9/30/2019 Lecture 10 - Determinants

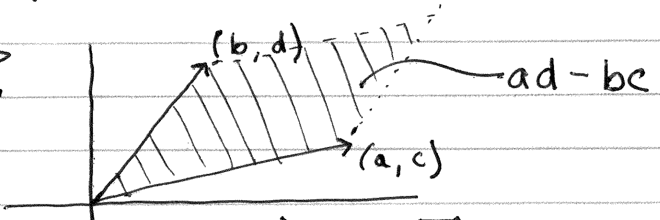
Recall that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\det \neq 0 \Rightarrow A^{-1} = \dfrac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

What is $ad - bc$?

visualized in 2D

geometric definition



say $\vec{AC}$ and $\vec{BD}$ are vectors

dotted lines are parallel to vectors

ad-bc is area of cross section between $\vec{AC}$ and $\vec{BD}$

algebraic definition of the determinant

1) Permutations Let $[n] = \{1, 2, 3, \ldots, n\}$

a permutation is a mapping $\sigma : [n] \to [n]$ such that each element $a \in [n]$ is mapped to a distinct element $\sigma(a) \in [n]$, i.e.

$$\forall a, b \in [n] : (\sigma(a) = \sigma(b) \Rightarrow a = b)$$

Example For $n = 3$ $\sigma = (2, 3, 1) \Rightarrow$

$$\Rightarrow \sigma(1) = 2 \wedge \sigma(2) = 3 \wedge \sigma(3) = 1$$

1 is like an index

the actual value at that index

9/30/2019 **Determinants, Algebraically - Permutations**

Remarks S_n = the set of all permutations on $[n]$

$|S_n| = n(n-1)(n-2)\dots(3)(2)(1)$ number of all possible permutations

also called factorial, $= n!$

grows faster than exponential

★ Project idea: shuffling a deck of cards, you have 52! possibilities, how likely are you to get same result twice, how soon will it take to get it? (consider time to shuffle, 10 se

2) Parity of permutation Let $\sigma \in S_n$ be a permutation we define parity $S(\sigma)$ such that

$$S(\sigma) = \text{sign}\left[\prod_{b=1}^{n-1} \prod_{a=b+1}^{n} (\sigma(a) - \sigma(b))\right]$$

$$\text{sign}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

σ even $\Leftrightarrow S(\sigma) = 1$

σ odd $\Leftrightarrow S(\sigma) = -1$

9/30/2019 Parity of Permutations

Example For $\sigma = (3,1,4,2) \in S_4$

$(1,2,3,4) \to (3,2,1,4) \to (3,1,2,4) \to$

$\to (3,1,4,2) = \sigma$, therefore $S(\sigma) = -1$

each change = called a transposition

3 transpositions $\Rightarrow \sigma$ odd

σ odd $\Leftrightarrow S(\sigma) = -1$

3) If $A \in M_n(\mathbb{R})$, then we define

$$\det(A) = \sum_{\sigma \in S_n} \left[S(\sigma) \prod_{a=1}^{n} A_{a,\sigma(a)} \right]$$

very useful for $n=2$, 2 terms
& $n=3$, 6 terms

$n=4$, 24 terms
$n=5$, 100+ terms
$n=6$, lots of calculations

Example For $n=2$ $S_2 = \{\overset{\text{even}}{(1,2)}, \overset{\text{odd}}{(2,1)}\}$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \Rightarrow \det(A) = A_{11}A_{22} - A_{12}A_{21}$$

9/30/2019 **Algebraic Definition of Determinant**

Example For $n=3$

$$S_3 = \{(1,2,3), (2,3,1), (3,1,2)$$ ← even (based on transpositio

odd → $(3,2,1), (2,1,3), (1,3,2)\}$

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} =$$

Note: first index 1, 2, 3
A A A is
second is based on permutation

$$= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32}$$
$$- A_{13}A_{22}A_{31} - A_{12}A_{21}A_{33} - A_{11}A_{23}A_{32}$$

How to remember?

$$\begin{array}{cc|ccc|cc} A_{12} & A_{13} & A_{11} & A_{12} & A_{13} & A_{11} & A_{12} \\ A_{22} & A_{23} & A_{21} & A_{22} & A_{23} & A_{21} & A_{22} \\ A_{32} & A_{33} & A_{31} & A_{32} & A_{33} & A_{31} & A_{32} \end{array}$$

subtract → | odd terms products | | even terms products | ← ad

$-A_{13}A_{22}A_{31}$ | $+A_{11}A_{22}A_{33}$

$-A_{12}A_{21}A_{33}$ | $+A_{12}A_{23}A_{31}$

$-A_{11}A_{23}A_{32}$ | $+A_{13}A_{21}A_{32}$

Homework: 1, 2, 4, 5, 6

↑ sarrus rule

9/30/2019 <u>n=3 determinant Examples</u>

<u>Example</u>

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \quad \text{find } \det(A)$$

<u>Solution</u>

$$\det(A) = \left| \begin{matrix} 1 & 2 & -2 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{matrix} \right| \begin{matrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{matrix} =$$

subtract add

$$= 1\cdot1\cdot1 + 2\cdot0\cdot0 + (-2)\cdot1\cdot3$$

$$-(-2)\cdot1\cdot0 - 1\cdot0\cdot3 - 2\cdot1\cdot1$$

$$\left.\begin{matrix} = 1 + 0 - 6 \\ -0 - 0 - 2 \end{matrix}\right\} = 1 - 6 - 2 \; \boxed{= -7}$$

det(A) =

10/02/2019 Lecture 11, Triangular matrices

▶ Determinant of upper/lower triangular matrices:

Definition Let $A \in M_n(\mathbb{R})$, then

A Upper triangular $\Leftrightarrow \forall a,b \in [n] : (a > b \Rightarrow A_{ab} = 0)$

A Lower triangular $\Leftrightarrow \forall a,b \in [n] : (a < b \Rightarrow A_{ab} = 0)$

A triangular $\Leftrightarrow$ (A upper triangular $\vee$ A lower triangular

Upper $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}$ Lower $\begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

Theorem Let $A \in M_n(\mathbb{R})$, then

A triangular $\Rightarrow \det(A) = \prod_{\alpha=1}^{n} A_{\alpha\alpha}$

Example $A = \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ Find $\det(A)$

Solution Since A upper triangular $\Rightarrow$ it follows that

$\Rightarrow \det A = A_{11} A_{22} A_{33} A_{44} = (1)(-1)(2)(7) = \boxed{-14}$

10/02/2019 <u>Properties of determinants</u>

1) $I \in M_n(\mathbb{R})$ identity matrix $\Rightarrow \det(I) = 1$

show $n=2$ then induction

2) $\forall A \in M_n(\mathbb{R}): \det(A^T) = \det(A)$

Proof - potential course project

3) $\forall A, B \in M_n(\mathbb{R}): \det(AB) = \det(A)\det(B)$

4) $\forall A \in M_n(\mathbb{R}): (\det(A) \neq 0 \Leftrightarrow A \in GL(n, \mathbb{R}))$
$\hookrightarrow \forall A \in M_n(\mathbb{R}): (\det(A) = 0 \Leftrightarrow A \notin GL(n, \mathbb{R}))$ contrapositive

5) $\forall A \in GL(n, \mathbb{R}): \det(A^{-1}) = \dfrac{1}{\det(A)}$

$A = LU$ + computer calculation of matrix A, lower $\times$ upper

<u>Application</u> $\forall A, B \in M_n(\mathbb{R}): (AB = I \Rightarrow BA = I)$

<u>Solution</u> Let $A, B \in M_n(\mathbb{R})$ be given and assume $AB = I$
Then $\det(A)\det(B) = \det(AB)$ [determinant property]
$= \det(I)$ [hypothesis $AB = I$]
$= 1 \Rightarrow$ [determinant property]
$\Rightarrow \det(A)\det(B) \neq 0$
$\Rightarrow \det(A) \neq 0 \wedge \det(B) \neq 0 \Rightarrow \det(A) \neq 0$
$\Rightarrow A \in GL(n, \mathbb{R})$ [determinant property]
Let A^{-1} be the inverse of A, then $BA = I(BA)$ [I identity] $\Rightarrow$
$\Rightarrow = (A^{-1}A)(BA)$ [A^{-1} inverse of A] $\Rightarrow$
$\Rightarrow = A^{-1}[A(BA)]$ [associative] $\Rightarrow = A^{-1}[(AB)A]$ [associative] $\Rightarrow$
$\Rightarrow = A^{-1}(IA)$ [hypothesis $AB = I$] $\Rightarrow$
$\Rightarrow = A^{-1}A$ [I identity] $\Rightarrow = I$ [A^{-1} inverse of A]
We conclude that $\forall A, B \in M_n(\mathbb{R}): (AB = I \Rightarrow BA = I)$

10/02/2019 Co-Factor expansion

Homework: $\boxed{1-7}$

Probability $A=LU$ not work due to division by zero? How to prove always reliab

Note $\mathcal{M}$ = minor matrix

Definition Let $A \in M_n(\mathbb{R})$ and let $a,b \in [n]$ we define $\mathcal{M}_{ab}(A) \in M_{n-1}(\mathbb{R})$ by deleting row a and column b such that

$$\forall c,d \in [n-1]: [\mathcal{M}_{ab}(A)]_{cd} = \begin{cases} A_{cd}, & \text{if } c<a \wedge d<b \\ A_{c+1,d}, & \text{if } c \geq a \wedge d<b \\ A_{c,d+1}, & \text{if } c<a \wedge d \geq b \\ A_{c+1,d+1}, & \text{if } c \geq a \wedge d \geq b \end{cases}$$

Formal definition for Minor matrix

Example $A = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 1 & 5 & 7 & 2 \\ 3 & 1 & 5 & 2 \\ 1 & 4 & 7 & 3 \end{bmatrix} \Rightarrow \mathcal{M}_{23}(A) = ?$

delete row 2, column 3

$$\mathcal{M}_{23}(A) = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 1 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

Property: Let $A \in M_n(\mathbb{R})$, Then

$$\forall a \in [n]: \det(A) = \sum_{b=1}^{n} (-1)^{a+b} A_{ab} \det(\mathcal{M}_{ab}(A))$$

$$\forall b \in [n]: \det(A) = \sum_{a=1}^{n} (-1)^{a+b} A_{ab} \det(\mathcal{M}_{ab}(A))$$

Easy to code recursively

if Aab is zero, easy to calculate
can manipulate matrix to have more z