9/23/2019 Lecture 8 Application to 2x2 linear syst

☆ Exam will be given weds - due Mon ☆

Recall that: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$

$ad - bc \neq 0 \Rightarrow A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$ad - bc = 0 \Rightarrow A$ singular

2x2 linear systems application

assume non-singular

$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \Leftrightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Leftrightarrow$

$\Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Proposition $\forall A \in GL(n,\mathbb{R}) : \forall x, b \in M_{n1}(\mathbb{R}) : (Ax = b \Leftrightarrow x$

Proof Let $A \in GL(n,\mathbb{R})$ and $x, b \in M_{n1}(\mathbb{R})$ be given

Then:

$Ax = b \Leftrightarrow A^{-1}(Ax) = A^{-1}b$ [cancellation law]

$\Leftrightarrow (A^{-1}A)x = A^{-1}b$ [associative property]

$\Leftrightarrow Ix = A^{-1}b$ [A^{-1} inverse of A]

$\Leftrightarrow \underline{x = A^{-1}b}$ [I identity matrix]

It follows that $\forall A \in GL(n,\mathbb{R}) : \forall x, b \in M_{n1}(\mathbb{R}) : (Ax = b \Leftrightarrow x = A^{-1}b)$

Proof holds for both $Ax = b \Rightarrow x = A^{-1}b \wedge x = A^{-1}b \Rightarrow Ax = b$

9/23/2019 Application to 2x2 linear systems

Example a) $\begin{cases} 2x+5y=12 \\ 3x-y=1 \end{cases} \iff \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \iff$

$\iff \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ 1 \end{bmatrix} \iff \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{(2)(-1)-(5)(3)} \begin{bmatrix} -1 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 12 \\ 1 \end{bmatrix}$

Recall $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ so $\begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}^{-1} = \frac{1}{2(-1)-(5)(3)} \begin{bmatrix} -1 & -5 \\ -3 & 2 \end{bmatrix}$

$\iff \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-2-15} \begin{bmatrix} -1 \cdot 12 + -5 \cdot 1 \\ -3 \cdot 12 + 2 \cdot 1 \end{bmatrix} \iff \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-17} \begin{bmatrix} -12 + -5 \\ -36 + 2 \end{bmatrix}$

$\iff \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-17} \begin{bmatrix} -17 \\ -34 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \iff \begin{cases} x=1 \\ y=2 \end{cases}$

$\iff (x,y) = (1,2)$

Example b) $\begin{cases} (a+1)x + (a-1)y = 4a+2 \\ (2a)x + (a-1)y = 7a-1 \end{cases}$. Solve with respect to x, y

Determinant notation

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det A = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

9/23/2019 Application to 2x2 linear systems

Example b) $\left\{ \begin{array}{l} (a+1)x + (a-1)y = 4a+2 \\ (2a)x + (a-1)y = 7a-1 \end{array} \right\} \longleftrightarrow$ Eq

Solution Since $\begin{vmatrix} a+1 & a-1 \\ 2a & a-1 \end{vmatrix} = (a+1)(a-1) - (a-1)(2a) =$

$= (a-1)[(a+1) - (2a)] = (a-1)(-a+1) = -(a-1)^2$

we distinguish between the following case

Case 1 Assume that $a \in \mathbb{R} - \{1\}$ Then

$$\text{Eq. (1)} \iff \begin{bmatrix} a+1 & a-1 \\ 2a & a-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4a+2 \\ 7a-1 \end{bmatrix} \iff$$

$$\iff \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+1 & a-1 \\ 2a & a-1 \end{bmatrix}^{-1} \begin{bmatrix} 4a+2 \\ 7a-1 \end{bmatrix} = \frac{1}{-(a-1)^2} \begin{bmatrix} a-1 & -(a-1) \\ -(2a) & a+1 \end{bmatrix} \begin{bmatrix} 4a \\ 7a \end{bmatrix}$$

$$\iff \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-(a-1)^2} \begin{bmatrix} (a-1)(4a+2) - (a-1)(7a-1) \\ (-2a)(4a+2) + (a+1)(7a-1) \end{bmatrix}$$ look for factors if none, simplify

$$\iff \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-(a-1)^2} \begin{bmatrix} (a-1)[(4a+2) - (7a-1)] \\ -8a^2 - 4a + 7a^2 - a + 7a - 1 \end{bmatrix} = \frac{1}{-(a-1)^2} \begin{bmatrix} (a-1)[4a+2-7a \\ (-8+7)a^2 + (-4-1 \end{bmatrix}$$

$$\iff \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-(a-1)^2} \begin{bmatrix} (a-1)(-3a+3) \\ -a^2 + 2a - 1 \end{bmatrix} = \frac{1}{-(a-1)^2} \begin{bmatrix} -3(a-1)^2 \\ -(a-1)^2 \end{bmatrix}$$

$$\iff \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-(a-1)^2} [-(a-1)^2] \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \iff (x, y) = (3, 1)$$

Case 2 assume $a = 1$

9/23/2019 Application to 2x2 Linear Systems

Example b) $\begin{cases} (a+1)x + (a-1)y = 4a+2 \\ (2a)x + (a-1)y = 7a-1 \end{cases}$

Solution Case 2 Assume that $a=1$, then

$$\begin{cases} (a+1)x + (a-1)y = 4a+2 \\ (2a)x + (a-1)y = 7a-1 \end{cases} \Leftrightarrow \begin{cases} (1+1)x + (1-1)y = 4(1)+2 \\ (2\cdot 1)x + (1-1)y = 7(1)-1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2x + 0y = 6 \\ 2x + 0y = 6 \end{cases} \Leftrightarrow 2x = 6 \Leftrightarrow x = 3 \Leftrightarrow$$

$$\Leftrightarrow (x,y) = (3,y) \Leftrightarrow (x,y) \in \{(3,b) \mid b \in \mathbb{R}\}$$

Say $7a-2$, then there is no solution at this case, inconsistent solution set

Homework: 18-25

9/25/2019 Lecture 9 - Matrix transpose

- Take Home Exam given

Lost exam? Email for another copy ASAP

Due Monday - in person / scanned if necessary

Matrix transpose

Definition Let $A \in M_{nm}(\mathbb{R})$, we define $A^T \in M_{mn}$ such t

$$\forall a \in [m] : \forall b \in [n] : (A^T)_{ab} = A_{ba} = \text{transpose of } A$$

Example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 7 & 5 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 4 & 5 \end{bmatrix}$

possibility for $A = A^T$ if A is square, else $A \neq A^T$

Definition Let $A \in M_n(\mathbb{R})$, we say A symmetric $\Leftrightarrow A^T = A \Leftrightarrow$
$\Leftrightarrow \forall a, b \in [n] : A_{ab} = A_{ba}$

Example $A = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 3 & 7 \\ 2 & 7 & 9 \end{bmatrix} = A^T$, is symmetric

Properties of matrix transpose

1. $\forall A \in M_{nm}(\mathbb{R}) : (A^T)^T = A$
2. $\forall A, B \in M_{nm}(\mathbb{R}) : (A+B)^T = A^T + B^T$
3. $\forall \lambda \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : (\lambda A)^T = \lambda(A^T)$
4. $\forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{km}(\mathbb{R}) : (AB)^T = B^T A^T$

Note $A^T \cdot B^T$ would not have compatible sizes for matrix multiplication (unless square) $\rightarrow A_{km} \cdot B_{n}$ $m \neq n$

9/25/2019 Properties of Matrix transpose

④ $\forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{km}(\mathbb{R}) : (AB)^T = B^T A^T$

⑤ $\forall A \in GL(n, \mathbb{R}) : (A^T)^{-1} = (A^{-1})^T$

Proof of ④ $(AB)^T = B^T A^T$

Let $A \in M_{nk}(\mathbb{R})$ and $B \in M_{km}(\mathbb{R})$ be given

Let $a \in [m]$ and $b \in [n]$ be given

Then $\left[(AB)^T\right]_{ab} = (AB)_{ba} = \sum_{c=1}^{k} A_{bc} B_{ca} =$

$= \sum_{c=1}^{k} A^T_{cb} B^T_{ac} = \sum_{c=1}^{k} B^T_{ac} A^T_{cb} = (B^T A^T)_{ab}$

It follows that

$\forall a \in [m] : \forall b \in [n] : \left[(AB)^T\right]_{ab} = (B^T A^T)_{ab} \Rightarrow$

$\Rightarrow (AB)^T = B^T A^T$ we conclude that

$\forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{km}(\mathbb{R}) : (AB)^T = B^T A^T$

Project: product of symmetric matrices $AB \neq BA$ necessarily

"What is easy to ask & difficult to answer?"

Probability of randomly generated matrices - Matlab

How often is $AB = BA$? Change variables, matrix size etc

$\forall A, B \in M_n(\mathbb{R}) : \begin{cases} A, B \text{ symmetric} \\ AB = BA \end{cases} \Rightarrow AB \text{ symmetric}$

Proof follows

9/25/2019 Properties of Matrix transpose

$\forall A, B \in M_n(\mathbb{R}) : \begin{cases} A, B \text{ symmetric} \\ AB = BA \end{cases} \Rightarrow AB$ symmetric

Proof Let $A, B \in M_n(\mathbb{R})$ be given
Assume that AB symmetric & $AB = BA$

Then $(AB)^T = B^T A^T$ [transpose of product]
$= B A^T$ [hypothesis B symmetric]
$= BA$ [hypothesis A symmetric]
$= AB$ [hypothesis $AB = BA$]

$\Rightarrow$ AB symmetric

we conclude that the claim is satisfied

(28) $\forall A, B \in M_n(\mathbb{R}) : (A, B, AB \text{ symmetric} \Rightarrow AB = BA)$

Proves if $AB = BA$ is not satisfied, AB cannot be symmetr

Homework: 26-33 *

Course Project topics due weeks 7-10 - we are on v
can discuss topics during office hours

theory - determinant theorys, skew-symmetric operations
application - computational, in a field,
randomly generated matrices & their properties
"how often do matrices commute?"

note A symmetric $\Leftrightarrow A^T = A$
A skew-symmetric $\Leftrightarrow A^T = -A \Leftrightarrow$
$\Leftrightarrow \forall a, b \in [n] : A_{ab} = -A_{ba}$