10/07/2019 **Lecture 12**, CoFactor Expansion

<u>Recall that</u>, $\forall \overset{\text{row}}{a} \in [n] : \det A = \sum_{b=1}^{n} (-1)^{a+b} A_{ab} \det(M_{ab}(A))$

$\forall \overset{\text{column}}{b} \in [n] : \det A = \sum_{a=1}^{n} (-1)^{a+b} A_{ab} \det(M_{ab}(A))$

Co-Factor Expansion can take just as long to calculate, but depends on matrix A

Note $(-1)^{a+b} \rightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ alternates

a) <u>Example</u> $\begin{vmatrix} 3 & 1 & 2 \\ 1 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix} =$

say we pick row 2

$$\begin{aligned}
&\overset{a}{(-1)} \cdot \overset{b}{(1)} \cdot \overset{c}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} \\
+ &(1) \cdot (5) \cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} \\
+ &(-1) \cdot (1) \cdot \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}
\end{aligned}$$

a = placement sign ~~at~~ position
b = number at position
c = minor matrix det

when expanded, 3 more terms

$= -((1 \cdot 1) - (2 \cdot 3)) + 5((3 \cdot 1) - (2 \cdot 2)) - ((3 \cdot 3) - (1 \cdot 2))$

$= -(1 - 6) + 5(3 - 4) - (9 - 2)$

$= -(-5) + 5(-1) - 7 = 5 - 5 - 7 = \boxed{-7 = \det}$

Pick any row/col to expand, det will be same

Use zeros to advantage

10/07/2019 Co-Factor Expansion

b) Example $\begin{vmatrix} 4 & 1 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 1 \end{vmatrix}$ = Find det

best to expand row 3 or col 1

lets expand col 1

$$= (1)\cdot(4)\cdot\begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 1 & 3 & 1 \end{vmatrix} + 0 + 0 + 0$$

Best to expand row 2

$$= 4\cdot\left[(-1)\cdot(2)\cdot\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}\right] = 4\cdot -1\cdot 2\cdot\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= -8\left((2\cdot 1)-(3\cdot 3)\right) = -8(2-9) = -8(-7) = \boxed{56 = \det}$$

How to add zeros?

Homework (34)

Simplifying determinants

① Transpose two rows (or two columns) gives additive inverse

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_2 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

switch row

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 \\ c_3 & c_2 & c_1 \end{vmatrix}$$

switch column

what if you switch identical col/row?

10/07/2019 Simplifying determinants

(2) Two identical rows (or columns) gives zero determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$$

because switching yields a sign change only number where $x = -x$ is number zero, so det $= 0$

$$\begin{vmatrix} a_1 & a_2 & a_2 \\ b_1 & b_2 & b_2 \\ c_1 & c_2 & c_2 \end{vmatrix} = 0$$

can be identical rows <u>OR</u> identical columns

(3) Factor out common factor on row or column

$$\lambda \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} \lambda a_1 & \lambda a_2 & \lambda a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & \lambda a_2 & a_3 \\ b_1 & \lambda b_2 & b_3 \\ c_1 & \lambda c_2 & c_3 \end{vmatrix}$$

(4) <u>Linearity</u>: one row or column is a sum of two numbers

$$\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

split up big numbers can create identical rows/columns

Property 4 creates property 5

10/07/2019 Simplifying determinants

5 Nothing changes if we add x multiple of a row or column to another row or column

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \lambda \in \mathbb{R} \quad = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + \lambda a_1 & b_2 + \lambda a_2 & b_3 + \lambda a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

→ break up with linearity, factor out λ, creates identical

→ zero maker.

tips: properties work with any size matrix
try to stick with integers, easier

Example a)

$$\begin{vmatrix} 1 & 2 & -1 & 2 \\ 2 & -4 & -3 & 3 \\ 0 & 4 & 0 & 1 \\ 1 & 6 & 0 & 1 \end{vmatrix} \quad (-2) \quad (-1)$$

can we make these zero?

$$= \begin{vmatrix} 1 & 2 & -1 & 2 \\ 0 & -4+(-2)(2) & -3+(-2)(-1) & 3+(-2)(2) \\ 0 & 4 & 0 & 1 \\ 0 & 6+(-1)(2) & 0+(-1)(-1) & 1+(-1)(2) \end{vmatrix}$$

simplify
expand
repeat

$$= \begin{vmatrix} 1 & 2 & -1 & 2 \\ 0 & -8 & -1 & -1 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 1 & -1 \end{vmatrix} = (1)\cdot(1)\cdot \begin{vmatrix} -8 & -1 & -1 \\ 4 & 0 & 1 \\ 4 & 1 & -1 \end{vmatrix}$$

10/07/2019 Example - simplifying determinants

a) $(1)\cdot(1)\cdot\begin{vmatrix} -8 & -1 & -1 \\ 4 & 0 & 1 \\ 4 & 1 & -1 \end{vmatrix} = (1)\cdot(1)\cdot(4)\begin{vmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

$= 4\begin{vmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 1+(-2) & 0 & \cancel{1}-1+(-1) \end{vmatrix} = 4\begin{vmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & -2 \end{vmatrix}$

$= 4\cdot\left[(-1)\cdot(-1)\cdot\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix}\right] = 4\cdot\left[1\cdot\left((-2)(1)-(1)(-1)\right)\right]$

$= 4\cdot(-2+1) = 4(-1) \boxed{= -4 = \det}$

Homework 9

recommended method to solve determinants

$\begin{vmatrix} 10 & 12 & 11 \\ 9 & 10 & 10 \\ 10 & 9 & 10 \end{vmatrix} = ?$

use linearity to make this smaller

10/09/2019 <u>Lecture 13, Simplifying determinants</u>

<u>Example</u> Show that

$$\begin{vmatrix} a+b & b+c & c+a \\ c+a & a+b & b+c \\ b+c & c+a & a+b \end{vmatrix} = 2(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

<u>Note</u> $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a^3+b^3+c^3-3abc$ Euler disco

$(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

also equals

$= \frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$

What about $\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix}$? circulant matrix/determ

☆ project idea

<u>Solution</u> $\begin{vmatrix} a+b & b+c & c+a \\ c+a & a+b & b+c \\ b+c & c+a & a+b \end{vmatrix} = \begin{vmatrix} (a+b)+(b+c)+(c+a) & b+c & c+a \\ (c+a)+(a+b)+(b+c) & a+b & b+c \\ (b+c)+(c+a)+(a+b) & c+a & a+b \end{vmatrix}$

add 2nd, 3rd columns to 1st column

$$= \begin{vmatrix} 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & c+a & a+b \end{vmatrix} = 2(a+b+c)\begin{vmatrix} 1 & b+c & c+a \\ 1 & a+b & b+c \\ 1 & c+a & a+b \end{vmatrix} (-1)$$

factorization creates 1s

1s are easier to make 0s

10/09/2019 **Simplifying determinants**

Example Show that
$$\begin{vmatrix} a+b & b+c & c+a \\ c+a & a+b & b+c \\ b+c & c+a & a+b \end{vmatrix} =$$
$$= 2(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

Solution
$$2(a+b+c)\begin{vmatrix} 1 & b+c & c+a \\ 1 & a+b & b+c \\ 1 & c+a & a+b \end{vmatrix} =$$

take row 1 subtract from rows 2, 3

take minor matrix

$$2(a+b+c)\begin{vmatrix} 1 & b+c & c+a \\ 1-1 & (a+b)-(b+c) & (b+c)-(c+a) \\ 1-1 & (c+a)-(b+c) & (a+b)-(c+a) \end{vmatrix} = 2(a+b+c)\begin{vmatrix} 1 & b+c & c+a \\ 0 & a-c & b-a \\ 0 & a-b & b-c \end{vmatrix}$$

$$= 2(a+b+c)\cdot(+1)\cdot(1)\cdot\begin{vmatrix} a-c & b-a \\ a-b & b-c \end{vmatrix} =$$

$$= 2(a+b+c)\cdot\left[(a-c)(b-c)-(a-b)(b-a)\right] = 2(a+b+c)\cdot\left[(a-c)(b-c)-\left[-(a-b)^2\right]\right]$$

$$= 2(a+b+c)(ab-ac-bc+c^2+a^2-2ab+b^2)$$

$$= 2(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

For course project research

JSTOR, look up papers, see whats been done (on library site)

$\boxed{\text{HW: 9, 10, 11}}$

10/09/2019 **Matrix Inverses**

Recall that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, if $ad - bc \neq 0$

Definition Let $A \in M_n(\mathbb{R})$ then we define $\mathrm{adj}(A) \in M_n(\mathbb{R})$ such that

$$\forall a, b \in [n] : [\mathrm{adj}(A)]_{ab} = (-1)^{a+b} \det(M_{ba}(A))$$

Theorem Let $A \in GL(n, \mathbb{R})$, then

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$$

Project Ideas = LU code

If you have LU composition, can calculate $\det(A)$, A^{-1} faster
Recall: $A = LU \Rightarrow \det(A) = \det(L)\det(U)$

Example $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix} \leftarrow A^{-1}$ find it

remember if $\det(A) = 0$ there is no A^{-1}

Solution Since

$$\det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & (-1)+(1) \\ 2 & 1 & (-1)+(2) \\ 1 & 2 & (5)+(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 6 \end{vmatrix} =$$

make 0; add; take minor matrix

$$= (+1)(1)\begin{vmatrix} 1 & 1 \\ 2 & 6 \end{vmatrix} = (1 \cdot 6) - (1 \cdot 2) = 4 = \det(A)$$

and

$$[\mathrm{adj}(A)]_{11} = (-1)^{1+1} \det(M_{11}(A)) = (1)\begin{vmatrix} 1 & -1 \\ 2 & 5 \end{vmatrix} = (1)(5) - (-1)(2) = 5 + 2 = \boxed{7}$$

Keep finding locations...

10/09/2019 <u>Matrix Inverse</u>

<u>Solution</u> (cont) <u>Example</u> Find A^{-1} of $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix}$

Since $\det(A) = 4$, inverse exists

nd $\operatorname{adj}(A)_{11} = 7$

$[\operatorname{adj}(A)]_{12} = (-1)^{1+2} \det(M_{21}(A)) =$

$= -\begin{vmatrix} 0 & -1 \\ 2 & 5 \end{vmatrix} = -((0)(5) - (-1)(2)) = -(2) = -2 = \operatorname{adj}(A)_{12}$

$[\operatorname{adj}(A)]_{13} = (-1)^{1+3} \det(M_{31}(A)) = \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} =$

$= (0)(-1) - (-1)(1) = 1 = \operatorname{adj}(A)_{13}$ $\rightarrow$ repeat 6 times

results

$\operatorname{Adj}(A)_{21} = -11 \quad \operatorname{Adj}(A)_{22} = 6 \quad \operatorname{Adj}(A)_{23} = -1$

$\operatorname{Adj}(A)_{31} = 3 \quad \operatorname{Adj}(A)_{32} = -2 \quad \operatorname{Adj}(A)_{33} = 1$

It follows that $A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A) = \dfrac{1}{4} \begin{bmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{bmatrix}$

How to verify correct? - online, Maxima
or multiply $A \cdot \operatorname{adj}$, if you get $= \begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix}$
then it's right

Recall $A \cdot A^{-1} = I$

$\boxed{\text{Homework}}$ 12-16

Note
$\forall A, B \in GL(n, \mathbb{R}) : (AB)^{-1} = B^{-1} A^{-1}$
may be useful for homework