambda b_1 & c_2+\lambda c_1 & d_2+ \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

Remark if we encounter $0 \; 0 \; \ldots \; 0 \mid a$, then

a) if $a \neq 0$, system has no solutions

b) if $a = 0$, equation is an identity therefore it can be deleted

10/16/2019 <u>Gaussian Elimination Examples</u>

a) $$(1)\ \begin{cases} 2x - y = 1 \\ x + y = 3 \\ 3x + y = 0 \end{cases}$$ ← overdetermined system (more equations 3 than unknowns 2)

<u>Solution</u> The augmented matrix for Eq(1) is

$$M = \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 1 & 3 \\ 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{array}\right] (-2)(-3) \sim$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1+(-2)\cdot 1 & 1+(-2)\cdot 3 \\ 0 & 1+(-3)\cdot 1 & 0+(-3)\cdot 3 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -3 & -5 \\ 0 & -2 & -9 \end{array}\right] \begin{array}{l} \\ \cdot 2 \\ \cdot 3 \end{array} \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -6 & -10 \\ 0 & -6 & -27 \end{array}\right] (-1) \sim$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -6 & -10 \\ 0 & 0 & -27+(-1)(-10) \end{array}\right] \Rightarrow$$ Eq(1) has no solution

10/16/2019 Gaussian Elimination Examples

b) $\begin{cases} x + z + 4w + 2v = 3 \\ y + 2w - v = -1 \\ -x + 3y + 2z = -2 \end{cases}$ ← underdetermined system (less equations 3 than unknowns 5)

$\Updownarrow$

$$\begin{cases} 1x + 0y + 1z + 4w + 2v = 3 \\ 0x + 1y + 0z + 2w - 1v = -1 \\ -1x + 3y + 2z + 0w + 0v = -2 \end{cases} \quad (1)$$

Solution

Eq(1) has augmented matrix

$$M = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ -1 & 3 & 2 & 0 & 0 & -2 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 3 & 3 & 4 & 2 & 1 \end{array}\right] (-3) \sim \left[\begin{array}{ccccc} 1 & 0 & 1 & 4 & 2 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 3 & 4+(-3)2 & 2+(-3)(-1 \end{array}\right.$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -2 & 5 & 4 \end{array}\right] \cdot 3 \sim \left[\begin{array}{ccccc|c} 3 & 0 & 3 & 12 & 6 & 9 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -2 & 5 & 4 \end{array}\right] (-1) \sim$$

$$\sim \left[\begin{array}{ccccc|c} 3 & 0 & 0 & 12+(-1)(-2) & 6+(-1)5 & 9+(-1)4 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -2 & 5 & 4 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 3 & 0 & 0 & 14 & 1 & 5 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -2 & 5 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 14/3 & 1/3 & 5/3 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -2/3 & 5/3 & 4/3 \end{array}\right]$$

← row-echelon form of augmented ma

it follows that Eq(1) $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} x + (14/3)w + (1/3)v = 5/3 \\ y + 2w - v = -1 \\ z - (2/3)w + (5/3)v = 4/3 \end{cases}$$

Now solve for x, y, z

10/16/2019 Gaussian Elimination Examples

Solution

Eq(1). $\Leftrightarrow \begin{cases} x + (14/3)w + (1/3)v = 5/3 \\ y + 2w - v = -1 \\ z - (2/3)w + (5/3)v = 4/3 \end{cases} \Leftrightarrow \begin{cases} x = -(14/3)w - (1/3)v + 5/3 \\ y = -2w + v - 1 \\ z = (2/3)w - (5/3)v + 4/3 \end{cases} \Leftrightarrow$

$\Leftrightarrow (x,y,z,w,v) = (\underbrace{-(14/3)w - (1/3)v + 5/3}_{x}, \underbrace{-2w + v - 1}_{y}, \underbrace{(2/3)w - (5/3)v + 4/3}_{z}, w, v) =$

$= \underbrace{(5/3, -1, 4/3, 0, 0)}_{A} + w\underbrace{(-14/3, -2, 2/3, 1, 0)}_{B} + v\underbrace{(-1/3, 1, -5/3, 0, 1)}_{C} \Leftrightarrow$

$\Leftrightarrow (x,y,z,w,v) \in \{a + tb + sc \mid t,s \in \mathbb{R}\}$ with

affine space

$a = (5/3, -1, 4/3, 0, 0)$

$b = (-14/3, -2, 2/3, 1, 0)$

$c = (-1/3, 1, -5/3, 0, 1)$

$\boxed{\text{Homework}}$ 19