9/16/2019 Lecture 6 - Matrix Multiplication

Recall that $A \in M_{nk}(\mathbb{R})$ and $B \in M_{km}(\mathbb{R})$ $\to AB \in M$

$$\forall a \in [n] : \forall b \in [m] : (AB)_{ab} = \sum_{c=1}^{k} A_{ac} B_{cb}$$

▸ Matrix powers

Let $A \in M_n(\mathbb{R})$ be given

then: $A^n = \underbrace{A \cdot A \cdot \ldots \cdot A}_{n \text{ times}} = \prod_{k=1}^{n} A$

$A^0 = I$

Example Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

Find all $(x, y) \in \mathbb{R}^2$ such that $A^2 = xA - yI$

↳ $A^2 = xA - yI \Leftrightarrow (x, y) \in S$

Solution since $A^2 = A \cdot A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot 2 + 1\cdot 3 & 2\cdot 1 + 1\cdot 2 \\ 3\cdot 2 + 2\cdot 3 & 3\cdot 1 + 2\cdot 2 \end{bmatrix} = \begin{bmatrix} 4+3 & 2+2 \\ 6+6 & 3+4 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 12 & 7 \end{bmatrix}$$

and $xA - yI = x\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} - y\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2x & x \\ 3x & 2x \end{bmatrix} + \begin{bmatrix} -y & 0 \\ 0 & -y \end{bmatrix}$

$$= \begin{bmatrix} 2x-y & x \\ 3x & 2x-y \end{bmatrix}$$

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Solution since $A^2 = \begin{bmatrix} 7 & 4 \\ 12 & 7 \end{bmatrix}$ and $xA - yI = \begin{bmatrix} 2x-y & x \\ 3x & 2x-y \end{bmatrix}$

It follows that $A^2 = xA - yI \Leftrightarrow$

$$\Leftrightarrow \begin{bmatrix} 7 & 4 \\ 12 & 7 \end{bmatrix} = \begin{bmatrix} 2x-y & x \\ 3x & 2x-y \end{bmatrix}$$

$$\Leftrightarrow 2x-y=7 \wedge x=4 \wedge 3x=12 \wedge 2x-y=7$$

$$\Leftrightarrow \begin{cases} 2x-y=7 \\ x=4 \end{cases} \Leftrightarrow \begin{cases} 2\cdot 4-y=7 \\ x=4 \end{cases}$$

$$\Leftrightarrow \begin{cases} y=2\cdot 4-7=8-7=1 \\ x=4 \end{cases} \Leftrightarrow \boxed{(x,y)=(4,1)}$$

▶ Properties of Matrix Multiplication

$$\forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{kl}(\mathbb{R}) : \forall C \in M_{lm}(\mathbb{R}) :$$

$$: (AB)C = A(BC)$$

$$\forall A \in M_{nk}(\mathbb{R}) : \forall B, C \in M_{km}(\mathbb{R}) : A(B+C) = AB + AC$$

$$\forall B, C \in M_{nk}(\mathbb{R}) : \forall A \in M_{km}(\mathbb{R}) : (B+C)A = BA + CA$$

$$\forall \lambda \in \mathbb{R} : \forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{km}(\mathbb{R}) : \lambda(AB) = (\lambda A)B = A(\lambda B)$$

(mcknares) psuedo associative ghost

$$\forall A \in M_n(\mathbb{R}) : AI = IA = A$$

9/16/2019 Properties of Matrix Multiplication

FAL

Note it is not true that $\forall A, B \in M_n(\mathbb{R}) : AB = BA$

★ Potential course project generalize counterexample for all values of n where $AB \neq BA$

Keep in mind, if $n \neq 1$ it's false, if $n = 1$, this is true

also, identity matrix is commutative with all matrices

★ another course project does there exist a matrix other than identity matrix that is commutative with all matrices?

$\forall a, b, \in \mathbb{N} : A^a A^b = A^{a+b}$ $\quad \forall a, b \in \mathbb{N} : (A^a)^b = A^{ab}$

Power properties of matrixes

Note it is not true that $\forall A, B \in M_n(\mathbb{R}) : \forall a \in \mathbb{N} : (AB)^a = A^a B^a$

FALSE

▶ Manipulation properties

$\forall A, B, C \in M_{nm}(\mathbb{R}) : A = B \iff A + C = B + C$

$\forall A, B, C \in M_{nm}(\mathbb{R}) : A + B = C \iff A = C - B$

$\forall A, B \in M_{nk}(\mathbb{R}) : \forall C \in M_{km}(\mathbb{R}) : A = B \Rightarrow AC = BC$

$\forall C \in M_{nk}(\mathbb{R}) : \forall A, B \in M_{km}(\mathbb{R}) : A = B \Rightarrow CA = CB$

Recall that $0 \cdot 2 = 0 \cdot 3 \Rightarrow 2 = 3$ → False

★ Project idea: find matrices with properties or projection matrix

9/16/2019 Matrix Multiplication Properties

Example Let $B \in M_n(\mathbb{R})$ such that $\boxed{B^2 = B}$ ← projection matrix
Show that $(2B-I)^2 = I$

Solution Let $B \in M_n(\mathbb{R})$ and assume that $B^2 = B$

$(2B-I)^2 = (2B-I)(2B-I) = (2B-I)(2B) + (2B-I)(-I)$

$= (2B)(2B) - I(2B) + 2B(-I) + (-I)(-I)$

$= 4B^2 - 2B - 2BI + I^2 = 4B^2 - 2B - 2B + I$

$= 4B^2 - 4B + I = 4B - 4B + I$ [hypothesis $B^2 = B$]

$= 0B + I = 0 + I = I$

$\forall A, B \in M_n(\mathbb{R})$: $(A+B)^2 = A^2 + AB + BA + B^2$
this can be proven
But without commutative property, cannot prove
this is not always true: $A^2 + 2AB + B^2$

HW: 6, 7, 8, 9 (not 10 yet)

11 – 17

after matrix inverse, transpose
exam could be Weds next week at earliest

9/18/2019 **Lecture 7** Properties of Matrix Multiplication

Recall that $A \in M_{nk}(\mathbb{R})$ and $B \in M_{km}(\mathbb{R}) \to AB \in M_{nm}(\mathbb{R})$

$$\forall a \in [n] : \forall b \in [m] : (AB)_{ab} = \sum_{c=1}^{k} A_{ac} B_{cb}$$

Prove $\forall B, C \in M_{nk}(\mathbb{R}) : \forall A \in M_{km}(\mathbb{R}) : (B+C)A = BA + CA$

Solution Let $B, C \in M_{nk}(\mathbb{R})$ and $A \in M_{km}(\mathbb{R})$ be given

Let $a \in [n]$ and $b \in [m]$ be given

Then $[(B+C)A]_{ab} = \sum_{c=1}^{k} (B+C)_{ac} A_{cb} = \sum_{c=1}^{k} (B_{ac} + C_{ac}) A_{cb} =$

$= \sum_{c=1}^{k} (B_{ac}A_{cb} + C_{ac}A_{cb}) = \sum_{c=1}^{k} (B_{ac}A_{cb}) + \sum_{c=1}^{k} (C_{ac}A_{cb}) =$

$= (BA)_{ab} + (CA)_{ab} = (BA + CA)_{ab}$

It follows that $\forall a \in [n] : \forall b \in [m] : [(B+C)A]_{ab} = [BA + CA]_{a}$

$\Rightarrow \underline{(B+C)A = BA + CA}$ we conclude that the original claim is $\forall B, C \in M_{nk}(\mathbb{R}) : \forall A \in M_{km}(\mathbb{R}) : (B+C)A = B$

HW: 10 *

18/2019

Matrix inverses

Definition Let $A \in M_n(\mathbb{R})$ and $B \in M_n(\mathbb{R})$
we say that B inverse of $A \Leftrightarrow AB = BA = I$

notation $GL(n, \mathbb{R}) = \{ A \in M_n(\mathbb{R}) \mid \exists B \in M_n(\mathbb{R}) : AB = BA = I \}$

A non-singular $\Leftrightarrow A \in GL(n, \mathbb{R})$

A singular $\Leftrightarrow A \notin GL(n, \mathbb{R})$

▶ Uniqueness of matrix inverse

$$\forall A, B, C \in M_n(\mathbb{R}) : \left(\begin{cases} AB = BA = I \\ AC = CA = I \end{cases} \Rightarrow B = C \right)$$

Proof Let $A, B, C \in M_n(\mathbb{R})$ be given
Assume that $AB = BA = I$ and $AC = CA = I$

Then $B = IB$ [I identity]
$= (CA)B$ [hypothesis $CA = I$]
$= C(AB)$ [associative property]
$= CI$ [hypothesis $AB = I$]
$= C$ [I identity]

The claim follows $\forall A, B, C \in M_n(\mathbb{R}) : \left(\begin{cases} AB = BA = I \\ AC = CA = I \end{cases} \Rightarrow B = C \right)$

Notation if A non-singular, then the unique inverse of A is denoted by A^{-1}

9/18/2019 <u>Matrix Inverse Property</u>

▸ <u>Cancellation property</u>

$$\forall A, B \in M_n(\mathbb{R}) : \forall C \in GL(n, \mathbb{R}) : \begin{cases} CA = CB \iff A = B \\ AC = BC \iff A = B \end{cases}$$

<u>Proof</u> Let $A, B \in M_n(\mathbb{R})$ and $C \in GL(n, \mathbb{R})$ be given

<u>$(\Leftarrow)$</u>: Assume that $A = B$. Then it immediately follows that $CA = CB$

$(\Rightarrow)$: Assume that $CA = CB$. Then...

$$\begin{aligned} A &= IA && [\text{identity } I] \\ &= (C^{-1}C)A && [C^{-1} \text{ inverse of } C] \\ &= C^{-1}(CA) && [\text{associative property}] \\ &= C^{-1}(CB) && [\text{hypothesis } CA = CB] \\ &= (C^{-1}C)B && [\text{associative property}] \\ &= IB && [C^{-1} \text{ inverse of } C] \\ &= B && [\text{identity } I] \end{aligned}$$

It follows that $\forall A, B \in M_n(\mathbb{R}) : \forall C \in GL(n, \mathbb{R}) : \begin{cases} CA = CB \iff A = B \\ AC = BC \iff A = B \end{cases}$

9/18/2019 Intro to determinants

▶ Inverse of 2x2 matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then a) $ad - bc \neq 0 \Rightarrow A$ nonsingular with

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

b) $ad - bc = 0 \Rightarrow A$ singular

Notation $\det(A) = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

HW: #19, 20, 21, 22, 23, 25