Monday week 5

Example

Evaluate the integral $I = \int_C (x^2+y^2)\,dx + 2xy\,dy$ over the curve $C: \alpha(t) = (t^2, t^3)$, $\forall t \in [0,1]$

Solution   Define $f(x,y) = (x^2-y^2, 2xy)$

We note that $\dot{\alpha}(t) = (2t, 3t^2)$, $\forall t \in [0,1]$ it follows that

$$I = \int_C (x^2+y^2)\,dx + 2xy\,dy = \int_0^1 dt\, f(\alpha(t)) \cdot \dot{\alpha}(t) = \int_0^1 dt\, f(t^2, t^3) \cdot (2t, 3t^2)$$

$$= \int_0^1 ((t^2)^2 - (t^3)^2, 2(t^2)(t^3)) \cdot (2t, 3t^2)\, dt$$

$$= \int_0^1 (t^4 - t^6, 2t^5) \cdot (2t, 3t^2)\, dt$$

$$= \int_0^1 [2t(t^4 - t^6) + (2t^5)(3t^2)]\, dt$$

$$= \int_0^1 (2t^5 - 2t^7 + 6t^7)\, dt$$

$$= \int_0^1 (2t^5 + 4t^7)\, dt$$

$$= \left[\frac{2t^6}{6} + \frac{4t^8}{8}\right]_0^1$$

$$= \left[\frac{t^6}{3} + \frac{t^8}{2}\right]_0^1$$

$$= \frac{1^6 - 0^6}{3} + \frac{1^8 - 0^8}{2} = \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = \underline{\frac{5}{6}}$$

Monday week 5

Basic Properties of line integrals

Linearity

Let f, g be two vector fields, $\lambda_1, \lambda_2 \in \mathbb{R}$ and let C be a path. Then

$$\int_C (\lambda_1 f + \lambda_2) \cdot dl = \lambda_1 \int_C f \cdot dl + \lambda_2 \int_C g \cdot dl$$

Path Equivalence

Consider two paths $a : [a_1, a_2] \to \mathbb{R}^n$ and $b : [b_1, b_2] \to \mathbb{R}^n$. It is possible for a and b to sketch the same curve but with different velocities. In that case we want to be able to say that a and b are equivalent (notation: $a \equiv b$). Furthermore, we would like equivalent paths over the same function to give equal line integrals. We give a formal definition as follows.

Def: We say that $a \equiv b$ (a is equivalent to b) if and only if there is a function $u : [b_1, b_2] \to [a_1, a_2]$ such that

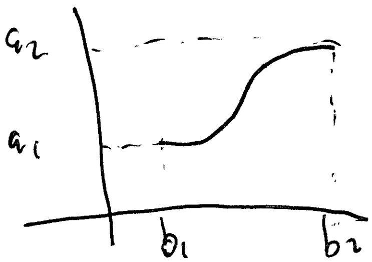
a) $u([b_1, b_2]) = [a_1, a_2]$

b) u differentiable on $[b_1, b_2]$

c) $\forall t \in [b_1, b_2] : u'(t) > 0$

d) $\forall t \in [b_1, b_2] : b(t) = a(u(t))$

Monday Week 5



▶ Note that an immediate consequence of the above conditions is that

$$\begin{cases} u(b_1) = a_1 \\ u(b_2) = a_2 \end{cases}$$

Then: let $a:[a_1, a_2] \to \mathbb{R}^n$ and $b:[b_1, b_2] \to \mathbb{R}^n$

be two paths then

$$a \equiv b \Rightarrow \int f \cdot da = \int f \cdot db$$

<u>Proof</u>

Since $a \equiv b$, there is a function $u: [b_1, b_2] \to [a_1, a_2]$ such that $u(b_1) = a_1 \wedge u(b_2) = a_2$ and

$$\forall t \in [b_1, b_2]: b(t) = a(u(t)) \Rightarrow$$

$$\Rightarrow \forall t \in [b_1, b_2]: \dot{b}(t) = \dot{a}(u(t))u'(t)$$

$$\int f \cdot db = \int_{b_1}^{b_1} f(b(t)) \cdot \dot{b}(t)\, dt = \int_{b_1}^{b_2} f(a(u(t))) \cdot \dot{a}(u(t))u'(t)dt$$

Let $\tau = u(t) \Rightarrow \begin{cases} d\tau = u'(t)dt \\ u(b_1) = a_1 \\ u(b_2) = a_2 \end{cases} \Rightarrow$

$$\Rightarrow \int f \cdot db = \int_{a_1}^{a_2} f(a(\tau)) \cdot \dot{a}(\tau)d\tau = \int f \cdot da$$

Monday week 5

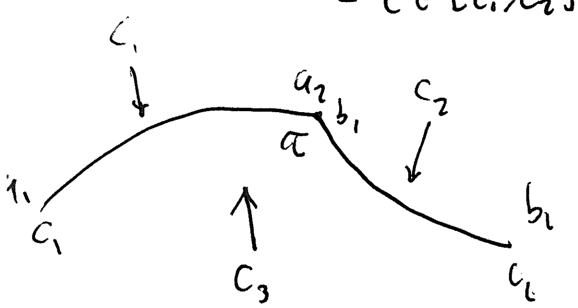
Path Merging

Consider 3 paths defined as

$C_1 : a(t) \; ; \; \forall t \in [a_1, a_2]$
$C_2 : b(t) \quad \forall t \in [b_1, b_2]$
$C_3 : c(t) : \; \forall t \in [c_1, c_2]$

we say that

$$C_3 = C_1 \cup C_2 \Rightarrow \exists \tau \in [c_1, c_2] = \begin{cases} a = c \restriction [c_1, \tau] \\ b = c \restriction [\tau, c_2] \end{cases}$$



- let F be a vector field. Then it can be shown that

$$\int_{C_1 \cup C_2} F \cdot dl = \int_{C_1} F \cdot dl + \int_{C_2} F \cdot dl$$

Path reversal

- Let $C : a(t), \forall t \in [t_1, t_2]$ be a path line we define the reverse path $-C_1$ as follows

  $-C_1 : b(t) = a(t_1 + t_2 - t), \; \forall t \in [t_1, t_2]$

- The reverse path $-C$ traverses the same points in space as C but in the reverse direction
- Given a vector field F, we can show that

$$\int_{-C} F \cdot dl = -\int_C F \cdot dl$$

Monday week 5

Conservative fields and potential functions

Thm: let $\phi : A \to \mathbb{R}$ with $A \subset \mathbb{R}^n$. Let $x, y \in A$

Assume that

$$\begin{cases} A \text{ open and path-connected} \\ \phi \text{ differentiable on } A \\ \nabla\phi \text{ continuous on } A \end{cases}$$

Then

$$\forall C \in P_A(x,y) : \int_C \nabla\phi \cdot d\ell = \phi(y) - \phi(x)$$

Proof

Let $C \in P_A(x,y)$ ~~with~~ $C: a(t)\ \forall t \in [a_1, a_2]$ be given such that $a(a_1) = x$ and $a(a_2) = y$. Define

$$\forall t \in [a_1, a_2] : g(t) = \phi(a(t))$$

Then : $\forall t \in [a_1, a_2] : \dot{g}(t) = \nabla\phi(a(t)) \cdot \dot{a}(t)$

Since $a(t)$ is piecewise smooth, let $a_1 = t_0 < t_1 < t_2 < \cdots < t_n = a_2$

be a partition of the interval $[a_1, a_2]$ such that $\forall k \in [n] : a \restriction [t_{k-1}, t_k]$ is a smooth path

Monday week 5

It follows that

$$\int_C \nabla\varphi \cdot dl = \int_{a_1}^{a_2} \nabla\varphi(a(t))\cdot \dot{a}(t)\,dt = \int_{a_1}^{a_2} g'(t)\,dt$$

$$= \sum_{k\in[n]} \int_{t_{k-1}}^{t_k} g'(t)\,dt = \sum_{k\in[n]} [g(t_k) - g(t_{k-1})]$$

$$= g(t_n) - g(t_0) = g(a_2) - g(a_1)$$

$$= \phi(a(a_2)) - \varphi(a(a_1))$$

$$= \phi(y) - \phi(x)$$

. we see that the line integral depends only on the initial and final points. and is independent of the path connecting the two points

Potential functions

Def $f: A \to \mathbb{R}^n$ be a vector field with $A \subseteq \mathbb{R}^n$ we say that

$$f \text{ conservative} \iff \exists \phi : \begin{cases} \varphi : A \to \mathbb{R} \\ \forall x \in A : f(x) = \nabla\phi(x) \end{cases}$$

Remark

f conservative $\to f = \nabla\phi$

f not conservative $\to f = \nabla\varphi + \nabla \times A$

scalar field vector potential

Mondy week 5

If f is a conservative vecta field and $f = \nabla \phi$ then ϕ = potential fon fion if f and furthermore

$$C \in P_A(a\ b) \Rightarrow \int_C f \cdot dl = \phi(b) - \phi(a)$$

If a vecta field is conservative, the fastest way to find its potential function is as follows

How to find the potential

Consider for example a three-dimensional function vecta field $f = (f_1, f_2, f_3)$. If $f = \nabla \phi$, then

$$\frac{\partial \phi}{\partial x} = f_1 \quad \text{and} \quad \frac{\partial \phi}{\partial y} = f_2 \quad \text{and} \quad \frac{\partial f}{\partial z} = f_3$$

Integrating with respect to x, y, z gives

$$\phi(x, y, z) = \int f_1(x, y, z)\, dx + A(y, z) \qquad (1)$$

$$\phi(x, y, z) = \int f_2(x, y, z)\, dy + B(z, x) \qquad (2)$$

$$\phi(x, y, z) = \int f_3(x, y, z)\, dz + C(x, y) \qquad (3)$$

Here $A(y, z)$, $B(z, x)$, $C(x, y)$ are integration constants to find ϕ it is sufficient to define A, B, C such that equations (1), (2), (3) given each other. Then any equations (1) (2) (3) yields the potential function

Example

Monday week 5

$F(x,y,z)=(2xyz+z^2-2y^2+1,\ x^2z-4xy,\ x^2y+2xz-2)$

Find ϕ such that $F=\nabla\phi$

Solution

Define

$F_1(x,y,z)=2xyz+z^2-2y^2+1$

$F_2(x,y,z)=x^2z-4xy$

$F_3(x,y,z)=x^2y+2xz-2$

Then

$\phi(x,y,z)=\int F_1(x,y,z)\,dx+A(y,z)=x^2yz+xz^2-2xy^2+x+A(y,z)$

$\phi(x,y,z)=\int F_2(x,y,z)\,dy+B(z,x)=x^2yz-2xy^2+B(z,x)$

$\phi(x,y,z)=\int F_3(x,y,z)\,dz+C(x,y)=x^2yz+xz^2-2z+C(x,y)$

Define

$A(y,z)=-2z$

$B(z,x)=xz^2+x-2z$

$C(x,y)=-2xy^2+x$

in order satisfy Eq(1) Eq(2) Eq(3)

therefore $\phi(x,y,z)=x^2yz+xz^2-2xy^2+x-2z$

Potential as a line integral Tuesday week 5

Def: Let $f: A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^n$ f path-independent in A $\Leftrightarrow$

$\Leftrightarrow \forall x,y \in A: \exists I \in \mathbb{R}: \forall c: P_A(x,y): \int_c f \cdot dl = I$

$\hookrightarrow$ notation

if f is path-independent on A then

$I = \int_x^y f \cdot dl$

<u>Thm</u>: let $f: A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^n$. Assume that

a) f path independent on A
b) A open and path-connected
c) $\phi(x) = \int_a^x f \cdot dl, \ \forall x \in A$

Then $\forall x \in A \quad \nabla\phi(x) = f(x)$

- This theorem is the generalization of the first fundamental theorem of calculus to line integrals.

2nd method: How to find the potential ϕ

- For each $x \in A$, choose a convenient path from some $a \in A$ to x and calculate the line integral

$$\phi(x) = \int_a^x f \cdot dl$$

- Check whether $\nabla\phi = f$. If yes, then f is conservative with potential function ϕ. if no, then f is not conservative.

Tuesday Week 5

Set of all loops in A

$$\text{Loop}(A) = \bigcup_{x \in A} P_A(x,x)$$

<u>Thm</u>: Let $f: A \to \mathbb{R}^n$ Assume A is open and <u>path-connected</u>

The following are equivalent

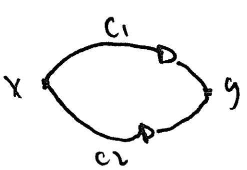
a) f conservative in A

b) f path independent in A

c) $\forall C \in \text{Loop}(A): \oint_C f \cdot dl = 0$

Proof

<u>(c) => (b)</u>: Assume that $\forall C \in \text{Loop}(A): \oint_C f \cdot dl = 0$ we will show that f is path independent. Let $x, y \in A$ be given. Let $C_1, C_2 \in P_A(x,y)$ be given. we define the closed path $C = C_1 \cup (-C_2)$



it follows that

$$\oint_C f \cdot dl = \oint_{C_1 \cup (-C_2)} f \cdot dl = \int_{C_1} f \cdot dl + \int_{-C_2} f \cdot dl = \int_C f \cdot dl - \int_{C_2} f \cdot dl \quad (1)$$

Since: $\oint_C f \cdot dl = 0 \Rightarrow \int_{C_1} f \cdot dl - \int_{C_2} f \cdot dl = 0 \Rightarrow \int_{C_1} f \cdot dl = \int_{C_2} f \cdot dl$

thus

$\forall x, y \in A \ \forall C_1, C_2 \in P_A(x,y): \int_{C_1} f \cdot dl = \int_{C_2} f \cdot dl \Rightarrow f$ path independent

Tuesday week 5

(b) $\Rightarrow$ (a) Assume that f is path-independent we will show that f is conservative

Choose some $a \in A$ and define the scalar field

$$\phi(x) := \int_a^x f \cdot d\ell \quad \forall x \in A$$

Since f is path-independent $\Rightarrow \forall x \in A: \nabla\phi(x) = f(x) \Rightarrow f$ conservative

(a) $\Rightarrow$ (c) Assume that f is conservative. Let $C \in \text{loop}(A)$ be given. Choose $x, y \in C$ and write $C = C_1 \cup (-C_2)$ with $C_1, C_2 \in P_A(x,y)$. Since f is conservative, there is a scalar field $\phi: A \to \mathbb{R}$ such that $\forall x \in A: \nabla\phi(x) = f(x)$ it follows that:

$$\int_{C_1} f \cdot d\ell = \int_{C_1} \nabla\phi \cdot d\ell = \phi(y) - \phi(x) \quad (1)$$

$$\int_{C_2} f \cdot d\ell = \int_{C_2} \nabla\phi \cdot d\ell = \phi(y) - \phi(x) \quad (2)$$

and therefore

$$\oint_C f \cdot d\ell = \int_{C_1} f \cdot d\ell - \int_{C_2} f \cdot d\ell = (\phi(y) - \phi(x)) - (\phi(y) - \phi(x)) = 0, \quad \forall C \in \text{loop}(A)$$

Tuesday Week 5

EXAMPLE

a) Show that the function $f(x,y) = (3x^2y, x^2y)$ is not conservative

Solution

Define the path $C(x,y)$: $a(t) = (xt, yt)$, $\forall t \in [0,1]$ from the point $(0,0)$ to (x,y). Then:

$\dot{a}(t) = (x,y)$ $\forall t \in [0,1]$

Define:

$$\phi(x,y) = \int_{C(x,y)} f \cdot dl = \int_0^1 f(xt, yt) \cdot \dot{a}(t)\,dt = \int_0^1 (3(xt)^2(yt), (xt)^2(yt)) \cdot (x,y)\,dt =$$

$$= \int_0^1 t^3 (3x^2y, x^2y) \cdot (x,y)\,dt =$$

$$= [(3x^2y)x + (x^2y)x] \int_0^1 t^3\,dt =$$

$$= (3x^3y + x^2y^2)\left[\frac{t^4}{4}\right]_0^1$$

$$= \frac{1}{4}(3x^3y + x^2y^2).$$

we note that

$$\frac{\partial\phi}{\partial x} = \frac{1}{4}\frac{\partial}{\partial x}(3x^3y + x^2y^2) = \frac{1}{4}(9x^2y + 2xy^2) \neq 3x^2y$$

$\Rightarrow \nabla\phi \neq f \Rightarrow f$ is not conservative

Tuesday Week 5

Greens theorem

Thm: Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ with $A \subset \mathbb{R}^2$

Let $C \in \underline{Jord(A)}$. Assum that A open $\wedge$ A simply connected

f, g differentiable on $C \cup int(C)$

$\nabla f, \nabla g$ continuous on $C \cup int(C)$

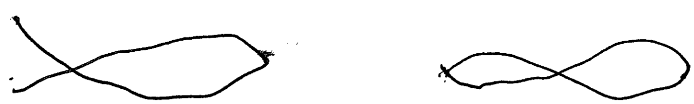
C, positive-oriented

Then $$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_{int(C)} dx\,dy \left(\frac{\partial g(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y} \right)$$

C simple $\Leftrightarrow \forall t_1, t_2 \in [\alpha_1, \alpha_2) : (t_1 \neq t_2 \Rightarrow \alpha(t_1) \neq \alpha(t_2))$

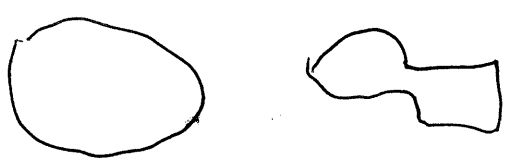
Simple:

Not simple:

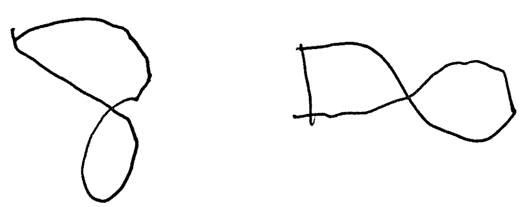


C Jordan curve $\Leftrightarrow \begin{cases} C \in Loop(\mathbb{R}^2) \\ C \text{ simple} \end{cases}$

Jordan curves:



Not Jordan curves



Jordan's Theorem

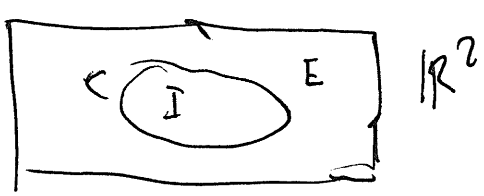
Thm: For every Jordan curve C there are two sets $I \subseteq \mathbb{R}$ and $E \subseteq \mathbb{R}^2$ such that all of the following statements are true.

a) $I \cup C \cup E = \mathbb{R}^2$

b) I, E are open sets

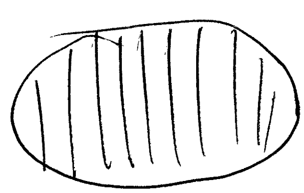
c) I bounded $\wedge$ E Not bounded

d) $\partial I = \partial E = C$ (i.e. C is the boundary set for both I and E)

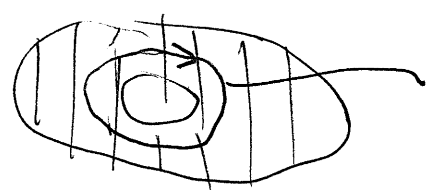


A simply connected $\iff \begin{cases} A \text{ path - connected} \\ \forall C \in \text{Jord}(A) : \text{int}(C) \subseteq A \end{cases}$

Simply connected



not simply connected



$\text{Int}(C) \subseteq A$

Tuesda week 5

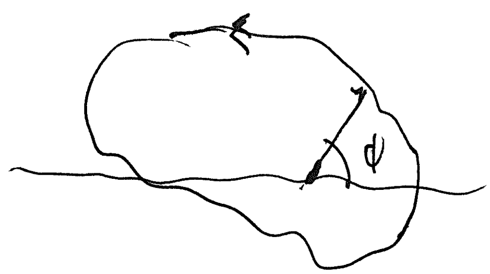
Orientation of Jorden curves

Def: Let $C \in Jord(\mathbb{R}^2)$ be a Jorden curve and let $(x_0, y_0) \in Int(C)$ be a point interior to the curve C. Consider the following polar coordinates presentation of C:

$$C: (x, y) = (x_0, y_0) + a(t)(\cos(\phi(t)), \sin(\phi(t))), \forall t \in [t_1, t_2]$$

we define the winding number $w(C)$ of C as

$$w(C) = \frac{1}{2\pi} \int_{t_1}^{t_2} \phi'(t)\, dt$$



Def Let $C \in Jord(\mathbb{R}^2)$ be a Jorden curve. We say that

a) C is positive oriented $\Leftrightarrow w(C) = +1$

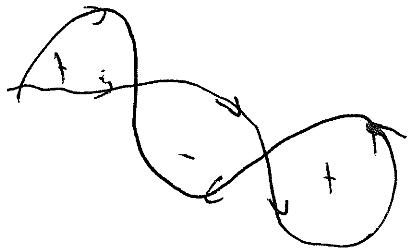
b) C is negative oriented? $\Longrightarrow w(C) = -1$

Another way to define the winding number

$$w(C|a) = \oint_C \frac{-(y-y_0)\, dx + (x-x_0)\, dy}{(x-x_0)^2 + (y-y_0)^2}$$

Tuesday week 5

For



C is a union of Jordan curves

$C = C_1 \cup C_2 \cup \cdots \cup C_n$

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \sum_{k=1}^{n} w(C_k) \iint_{\text{Int } C(k)} dx\,dy \left(\frac{\partial g(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y} \right)$$

Wednesday week 5

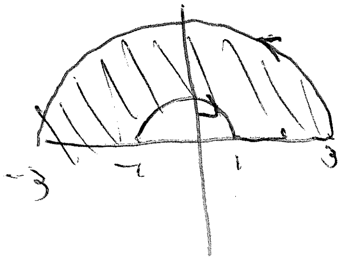
Example

$A = \{ (r\cos\theta, r\sin\theta) \mid r \in [1, 3] \wedge \theta \in [0, \pi] \}$

$C = \partial A$ ← positive oriented

$I = \oint_C y^2 dx + x^2 dy$

Solution



$$I = \oint_C y^2 dx + x^2 dy$$
$$= \iint_A dx\,dy \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2) \right]$$
$$= \iint dx\,dy \,[2x - 2y]$$
$$= 2 \iint_A dx\,dy\,(x - y)$$

Let $x = r\cos\theta$ and $y = r\sin\theta$ then $dx\,dy = r\,dr\,d\theta$

Define $B = \{ (r, \theta) \mid r \in [1, 3] \wedge \theta \in [0, \pi] \}$

$$I = 2\iint_B dr\,d\theta\, r(r\cos\theta - r\sin\theta) = 2\iint_B dr\,d\theta\, r^2(\cos\theta - \sin\theta)$$
$$= 2\int_1^3 dr \int_0^{\pi} d\theta\, r^2(\cos\theta - \sin\theta)$$
$$= 2\left[\int_1^3 dr\, r^2 \right]\left[\int_0^{\pi} d\theta\,(\cos\theta - \sin\theta) \right]$$
$$= 2 I_1 I_2$$

Wednesday week 5

with

$$I_1 = \int_1^3 dr\, r^2 = \left[\frac{r^3}{3}\right]_1^3 = \frac{3^3 - 1^3}{3} = \frac{26}{3}$$

$$I_2 = \int_0^{\pi} d\theta\, (\cos\theta - \sin\theta) = \left[\sin\theta + \cos\theta\right]_0^{\pi} = (0 + (-1)) - (0+1) = -2$$

therefore

$$I = 2 I_1 I_2 = 2\left(\frac{26}{3}\right)(-2) = -\frac{104}{3}$$

Applications of Green theorem

① Area calculation

Given $C \in \mathrm{Jord}(\mathbb{R}^2)$, the area $A(C)$ of $\mathrm{int}(C)$ is given by

$$A(C) = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

Proof

$$\frac{1}{2}\oint_C (x\,dy - y\,dx) = \frac{1}{2}\iint_{\mathrm{int}(C)} dx\,dy \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right]$$

$$= \frac{1}{2}\iint_{\mathrm{int}(C)} dx\,dy\,(1 - (-1))$$

$$= \iint_{\mathrm{int}(C)} dx\,dy$$

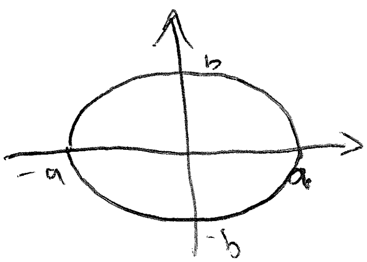
$$= A(C)$$

Wednesday week 5

EXAMPLE

Find the area of

$S = \{(x,y) \in \mathbb{R}^2 \mid x^2/a^2 + y^2/b^2 \leq 1\}$



<u>Solution</u>

Since ∂S: $(x,y) = (a\cos\theta, b\sin\theta)$, $\forall \theta \in [0, 2\pi]$

we have $(\dot{x}, \dot{y}) = (-a\sin\theta, b\cos\theta)$, $\forall \theta \in [0, 2\pi]$

therefore

$$A(S) = \iint_S dx\,dy = \frac{1}{2}\oint_{\partial S} (x\,dy - y\,dx) = \frac{1}{2}\int_0^{2\pi} d\theta\,(-y(\theta), x(\theta)) \cdot (-a\sin\theta, b\cos\theta)$$

$$= \frac{1}{2}\int_0^{2\pi} d\theta\,(-b\sin\theta, a\cos\theta)(-a\sin\theta, b\cos\theta)$$

$$= \frac{1}{2}\int_0^{2\pi} d\theta\,[(-b\sin\theta)(-a\sin\theta) + (a\cos\theta)(b\cos\theta)]$$

$$= \frac{1}{2}\int_0^{2\pi} d\theta\, ab\,[\sin^2\theta + \cos^2\theta]$$

$$= \frac{ab}{2}\int_0^{2\pi} d\theta$$

$$= \frac{ab}{2}[\theta]_{\theta=0}^{\theta=2\pi}$$

$$= \frac{ab}{2}(2\pi - 0)$$

$$= \pi ab$$

Wednesday week 5

Applications of Green's theorem

② Conservative vector fields in $\mathbb{R}^2$

Thm: Let $f(x,y) = (f_1(x,y), f_2(x,y))$, $\forall (x,y) \in A$. Assume that

$$\begin{cases} A \text{ open set} \wedge A \text{ simply connected} \\ f_1, f_2 \text{ differentiable on } A \\ \nabla f_1, \nabla f_2 \text{ continuous on } A \end{cases}$$

Then

$$f \text{ conservative} \Leftrightarrow \forall (x,y) \in A : \frac{\partial f_2(x,y)}{\partial x} = \frac{\partial f_1(x,y)}{\partial y}$$

Proof

$\Rightarrow$: Assume that f conservative. Choose $\phi(x,y)$ such that

$\forall (x,y) \in A : f(x,y) = \nabla \varphi(x,y)$

Then
$$\begin{cases} f_1(x,y) = \dfrac{\partial \varphi(x,y)}{\partial x}, \forall (x,y) \in A \\ f_2(x,y) = \dfrac{\partial \varphi(x,y)}{\partial y}, \forall (x,y) \in A \end{cases}$$

It follows that

$$\forall (x,y) \in A : \frac{\partial f_2(x,y)}{\partial x} = \frac{\partial}{\partial x}\left[\frac{\partial \varphi}{\partial y}(x,y)\right] \overset{\text{Clairots Theorem}}{=} \frac{\partial}{\partial y}\left[\frac{\partial \varphi}{\partial x}(x,y)\right] = \frac{\partial f_1(x,y)}{\partial y}$$

Wednesday Week 5

Proof

($\Leftarrow$): Assume that $\forall (x,y) \in A$. $\frac{\partial f_2(x,y)}{\partial x} = \frac{\partial_1 f_1(x,y)}{\partial y}$

Let $C \in Loop(A)$ be given

while $C = \bigcup_{\alpha \in I} C_\alpha$ such that $\forall \alpha \in I\, Jord(A)$

Then $\int_C f(x,y) \cdot d\ell = \sum_{\alpha \in I} \oint_{C_\alpha} f_1(x,y)\,dx + f_2(x,y)\,dy$

$$= \sum_{\alpha \in I} w(C_\alpha) \iint_{Int(C_\alpha)} dx\,dy \left[\frac{\partial}{\partial x} f_2(x,y) - \frac{\partial}{\partial y} f_1(x,y)\right]$$

$$= \sum_{\alpha \in I} w(C_\alpha) \iint_{int(C_\alpha)} dx\,dy(0) = 0 \quad \forall c \in Loop(C)$$

$\Rightarrow f$ conservative on A

f not conservative $\Leftrightarrow \exists (x,y) \in A$ $\frac{\partial f_2}{\partial x}(x,y) \neq \frac{\partial f_1}{\partial y}(x,y)$

Wednesday week 5

$f(x,y) = (x-y, x-2), \ \forall (x,y) \in \mathbb{R}^2$

Investigate if f is conservative

Solution

Define $f_1(x,y) = x-y$ and $f_2(x,y) = x-2$

Then

$$\frac{\partial f_2}{\partial x}(x,y) = \frac{\partial}{\partial x}(x-2) = 1 \quad \forall (x,y) \in \mathbb{R}^2$$

$$\frac{\partial f_1}{\partial y}(x,y) = \frac{\partial}{\partial y}(x-y) = -1 \quad \forall (x,y) \in \mathbb{R}^2$$

therefore

$$\exists (x,y) \in \mathbb{R} \cdot \frac{\partial f(x,y)}{\partial y} \neq \frac{\partial f_2(x,y)}{\partial x} \Rightarrow$$

$\Rightarrow f$ not conservative

Wednesday week 5

$f(x, y) = (3+2xy, x^2-3y^2) \quad \forall (x,y) \in \mathbb{R}^2$

Examine whether f is conservative

Solution

Define $\begin{cases} f_1(x,y) = 3+2xy & \forall (x,y) \in \mathbb{R}^2 \\ f_2(x,y) = x^2-3y^2 & \forall (x,y) \in \mathbb{R}^2 \end{cases}$

Then

$\frac{\partial f_1}{\partial y}(x,y) = \frac{\partial}{\partial y}(3+2xy) = 2x \quad \forall (x,y) \in \mathbb{R}^2$

$\frac{\partial f_2}{\partial x}(x,y) = \frac{\partial}{\partial x}(x^2-3y^2) = 2x \quad \forall (x,y) \in \mathbb{R}^2$

$\Rightarrow \frac{\partial f_1}{\partial y}(x,y) = \frac{\partial f_2}{\partial x}(x,y) \quad \forall (x,y) \in \mathbb{R}^2$

$\Rightarrow$ f conservative on $\mathbb{R}^2$

wednesday week 5

Parametric surfaces

Def: For $x: A \to \mathbb{R}$ $y: A \to \mathbb{R}$ $z: A \to \mathbb{R}$
with $A \subseteq \mathbb{R}^2$, we define

$S = \{(x(t,s), y(t,s), z(t,s)) \mid (t,s) \in A\}$

we say that

S surface $\Leftrightarrow \begin{cases} A \text{ simply connected } \wedge A \text{ closed} \\ x, y, z \text{ continuous on } A \end{cases}$

S differentiable surface $\Leftrightarrow \begin{cases} S \text{ surface} \\ x, y, z \text{ differentiable} \end{cases}$

(c) Given $a(t,s) = (x(t,s), y(t,s), z(t,s)) \ \forall (t,s) \in A$
we define the fundamental product

$$R(t,s \mid a) = \left(\frac{\partial a}{\partial t}\right) \times \left(\frac{\partial a}{\partial s}\right), \ \forall (t,s) \in A$$

(d) S smooth surface $\Leftrightarrow$

$\Leftrightarrow \begin{cases} S \text{ differentiable surface} \\ R(t,s \mid a) \neq 0, \ \forall (t,s) \in \operatorname{int}(A) \\ R(t,s \mid a) \text{ continuous on } A \end{cases}$

Thursday week5

Def: Let $(S): a(t,s) = (x(t,s), y(t,s), z(t,s))$ $\forall (t,s) \in A$
be a differentiable surface

a) we define the fundamental product $R(t,s|a)$
of the surface (S) as

$$R(t,s|a) = \frac{\partial a}{\partial t} \times \frac{\partial a}{\partial s}, \forall (t,s) \in A$$

b) we say that

$$S \text{ smooth surface} \iff \begin{cases} S \text{ differentiable surface} \\ R(t,s|a) \text{ continuous in } A \\ \forall (t,s) \in \operatorname{Int}(A): R(t,s|a) \neq 0 \end{cases}$$

Thursday week 5

Remark

$$R(t,s\,|\,a) = \frac{\partial a}{\partial t} \times \frac{\partial a}{\partial s}$$

$$= \left(\frac{\partial(y,z)}{\partial(t,s)}, \frac{\partial(z,x)}{\partial(t,s)}, \frac{\partial(x,y)}{\partial(t,s)} \right)$$

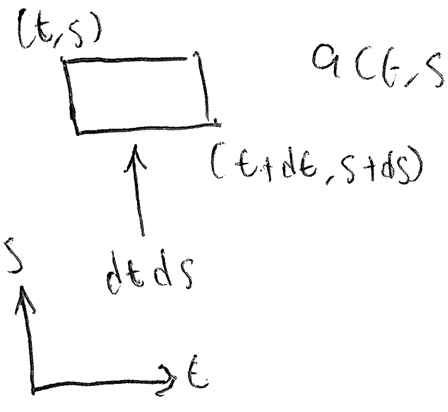
with

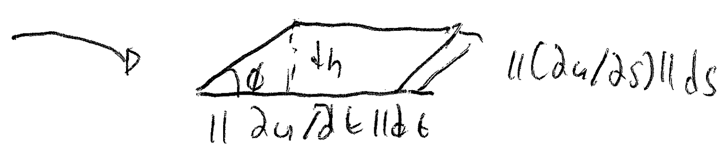
$$a(t,s) = (x(t,s), y(t,s), z(t,s))$$

$$\frac{\partial(y,z)}{\partial(t,s)} = \begin{vmatrix} \partial y/\partial t & \partial y/\partial s \\ \partial z/\partial t & \partial z/\partial s \end{vmatrix}$$

$$\frac{\partial(z,x)}{\partial(t,s)} = \begin{vmatrix} \partial z/\partial t & \partial z/\partial s \\ \partial x/\partial t & \partial x/\partial s \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(t,s)} = \begin{vmatrix} \partial x/\partial t & \partial x/\partial s \\ \partial y/\partial t & \partial y/\partial s \end{vmatrix}$$





$$dA = [\,\|\partial a/\partial t\|\,dt\,]\,dh$$

$$= [\,\|\partial a/\partial t\|\,dt\,][\,\|\partial a/\partial s\|\,ds\,\sin\phi\,]$$

$$= [\,\|\partial a/\partial t\|\,\|\partial a/\partial s\|\sin\phi\,]\,dt\,ds$$

$$= \left\| \frac{\partial a}{\partial t} \times \frac{\partial a}{\partial s} \right\| dt\,ds$$

$$= \|R(t,s\,|\,a)\|\,dt\,ds \Rightarrow$$

$$\Rightarrow \mathrm{Area}(S) = \iint_A dt\,ds\,\|R(t,s\,|\,a)\|$$

Thursday Week 5

Surface Integrals

Let $S = \{a(t,s) \mid (t,s) \in A\}$ and let $f : B \to \mathbb{R}$ with $B \subseteq \mathbb{R}^3$

Such that f continuous on B and $a(A) \subseteq B$ we define

$$\iint_S f\,ds = \iint_A dt\,ds\, \|R(t,s|a)\|\, f(a(t,s))$$

For $f : B \to \mathbb{R}^3$ with $B \subseteq \mathbb{R}^3$ such that f continuous on B and $a(A) \subseteq B$

$$\iint_S f \cdot ds = \iint_A dt\,ds\, f(a(t,s)) \cdot R(t,s|a)$$

$$\iint_S f \times ds = \iint_A dt\,ds\, f(a(t,s)) \times R(t,s|a)$$

Unit normal Vecta

$$n(t,s|a) = \frac{1}{\|R(t,s|a)\|} R(t,s|a)$$

$$\iint_S f \cdot ds = \iint_S (f \cdot n)\,ds$$

Given $a(t,s) = (x(t,s), y(t,s), z(t,s))$

$$\frac{\partial f(t,s)}{\partial n} = \nabla f(x(t,s), y(t,s), z(t,s)) \cdot n(t,s|a)$$

Thursday Week 5

Surface defined as $(S): z = f(x,y)$

$S = \{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y) \wedge (x,y) \in A\}$

$= \{(x,y,f(x,y)) \mid (x,y) \in A\}$

$a(x,y) = (x, y, f(x,y))$

$R(x,y \mid a) = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$

$\|R(x,y \mid a)\| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$

Proof

$\frac{\partial a}{\partial x} = \left(1, 0, \frac{\partial f}{\partial x}\right)$ and $\frac{\partial a}{\partial y} = \left(0, 1, \frac{\partial f}{\partial y}\right) \Rightarrow$

$\Rightarrow R(x,y \mid a) = \frac{\partial a}{\partial x} \times \frac{\partial a}{\partial y} = \left(1, 0, \frac{\partial f}{\partial x}\right) \times \left(0, 1, \frac{\partial f}{\partial y}\right)$

$$= \left|\begin{matrix} e_1 & e_2 & e_3 \\ 1 & 0 & \partial f/\partial x \\ 0 & 1 & \partial f/\partial y \end{matrix}\right| \begin{matrix} e_1 & e_2 \\ 1 & 0 \\ 0 & 1 \end{matrix}$$

$= e_3 - \frac{\partial f}{\partial x} e_1 - \frac{\partial f}{\partial y} e_2$

$= \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$

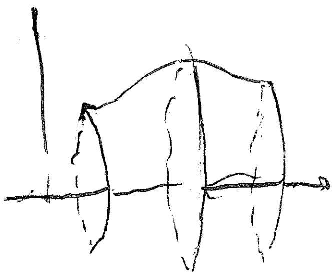
$\Rightarrow \|n(x,y \mid a)\| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$

Surface of revolution of f(x) around x-axis Thursday week 5

Given $f:[a,b]\to\mathbb{R}$ such that $\forall x\in[a,b]: f(x)>0$

Define $S=\{(x, f(x)\cos\theta, f(x)\sin\theta) \mid x\in[a,b] \wedge \theta\in[0,2\pi]\}$

$\hookrightarrow a(x,\theta)=(x, f(x)\cos\theta, f(x)\sin\theta), \forall (x,\theta)\in[a,b]\times[0,2\pi]$



$R(x,\theta|a) = (f(x)f'(x), -f(x)\cos\theta, -f(x)\sin\theta)$

$\|R(x,\theta|a)\| = f(x)\sqrt{1+[f'(x)]^2}$

Proof

$\frac{\partial a}{\partial x} = (1, f'(x)\cos\theta, f'(x)\sin\theta)$

$\frac{\partial a}{\partial \theta} = (0, -f(x)\sin\theta, f(x)\cos\theta)$

$$R(x,\theta|a) = \frac{\partial a}{\partial x}\times\frac{\partial a}{\partial \theta} = \left|\begin{matrix} e_1 & e_2 & e_3 \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{matrix}\right| \begin{matrix} e_1 & e_2 \\ 1 & f'(x)\cos\theta \\ 0 & -f(x)\sin\theta \end{matrix}$$

$= f(x)\cos^2\theta f'(x) e_1 - f(x)\sin\theta\, e_3 - [-f(x)\sin\theta][f'(x)\sin\theta]e_1 - [f(x)\cos\theta]e_2$

$= f(x)f'(x)e_1 - f(x)\cos\theta\, e_2 - f(x)\sin\theta\, e_3$

$= (f(x)f'(x), -f(x)\cos\theta, -f(x)\sin\theta)$

Thursday week 5

$$R(\phi,\theta\,|\,a) = \frac{\partial a}{\partial \phi} \times \frac{\partial a}{\partial \theta} = \begin{vmatrix} e_1 & e_2 & e_3 \\ l\cos\phi\cos\theta & l\cos\phi\sin\theta & -l\sin\phi \\ -l\sin\phi\sin\theta & l\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \sin\phi \begin{vmatrix} e_1 & e_2 & e_3 \\ l\cos\phi\cos\theta & l\cos\phi\sin\theta & -l\sin\phi \\ -l\sin\theta & l\cos\theta & 0 \end{vmatrix} \begin{matrix} e_1 & e_2 \\ l\cos\phi\cos\theta & l\cos\phi\sin\theta \\ -l\sin\theta & l\cos\theta \end{matrix}$$

$$= \sin\phi[e(-l\sin\theta)(-l\sin\phi) + e_3(l\cos\phi\cos\theta)(l\cos\theta)$$
$$- e_3[l\cos\phi\sin\theta][-l\sin\theta) - e_1(-l\sin\phi)(l\cos\theta)]$$

$$= e_1[l^2\sin^2\phi\cos\theta] + e_2[l^2\sin^2\phi\sin\theta]$$
$$+ e_3[l^2\sin\phi\cos\phi\cos^2\theta + l^2\sin\phi\cos\phi\sin^2\theta]$$

$$= l^2(\sin^2\phi\cos, \sin^2\phi\sin\theta, \cos\phi\sin\phi)$$

$$\Rightarrow \|R(\phi,\theta|a)\| = \|l^2(\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \cos\phi\sin\phi)\|$$

$$= |l^2|\sqrt{(\sin^2\phi\cos\theta)^2 + (\sin^2\phi\sin\theta)^2 + (\cos\phi\sin\phi)^2}$$

$$= |l^2|\sqrt{\sin^4\phi(\cos^2\theta + \sin^2\theta) + \sin^2\phi\cos^2\phi}$$

$$= l^2\sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi}$$

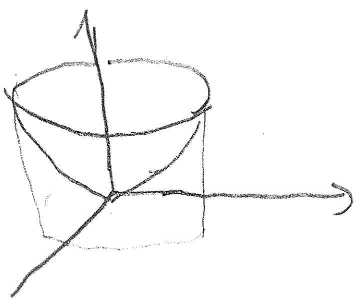
$$= l^2\sqrt{\sin^2\phi(\sin^2\phi + \cos^2\phi)}$$

$$= l^2\sqrt{\sin^2\phi} = l^2|\sin\phi| = l^2\sin\phi \quad \text{since } \phi\in[0,\pi]$$
$$\sin\phi \geq 0$$

Friday week 5

Example

Find the area of paraloid
(S) $z = x^2 + y^2$
restricted between
$(P_1): z = 0$ and $(P_2): z = 9$



Solution

The surface has representation
$S: \{(x, y, x^2+y^2) \mid 0 \le x^2 + y^2 \le 9\} = \{(x, y, x^2+y^2 \mid (x,y) \in A\}$
with
$A: \{(x, y) \in \mathbb{R}^2 \mid 0 \le x^2 + y^2 \le 9\}$
$= \{(r\cos\theta, r\sin\theta) \mid r \in [0, 3] \wedge \theta \in [0, 2\pi]\}$
Define $f(x,y) = x^2 + y^2$
The corresponding fundamental product is
$\|R(x, y \mid s)\| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$

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$= \sqrt{1+(2x)^2+(2y)^2} \implies$

$\implies Area(S) = \int_S dS = \iint_A dx\,dy\, \|R(x,y|S)\|$

$= \iint_A dx\,dy \sqrt{1+4(x^2+y^2}$

Change to $x = r\cos\theta$ and $y = r\sin\theta$

Then $dx\,dy = r\,dr\,d\theta$ and the new integral is over

$B = \{(r,\theta) \mid r \in [0,3] \wedge \theta \in [0, 2\pi]\}$

Therefore

$Area(S) = \iint_B dr\,d\theta\, r\sqrt{1+4r^2} = \int_0^3 dr \int_0^{2\pi} d\theta\, r\sqrt{1+4r^2}$

$= \int_0^3 dr\, r\sqrt{1+4r^2}\left[\int_0^{2\pi} d\theta\right]$

$= 2\pi \int_0^3 dr\, r\sqrt{1+4r^2}$

For $t = 1+4r^2 \implies dt = 8r\,dr \implies r\,dr = \left(\frac{1}{8}\right)dt$

For $r=0$: $t = 1+4\,0^2 = 1$

For $r=3$ $t = 1+4\cdot 3^2 = 37$

therefore

$Are(S) = 2\pi \int_1^{37} \sqrt{t}\,(1/8)\,dt = 2\pi \frac{1}{8}\left[\frac{t^{3/2}}{3/2}\right]_1^{37}$

$= 2\pi \frac{1}{8}\,\frac{2}{3}\left[t\sqrt{t}\right]_1^{37} = \frac{\pi}{6}\left[37\sqrt{37}-1\right]$

Example

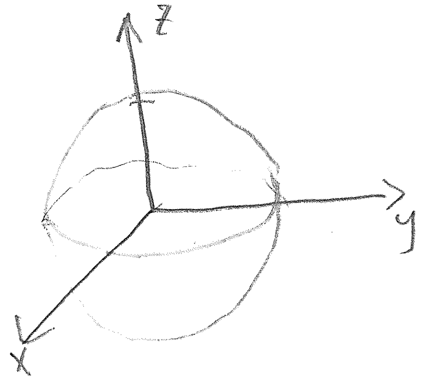
Evaluate

$$I = \iint_S x^2 \, ds$$

with S the unit sphere

$(S): x^2+y^2+z^2=1$

Solution



$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$$
$$= \{(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) \mid \phi \in [0,\pi] \wedge \theta [0,2\pi]\}$$
$$= \{(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) \mid (\phi,\theta) \in A\}$$

with

$$A = \{(\phi,\theta) \mid \phi \in [0,\pi] \wedge \theta \in [0, 2\pi]\}.$$

The fundamental product of S gives

$$\|R(\phi,\theta)\| = 1^2 \sin\phi = \sin\phi$$

Therefore

$$I = \iint_S x^2 \, ds = \iint_A d\phi \, d\theta \, [\sin\phi\cos\theta]^2 \, \|R(\phi,\theta \mid S)\|$$
$$= \iint_A d\phi \, d\theta \, [\sin^2\phi \cos^2\theta] \sin\phi$$
$$= \int_0^{\pi} d\phi \int_0^{2\pi} d\theta \sin^3\phi \cos^2\theta$$
$$= \left[\int_0^{\pi} d\phi \sin^3\phi\right]\left[\int_0^{2\pi} d\theta \cos^2\theta\right] = I_1 I_2$$

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with

$$I_1 = \int_0^{\pi} d\phi \sin^3\phi = \int_0^{\pi} d\phi \sin\phi \sin^2\phi$$

$$= \int_0^{\pi} d\phi \sin\phi [1 - \cos^2\phi]$$

Define $t = \cos\phi \Rightarrow dt = -\sin\phi d\phi$

$\Rightarrow \sin\phi \, d\phi = (-1)dt$

For $\phi = 0$ $t = \cos 0 = 1$

For $\phi = 0$ $t = \cos 0 = 1$

For $\phi = \pi$ $t = \cos\pi = -1$

Therefore

$$I_1 = \int_1^{-1} (1-t^2)(-1)dt = \int_{-1}^{1} (1-t^2)dt = \left[t - \frac{t^3}{3}\right]_{-1}^{1}$$

$$= \left[1 - \frac{1}{3}\right] - \left[(-1) - \frac{(-1)^3}{3}\right]$$

$$= 1 - \frac{1}{3} + 1 - \frac{1}{3}$$

$$= 2\left(1 - \frac{1}{3}\right)$$

$$= 2\,\frac{2}{3}$$

$$= \frac{4}{3}$$

Friday week 5

Likewise

$$I_2 = \int_0^{2\pi} d\theta \cos^2\theta = \int_0^{2\pi} d\theta \, \frac{1+\cos(2\theta)}{2} \left[\frac{\theta}{2} + \frac{1}{2}\frac{\sin(2\theta)}{2}\right]_0^{2\pi}$$

$$= \left[\frac{2\pi}{2} + \frac{1}{2}\frac{\sin(4\pi)}{2}\right] - \left[\frac{0}{2} + \frac{1}{2}\frac{\sin 0}{2}\right]$$

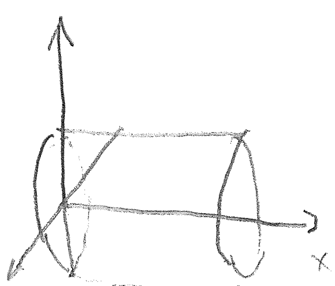
$$= \pi$$

Therefore $I = I_1 I_2 = \frac{4}{3}\pi = \frac{4\pi}{3}$

Example

Evaluate $I = \iint_S z \, ds$

where (S) is the cylinder (C): $y^2 + z^2 = 3$ bounded by (P_1): $x = 0$ and $(P_2) = 6$



Solution

S given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 = 3 \wedge x \in [0, 6]\}$$

$$= \{(x, \sqrt{3}\cos\theta, \sqrt{3}\sin\theta) \mid x \in [0, 6] \wedge \theta \in [0, 2\pi]\}$$

$$= \{(x, \sqrt{3}\cos\theta, \sqrt{3}\sin\theta \mid (x, \theta) \in A\}$$

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With $A = \{(x,\theta) \mid x \in [0,6] \wedge \theta \in [0, 2\pi]\}$.

Since S is generated by rotating $f(x) = \sqrt{3}$ around the x-axis
it follows that

$$\|R(x,\theta|S)\| = f(x)\sqrt{1+[f'(x)]^2}$$

$$\Rightarrow I = \iint_S z\,dS = \iint_A dx\,d\theta\,\|R(x,\theta|S)\|\,\sqrt{3}\sin\theta$$

$$= \iint_A dx\,d\theta\,\sqrt{3}\sqrt{3}\sin\theta$$

$$= 3\iint_A dx\,d\theta\,\sin\theta$$

$$= 3\int_0^6 dx\,[(-\cos(2\pi)) - (-\cos 0)]$$

$$= 3[\underbrace{\cancel{(-1)-(-1)}}_{0}]\int_0^6 dx$$

$$= 0$$

Friday week 5

Stokes theorem

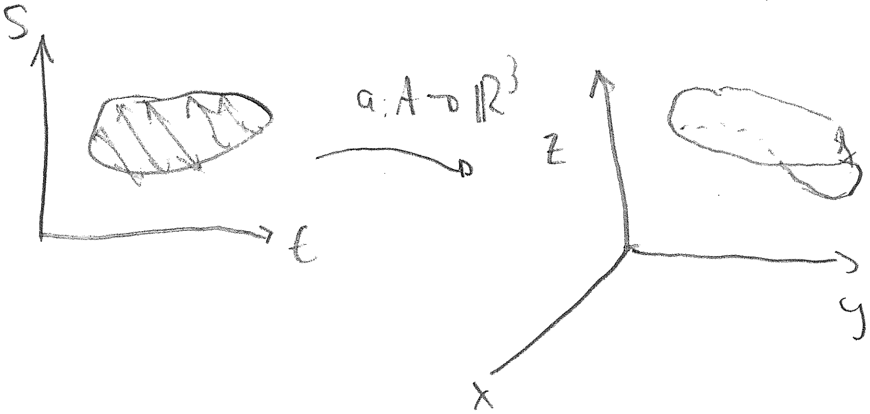
Def. Let $S = \{ a(t,s) \mid (t,s) \in A \}$

with $A \subseteq \mathbb{R}^2$ and $a : A \to \mathbb{R}^3$

Let ∂A be the boundary of A

Define $\partial S = \{ a(t,s) \mid (t,s) \in \partial A \}$

S Jordan bounded $\iff \begin{cases} a \text{ one-to-one mapping} \\ S \text{ smooth surface} \\ \partial A \in \text{Jord}(\mathbb{R}^2) \end{cases}$



Assume that ∂A is oriented with $w(\partial A) = +1$

Thm: Let $S = \{ a(t,s) \mid (t,s) \in A \}$ with $A \subseteq \mathbb{R}^2$ and let $f : B \to \mathbb{R}^3$ with $\subseteq \mathbb{R}^3$ and $S \subseteq B$. Assume that

$\begin{cases} S \text{ Jordan bounded surface} \\ \alpha(t,s) \text{ has continuous 2nd partial derivatives on } A \\ f \text{ differentiable on } B \\ \nabla f \text{ continuous on } B \end{cases}$

Then

$$\iint_S (\nabla \times f) \cdot ds = \oint_{\partial S} f \cdot dl$$