8/28/2019 Quantified Statements

② Existential quantifier $\exists x \in A : p(x)$

"There is at least one $x \in A$ such that $p(x)$ is true"

$(\exists x \in \{a,b\} : p(x) \Leftrightarrow (p(a) \vee p(b))$

$(\exists x \in \{a,b,c\} : p(x) \Leftrightarrow (p(a) \vee p(b) \vee p(c))$

9/4/2019 Lin Alg textbooks added to class website

1/4/2019 Lecture 3: Quantified Statements

rsal: $\forall x \in A : p(x)$ → For all $x \in A$, $p(x)$ is true

ntial: $\exists x \in A : p(x)$ → There exists at least one $x \in A$ such that $p(x)$ is true

$\exists! x \in A : p(x)$ → There is a unique $x \in A$ such that $p(x)$ is true
(one value works, the rest do not)

$$(\exists! x \in A : p(x)) \iff \left[\exists x \in A : (p(x) \wedge (\forall y \in A - \{x\} : \overline{p(y)}))\right]$$

Nested Quantified statements:

$\forall x \in A : \exists y \in B : p(x, y)$ ← what happens for y ← what happens for x

→ a loop

"For all $x \in A$, there is at least one $y \in B$ such that $p(x, y)$"

$\exists y \in B : \forall x \in A : p(x, y)$

"There is at least one $y \in B$ such that for all $x \in A$, $p(x, y)$ is true."

$$\exists y \in B : \forall x \in A : p(x, y) \Rightarrow \forall x \in A : \exists y \in B : p(x, y)$$

(implies)

Quantified Statements

9/4/2019

▶ Examples from algebra

$$\forall a,b \in \mathbb{R} : (ab=0 \iff (a=0 \vee b=0))$$

$$\forall a,b \in \mathbb{R} : (a^2+b^2=0 \iff (a=0 \wedge b=0))$$

$$\forall a,b \in [0,+\infty] : (a=b \iff a^2=b^2)$$

▶ Example from Trigonometry

$$\forall x,y \in \mathbb{R} : (\sin x = \sin y \iff \exists k \in \mathbb{Z} : x = 2k\pi +$$

Definition of sets:

1) <u>By listing</u> e.g. $A = \{1,3,5,9\}$ ← high level description

machine language → $x \in A \iff (x=1 \vee x=3 \vee x=5 \vee x=9)$

elements in a set can repeat, e.g. $A = \{1,3,3\} = \{1,$

$$x \in A \iff (x=1 \vee x=3 \vee x=3) \iff (x=1 \vee x=$$

2) <u>By selection/predicate</u> $A = \{x \in U \mid p(x)\}$

where U is a set, $p(x)$ is a pre

$$x \in A \iff (x \in U \wedge p(x))$$

4/2019 Definition of sets

2) By selection/ predicate - examples:

$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$

$[n] = \{1, 2, 3, \dots, n\} = \{x \in \mathbb{N} \mid 1 \leq x \leq n\}$

$\hookrightarrow \boxed{x \in [n] \iff x \in \mathbb{N} \wedge 1 \leq x \leq n}$

every set can be defined by belonging conditions

3) By mapping:

$A = \{\varphi(x) \mid x \in U \wedge p(x)\}$ where $\varphi(x)$ is an expression
U is a set
$p(x)$ is a predicate

$y \in A \iff \exists x \in U : (p(x) \wedge \varphi(x) = y)$

Examples

a) Set of complex numbers

$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$

$z \in \mathbb{C} \iff \exists a, b \in \mathbb{R} : a + bi = z$

$\iff \exists a \in \mathbb{R} : \exists b \in \mathbb{R} : a + bi = z$

9/4/2019 Examples, defining sets by Mapping

b) Set of rational numbers

$$\mathbb{Q} = \{ \tfrac{a}{b} \mid a \in \mathbb{Z} \wedge b \in \mathbb{N} \wedge b \neq 0 \}$$

a = numerator, can make fraction positive or negative

$$x \in \mathbb{Q} \iff \exists a \in \mathbb{Z} : \exists b \in \mathbb{N} : (b \neq 0 \wedge x = \tfrac{a}{b})$$

Nested quantified statement

c) Set of Even numbers

$$A = \{0, 2, -2, 4, -4, \dots\}$$

more precise

$$A = \{2k \mid k \in \mathbb{Z}\}$$

$$x \in A \iff \exists k \in \mathbb{Z} : x = 2k$$

definition of even numbers for proofs

d) Set of odd numbers

$$B = \{0, 1, -1, 3, -3, \dots\}$$

more precise

$$B = \{2k+1 \mid k \in \mathbb{Z}\}$$

$$x \in B \iff \exists k \in \mathbb{Z} : x = 2k+1$$

Homework: 1, 2, 3, 4