day 12

Exam 2

Email Monday night

Due by Email Thursday 5pm

(or in class)

400 dpi - pdf file black and white

black ink

day 12

Multiple Integrals

Double Integrals

Given $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ with

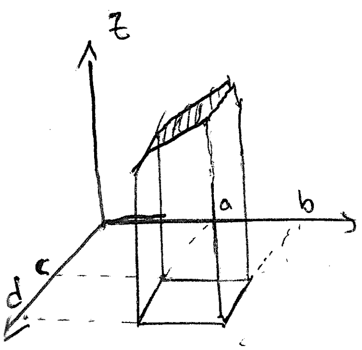
$A = [a, b] \times [c, d] = \{(x, y) \mid x \in [a, b] \wedge y \in [c, d]\}$

Thm: If f continuous on A

$$\int_a^b dx \int_c^d dy\, f(x, y) = \int_c^d dy \int_a^b dx\, f(x, y) = I$$

Then we define

$$\iint_A dx\,dy\, f(x, y) = I$$



Think of it as a volume under a surface

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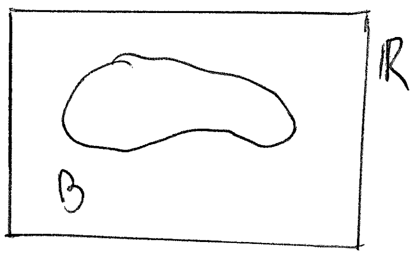
Let $B \subseteq A$ be a closed bounded region

Then there is a rectangle $\mathbb{R} = [a,b] \times [c,d]$

Such that $B \subseteq \mathbb{R}$



define $g(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in B \\ 0 & \text{if } (x,y) \in \mathbb{R} - B \end{cases}$

Then we define $\iint_B f(x,y)\,dx\,dy = \iint_{\mathbb{R}} g(x,y)\,dx\,dy$

and $\iint_{\emptyset} f(x,y)\,dx\,dy = 0$

empty set

Examples

a) Evaluate the Integral

$I = \iint_A \frac{1+x^2}{1+y^2}\,dx\,dy$ for $A = [0,1] \times [0,1]$

Solution

$$I = \iint_A \frac{1+x^2}{1+y^2}\,dx\,dy = \int_0^1 dx \int_0^1 dy\, \frac{1+x^2}{1+y^2} = \int_0^1 dx\,(1+x^2)\int_0^1 \frac{dy}{1+y^2} = \int_0^1 dx\,(1+x^2)\left[\operatorname{Arctan} y\right]_0^1$$

$$= \int_0^1 dx\,(1+x^2)\,(\operatorname{Arctan} 1 - \operatorname{Arctan} 0) = \left(\frac{\pi}{4} - 0\right)\int_0^1 dx\,(1+x^2) = \frac{\pi}{4}\left[x + \frac{x^3}{3}\right]_0^1$$

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$$= \frac{\pi}{4}\left[(1-0)+\frac{1^3-0^3}{3}\right] = \frac{\pi}{4}\frac{4}{3} = \frac{\pi}{3}$$

b) Find the volume of the solid under the surface

(S): $z = x\sqrt{x^2+y}$ and above the rectangle

$A = [0,1]\times[0,1]$ on the xy-plane

Solution

$$V = \iint_A dx\,dy\, x\sqrt{x^2+y} = \int_0^1 dx \int_0^1 dy\, x\sqrt{x^2+y} = \int_0^1 dx\, x\left[\int_0^1 dy\sqrt{x^2+y}\right]$$

Let $t = g(y) = \sqrt{x^2+y} \Leftrightarrow t^2 = x^2+y \Leftrightarrow y = t^2 - x^2$

thus: $dy = 2t\,dt$ and furthermore:

for $y=0$: $g(0) = \sqrt{x^2+0} = \sqrt{x^2}$

for $y=1$: $g(1) = \sqrt{x^2+1}$

it follows that

$$V = \int_0^1 dx\, x\left[\int_{\sqrt{x^2}}^{\sqrt{1+x^2}} dt(2t)t\right] = 2\int_0^1 dx\, x\left[\int_x^{\sqrt{1+x^2}} t^2\,dt\right] = 2\int_0^1 dx\, x\left[\frac{t^3}{3}\right]_x^{\sqrt{1+x^2}}$$

$$= 2\int_0^1 dx\, x\left[\frac{(\sqrt{1+x^2})^3 - x^3}{3}\right] = \frac{2}{3}\int_0^1 dx\, x\left[(1+x^2)\sqrt{1+x^2} - x^3\right]$$

$$= \frac{2}{3}\int_0^1 dx\, x(1+x^2)\sqrt{1+x^2} - \frac{2}{3}\int_0^1 dx\, x^4$$

$= \frac{2}{3}(I_1 - I_2)$ with

$I_1 = \int_0^1 dx\, x(1+x^2)\sqrt{1+x^2}$ and $I_2 = \int_0^1 x^4\, dx$

- For I_2:

$I_2 = \int_0^1 x^4 dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1^5 - 0^5}{5} = \frac{1}{5}$

- For I_1

Let $x = \tan\theta \Rightarrow \begin{cases} dx = \frac{d\theta}{\cos^2\theta} \\ \text{For } x=0:\ \theta = 0 \\ \text{for } x=1:\ \theta = \frac{\pi}{4} \end{cases} \Rightarrow$

$\Rightarrow I_1 = \int_0^{\frac{\pi}{4}} \frac{d\theta}{\cos^2\theta} \tan\theta(1+\tan^2\theta)\sqrt{1+\tan^2\theta}$

$= \int_0^{\frac{\pi}{4}} \frac{d\theta}{\cos^2\theta} \tan\theta \frac{1}{\cos^2\theta} \sqrt{\frac{1}{\cos\theta}}$

$= \int_0^{\frac{\pi}{4}} d\theta \frac{\tan\theta}{\cos^4\theta\,|\cos\theta|} = \int_0^{\frac{\pi}{4}} \frac{d\theta \tan\theta}{\cos^5\theta}$

$= \int_0^{\frac{\pi}{4}} d\theta \frac{\sin\theta}{\cos^5\theta \cos\theta} = \int_0^{\frac{\pi}{4}} d\theta \frac{\sin\theta}{\cos^6\theta}$

Let $s = \cos\theta : g(\theta) \Rightarrow \begin{cases} ds = -\sin\theta d\theta \Rightarrow \sin\theta d\theta = -ds \\ g(0) = \cos 0 = 1 \\ g(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \end{cases}$

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Double integrals over simple regions

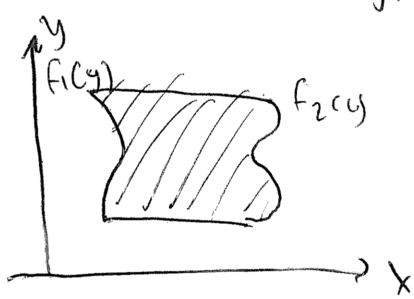
x-Simple region

- Let $A \subseteq \mathbb{R}^2$. we say that A is x-simple if and only if it can be written as

$$A = \{ (x,y) \in \mathbb{R}^2 \mid f_1(y) \leq x \leq f_2(y) \wedge y \in [a,b] \}$$

with f_1, f_2 function with $f_1 : [a,b] \to \mathbb{R}$ and $f_2 : [a,b] \to \mathbb{R}$

$$\iint_A f(x,y)\,dx\,dy = \int_a^b dy \int_{f_1(y)}^{f_2(y)} dx\, f(x,y)$$

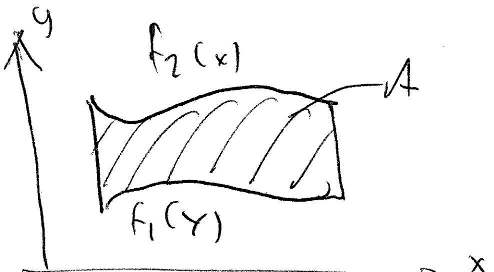


y-Simple region

- Let $A \subseteq \mathbb{R}^2$. we say that A is y-simple if and only if it can be written as

$$A = \{ (x,y) \in \mathbb{R}^2 \mid x \in [a,b] \wedge f_1(x) \leq y \leq f_2(x) \}$$

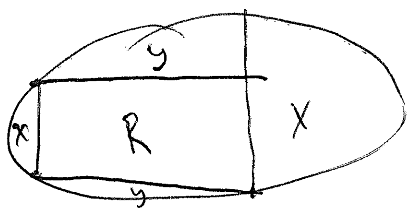
with f_1, f_2 function with $f_2 : [a,b] \to \mathbb{R}$ and $f_2 : [a,b] \to \mathbb{R}$



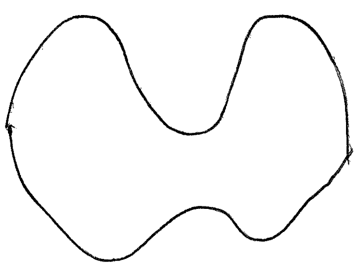
if A is y-Simple then

$$\iint_A f(x,y)\,dx\,dy = \int_a^b dx \int_{f_1(x)}^{f_2(x)} dy\, f(x,y)$$

we can always divide a convex region in x-simple and y-simple



Convex region too but we need to be careful



Examples

a) Evaluate the integral $I = \iint_A 2y e^x \, dx\, dy$ with
$A = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1 \wedge 0 \le x \le y^2\}$

$$I = \iint_A 2ye^x \, dx\,dy = \int_0^1 dy \int_0^{y^2} dx\, 2y\, e^x = \int_0^1 dy\, 2y \left[\int_0^{y^2} e^x dx\right]$$

$$= \int_0^1 dy\, 2y \left[e^x\right]_0^{y^2} = \int_0^1 dy\, 2y\,(e^{y^2} - e^0)$$

$$= \int_0^1 2y e^{y^2} dy - \int_0^1 2y\, dy = I_1 - I_2 \qquad (1)$$

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Example #1

find the min and max values for

$f(x,y,z) = x+2y+3z$ ① with constraint

$$\begin{cases} x^2+y^2=2 & ② \\ y+z=1 & ③ \end{cases}$$

Solution:

define $g_1(x,y,z) = 2-x^2-y^2$

$g_2(x,y,z) = y+z-1$

Note that

$\nabla f(x,y,z) = (1,2,3)$

$\nabla g_1(x,y,z) = (-2x,-2y,0)$

$\nabla g_2(x,y,z) = (0,1,1)$

for linear independence.

$$\nabla g_1 \times \nabla g_2 = (-2x,-2y,0) \times (0,1,1)$$

$$= \begin{vmatrix} i & j & k \\ -2x & -2y & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} i & j & -j+k \\ -2x & -2y & 2y \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 1 \cdot (-1)^{3+2} \cdot \begin{vmatrix} i & -j+k \\ -2x & 2y \end{vmatrix}$$

$$= -1\left(i \cdot 2y - (-j+k)(-2x)\right)$$

$$= -(2y, -2x, +2x)$$

$$= (-2y, +2x, -2x)$$

$\therefore \nabla g_1, \nabla g_2$ are linearly independent iff. $\nabla g_1 \times g_2 \neq \vec{0}$

$\therefore x \neq 0$ or $y \neq 0$ $(x \neq 0 \vee y \neq 0)$.

$\therefore$ this linear indepence fails at $(x,y,z) = (0,0,z)$.

by (4) $z = 1 - y$, and since $x = -\frac{1}{2\lambda_1}$, $y = \frac{1}{2\lambda_1}$

$$\therefore \begin{cases} x = \frac{-1}{2\lambda_1} \\ y = \frac{1}{2\lambda_1} \\ \lambda_1 = \frac{1}{2} \vee \lambda_1 = -\frac{1}{2} \\ z = 1 - y \end{cases}$$

$$\therefore \begin{cases} x = -1 \\ y = 1 \\ z = 0 \end{cases} \vee \begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases}$$

$\therefore (x, y, z) = (-1, 1, 0) \vee (x, y, z) = (1, -1, 2)$

Note that both point satisfy the linear independence condition

considering $f(x, y, z) = x + 2y + 3z$, then.

$f(-1, 1, 0) = -1 + 2 = 1.$

$f(1, -1, 2) = 1 - 2 + 6 = 5$

$\therefore (-1, 1, 0)$ is the minimum and.

$(1, -1, 2)$ is the maximum

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Example #2. find the stationary point.

$f(x,y,z) = xyz$, with $x,y,z \in (0,+\infty)$.

under constraint

$$xy + yz + zx = 1$$

$\therefore$ define $g(x,y,z) = xy + yz + xz - 1$.

$\therefore$ $\nabla f = (yz, xz, xy)$. $\because x,y,z \in (0,+\infty)$

$\nabla g = (y+z, x+z, y+x)$ $\therefore \nabla g \neq \vec{0}$ $\therefore \nabla g$ is linearly independent.

$\therefore$ (x,y,z) is stationary iff.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 0 \end{cases} \quad \therefore \begin{cases} (yz, xz, xy) = \lambda(y+z, x+z, y+x) \\ xy + yz + zx - 1 = 0 \end{cases}$$

$$\therefore \begin{cases} yz = \lambda(y+z) \\ xz = \lambda(x+z) \\ xy = \lambda(x+y) \\ xy + yz + zx - 1 = 0 \end{cases} \quad \therefore \begin{cases} xyz = \lambda x(y+z) & (7) \\ xyz = \lambda y(x+z) & (8) \\ xyz = \lambda z(x+y) & (9) \\ xy + yz + zx = 1 & (10) \end{cases}$$

by (7)(8),

$$\lambda x(y+z) - \lambda y(x+z) = 0$$

$$\therefore \lambda(xy + xz - yx - yz) = 0$$

$$\therefore \lambda y(x-z) = 0$$

$$\therefore \lambda = 0 \lor y = 0 \lor x = z$$

if $\lambda = 0$ then

$$\begin{cases} yz = 0 \\ xz = 0 \\ xy = 0 \\ xy + yz + zx = 1 \end{cases} \quad \therefore 0 + 0 + 0 = 1 \text{ inconsistent} \quad \therefore \lambda \neq 0$$

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optimization problem on a bounded set

Def $\forall A \subset \mathbb{R}^n$, A is said to be bounded iff.
$\exists p \in \mathbb{R}^n$, $\exists r \in (0,+\infty)$ s.t. $A \subset B(p,r)$

Thm Let $f: A \to \mathbb{R}$ with $A \subset \mathbb{R}^n$. Assume that
a) f is continuous on A
b) A is closed ($\partial A \subset A$)
c) A is bounded
then $\exists P_1, P_2 \in A$, such that $\forall x \in A$, $f(P_1) \le f(x) \le f(P_2)$

Example. find the min and max of
$f(x,y) = 2 + x^2 + y^2$.
under the constraint
$x^2 + \frac{y^2}{4} \le 1$

solution. Define $S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2/4 \le 1\}$ (interior points)
$\nabla f(x,y) = (2x, 2y)$.
let $\nabla f = 0$, $\therefore (x,y) = (0,0)$

boundary
let $g(x,y) = x^2 + \frac{y^2}{4} - 1$ $\because x^2 + \frac{y^2}{4} = 1$ $\therefore \nabla g \neq \vec{0}$
$\therefore \nabla g = (2x, \frac{y}{2})$ $\therefore \nabla g$ is linearly independent
$\therefore \begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}$ $\therefore \begin{cases} 2x = \lambda 2x & (1) \\ 2y = \lambda \cdot \frac{y}{2} & (2) \\ x^2 + \frac{y^2}{4} - 1 = 0 & (3) \end{cases}$

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2nd methord

$\because \ x^2 + \frac{y^2}{4} \le 1$

$\therefore$ let $\begin{cases} x(t) = \cos t \\ y(t) = 2\sin t \end{cases}$ $\quad t \in [0, 2\pi)$

$\therefore$ it follows that

$g(t) = f(\cos t, 2\sin t) = 2 + \cos^2 t + 4\sin^2 t$
$= 3 + 3\sin^2 t$

$g'(t) = (3 + 3\sin^2 t)' = (3\sin^2 t)' = 3 \cdot 2\sin t \cdot (\sin t)' = 6\sin t \cos t$.
$= 3\sin(2t)$

let $g'(t) = 0$ $\quad \therefore \ \sin(2t) = 0$

$\therefore 2t = n\pi$, $\quad \therefore t = \frac{n\pi}{2}$

$\because t \in [0, 2\pi)$ $\quad \therefore t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

$\therefore$ for $t = 0$ $\quad (x, y) = (\cos t, 2\sin t) = (1, 0)$.

$t = \frac{\pi}{2}$ $\quad (x, y) = \dots\dots = (0, 2)$

$t = \pi$ $\quad (x, y) = \dots\dots = (-1, 0)$

$t = \frac{3\pi}{2}$ $\quad (x, y) = \dots\dots = (0, -2)$.

Tuesday week 4

Change of variables in double Integrals

Given $x_1, x_2, \dots, x_n \in \mathbb{R}$ we define

$$\begin{cases} y_1 = f_1(x_1, x_2, \dots, x_n) \\ y_2 = f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ y_n = f_n(x_1, x_2, \dots, x_n) \end{cases}$$

Notation $x = (x_1, x_2, \dots, x_n)$

$$f(x) = (f_1(x), f_2(x), \dots, f_n(x))$$

Def: f differentiable on $x_0 \iff \forall k \in [n]: f_k$ differentiable at x_0

Derivative matrix:
$$\begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_n \\ \vdots & & & \\ \partial f_n/\partial x_1 & \partial f_n/\partial x_2 & \cdots & \partial f_n/\partial x_n \end{bmatrix}$$

Jacobian of $f(x)$

$$\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)} = \det(Df(x))$$

Def: f one-to-one $\iff \forall x, y \in A: (f(x) \Rightarrow$ with $f: A \to \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n \Rightarrow x = y)$

Tuesday week 4

Def. $f: A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^n$ is a smooth change of variables

if and only if $f_1, f_2, \dots, f_n$ continually differentiable on A

f one-to-one

$$\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)} \neq 0, \ \forall x \in A$$

Change of variables in $\mathbb{R}^2$

Consider $x = g_1(u,v) \wedge y = g_2(u,v)$ with $g(u,v) = (g_1(u,v), g_2(u,v))$ and $g: A \to \mathbb{R}^2$ with $A \subseteq \mathbb{R}^2$. Let $B \subseteq \mathbb{R}^2$ with $B \cup \partial B \subseteq A$ and let $f: g(B) \to \mathbb{R}$. Assume that g smooth chang of variables.

Then:

$$\iint_{g(B)} f(x,y)\,dx\,dy = \iint f(g_1(u,v), g_2(u,v)) \left| \frac{\partial(g_1, g_2)}{\partial(u,v)} \right| du\,dv.$$

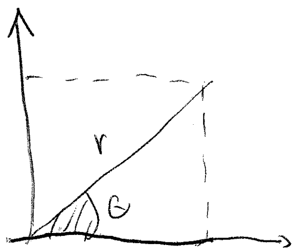
Note $dx\,dy = \left| \frac{\partial(g_1, g_2)}{\partial(u,v)} \right| du\,dv$

Tuesday week 4

Application to polar coordinates

- Consider the change of variables

$x = g_1(r,\theta) = r\cos\theta$
$y = g_2(r,\theta) = r\sin\theta$



or equivalently
$(x,y) = g(r,\theta) = r(\cos\theta, \sin\theta)$

- Jacobian : $\frac{\partial(g_1,g_2)}{\partial(r,\theta)} = \begin{vmatrix} \partial g_1/\partial r & \partial g_1/\partial\theta \\ \partial g_2/\partial r & \partial g_2/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r \neq 0$

$\forall r > 0$

Thus changing to polar coordinates is a smooth change of variables provided the origin is extended and

$dx\,dy = r\,dr\,d\theta$

- Double integral over a polar rectangle

Consider the polar rectangle

$R = [r_1, r_2] \times [\theta_1, \theta_2] = \{(r,\theta) \mid r \in [r_1, r_2] \wedge \theta \in [\theta_1, \theta_2]\}$

The image of R under change to cartesian coordinates given by
$g(r,\theta) = (r\cos\theta, r\sin\theta)$
gives the polar coordinates

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$g(R) = g([r_1, r_2] \times [\theta_1, \theta_2]) = \{(r\cos\theta, r\sin\theta) \mid r_1 \leq r \leq r_2 \wedge \theta_1 \leq \theta \leq \theta_2\}$

Let $f: R \to \mathbb{R}$ be a scalar field. It follows that

$$\iint_{g(R)} f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta = \int_{r_1}^{r_2} dr \int_{\theta_1}^{\theta_2} d\theta\, f(r\cos\theta, r\sin\theta)\, r$$

EXAMPLES

Evaluate the integral $I = \iint_A \cos(\pi x^2 + \pi y^2)\,dx\,dy$ with A given by
$A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4\}$

Solution

We note that

$A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 < 4\} = \{(r\cos\theta, r\sin\theta) \mid r \in [1,2] \wedge \theta \in [0, 2\pi]\}$
and it follows that:

$$I = \iint_A \cos(\pi x^2 + \pi y^2)\,dx\,dy = \int_1^2 dr \int_0^{2\pi} d\theta\, r\cos(\pi r^2) = \int_1^2 dr\, r\cos(\pi r^2)\left[\int_0^{2\pi} d\theta\right]$$

$$= \int_1^2 dr\, r\cos(\pi r^2)\, 2\pi = 2\pi \int_1^2 r\cos(\pi r^2)\,dr$$

let $t = \pi r^2 = g(r) \Rightarrow \begin{cases} dt = 2\pi r\,dr \\ g(1) = \pi \cdot 1^2 = \pi \\ g(2) = \pi \cdot 2^2 = 4\pi \end{cases}$

$$\Rightarrow I := \int_{0\pi}^{4\pi} \cos(t)\,dt = [\sin(t)]_{0\pi}^{4\pi} = \sin(4\pi) - \sin(0\pi)$$
$$= \sin(0) - \sin(0\pi)$$
$$= 0 - 0$$
$$= 0$$

Application: Gaussian Integral

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Proof

Define $I := \int_{-\infty}^{+\infty} e^{-x^2}\,dx$ and note that

$$\mathbb{I} := \iint_{\mathbb{R}^2} e^{(-x^2-y^2)}\,dx\,dy = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, e^{(-x^2-y^2)} = \int_{-\infty}^{+\infty} dx\, e^{-x^2}\left[\int_{-\infty}^{+\infty} e^{-y^2}\right]$$
$$= \int_{-\infty}^{+\infty} dx\, e^{-x^2}\, I$$
$$= I \int_{+\infty}^{+\infty} dx\, e^{-x^2}$$
$$= I \cdot I$$
$$= I^2$$

We now calculate $\mathbb{I}$

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$$I = \iint_{\mathbb{R}^2} e^{(-x^2-y^2)}\,dx\,dy = \int_0^{+\infty} dr \int_0^{2\pi} d\theta\, r e^{-r^2}$$

$$= \int_0^{+\infty} dr\, r e^{-r^2} \left[\int_0^{2\pi} d\theta \right]$$

$$= \int_0^{+\infty} 2\pi r e^{-r^2}\, dr$$

$$= 2\pi \int_0^{+\infty} r e^{-r^2}\, dr$$

Let $\rho = -r^2 = g(r) \Rightarrow \begin{cases} d\rho = -2r\,dr \Rightarrow r\,dr = (-1/2)\,d\rho \\ g(0) = 0 \\ g(+\infty) = \lim\limits_{r \to +\infty} (-r^2) = -\infty \end{cases}$

$$\Rightarrow I = 2\pi \int_0^{+\infty} e^{\rho}(-1/2)\, d\rho = -\pi \int_0^{+\infty} e^{\rho}\, d\rho = -\pi \left[e^{\rho} \right]_0^{-\infty}$$

$$= -\pi \left[\lim_{\rho \to -\infty} e^{\rho} - e^{0} \right]$$

$$= -\pi \left[0 - 1 \right] = \pi$$

$\Rightarrow I^2 = \pi \Rightarrow I = \sqrt{\pi} \vee I = -\sqrt{\pi}$

Since $\forall x \in \mathbb{R}: e^{-x^2} > 0 \Rightarrow \int_{-\infty}^{+\infty} e^{x^2}\,dx > 0 \Rightarrow I > 0$

and therefore $I = \sqrt{\pi}$

(the solution $I = -\sqrt{\pi}$ is rejected)

Tuesday week 4

Definition of the triple integral

- Let $A \subseteq \mathbb{R}^3$ be a rectangular region given by

$$A = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid x \in [a_1, a_2] \land y \in [b_1, b_2] \land z \in [c_1, c_2]\}$$

and consider a scaler field $f : A \to \mathbb{R}$, such as f continuous at A.

We define the toiple integral of f over A as,

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \int_{a_1}^{a_2} dx \int_{b_1}^{b_2} dy \int_{c_1}^{c_2} dz\, f(x,y,z)$$

It can be shown that permuting the order of the integrals gives the same result.

- Consider now a more general closed and bounded region $A \subseteq \mathbb{R}^3$ and let $f : A \to \mathbb{R}$ be a scaler field. Thus there is a rectangular region

$$A = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$$

Thus we may define the integral of f over A as,

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iiint_R f(x,y,z)\,dx\,dy\,dz$$

Tuesday week 3

EXAMPLE

Evaluate $I = \iiint_A x^2 y z \, dx\,dy\,dz$ with

$A = [1,2] \times [0,1] \times [0,2]$

Solution

$$I = \iiint_A x^2 y z \, dx\,dy\,dz = \int_1^2 dx\, x^2 \int_0^1 dy \int_0^2 dz\, x^2 y z$$

$$= \int_1^2 dx\, x^2 \left[\int_0^1 dy \int_0^2 dz\, y z \right]$$

$$= \left[\int_1^2 x^2 dx \right] \left[\int_0^1 dy\, y \left(\int_0^2 dz\, z \right) \right]$$

$$= \left[\int_1^2 x^2 dx \right] \left[\int_0^1 y\, dy \right] \left[\int_0^2 z\, dz \right]$$

with

$$I_1 = \int_1^2 x^2 dx = \left[\frac{x^3}{3} \right]_1^2 = \frac{2^3 - 1^3}{3} = \frac{8-1}{3} = \frac{7}{3}$$

$$I_2 = \int_0^1 y\, dy = \left[\frac{y^2}{2} \right]_0^1 = \frac{1^2 - 0^2}{2} = \frac{1}{2}$$

$$I_3 = \int_0^2 z\, dz = \left[\frac{z^2}{2} \right]_0^2 = \frac{2^2 - 0^2}{2} = 2$$

and therefore

$$I = I_1 I_2 I_3 = \frac{7}{3} \cdot \frac{1}{2} \cdot 2 = \frac{7}{3}$$

Tuesday week 8

In general, given a region

$A = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$

It can be shown that

$$I = \iiint_A f_1(x)\, f_2(y)\, f_3(z)\, dx\, dy\, dz$$

$$= \left[\int_{a_1}^{b_1} f_1(x)\, dx\right]\left[\int_{a_2}^{b_2} f_2(y)\, dy\right]\left[\int_{a_3}^{b_3} f_3(z)\, dz\right]$$

Evaluation of triple Integrals

① z-simple regions

For $A = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in S_{xy} \wedge g_1(x, y) \leq z \leq g_2(y, t)\}$

$$\Rightarrow \iiint_A f(x, y, z)\, dx\, dy\, dz = \iint_{S_{xy}} dx\, dy \int_{g_1(x,y)}^{g_2(x,y)} dz \cdot f(x, y, z)$$

② x-simple

For $A = \{(x, y, z) \in \mathbb{R}^3 \mid (y, z) \in S_{yz} \wedge g_1(y, z) \leq x \leq g_2(y, z)\}$

$$\Rightarrow \iiint_A f(x, y, z)\, dx\, dy\, dz = \iint_{S_{yz}} dy\, dz \int_{g_1(\cancel{x}, z)}^{g_2(x, z)} dy\, f(x, y, z)$$

Tuesday week 4

③ y-simple regions

For $A=\{(x,y,z)\in\mathbb{R}^3 \mid (x,z)\in S_{xz} \wedge g_1(x,z)\le y\le g_2(x,z)\}\Rightarrow$

$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \iint_{S_{xz}} dx\,dz \int_{g_1(x,z)}^{g_2(x,z)} dy\, f(x,y,z)$$

④ xy-simple regions

For $A=\{(x,y,z)\in\mathbb{R}^3 \mid z\in[a,b] \wedge (x,y)\in S_{xy}(z)\}\Rightarrow$

$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \int_a^b dz \iint_{S_{xy}(z)} dx\,dy\, f(x,y,z)$$

⑤ yz-simple regions

For $A=\{(x,y,z)\in\mathbb{R}^3 \mid x\in[a,b] \wedge (y,z)\in S_{yz}(x)\}\Rightarrow$

$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \int_a^b dx \iint_{S_{yz}(x)} dy\,dz\, f(x,y,z)$$

⑥ xz-simple regions

For $A=\{(x,y,z)\in\mathbb{R}^3 \mid y\in[a,b] \wedge \in S_{xz}(y)]\Rightarrow$

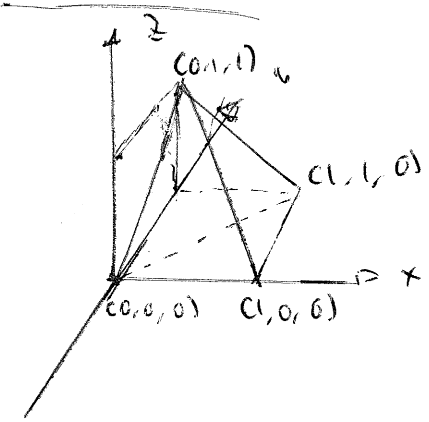
$$\Rightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz = \int_a^b dy \iint_{S_{xz}(y)} dx\,dz\, f(x,y,z)$$

Wednesday week 4

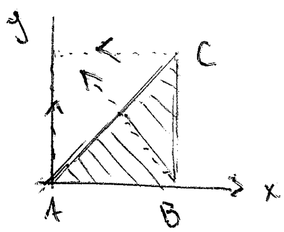
Evaluate $I = \iiint_A xy \, dx \, dy \, dz$

with A a tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(0,1,1)$

Solution



xy cross-section at $z=0$

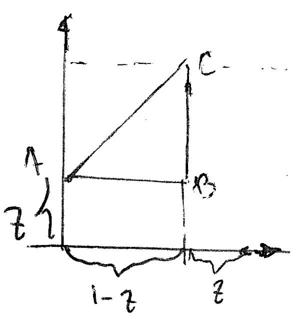


$A: (0,0,0) \to (0,1,1) \Rightarrow (0,z,z)$

$B: (1,0,0) \to (0,1,1) \Rightarrow (1-z, z, z)$

$C: (1,1,0) \to (0,1,1) \Rightarrow (1-z, 1, z)$

xy cross-section at $z \in (0,1)$



$A = \{(x,y,z) \in \mathbb{R}^3 \mid z \in [0,1] \wedge (x,y) \in S_{xy}(z)\}$

$S_{xy} = \{(x,y) \in \mathbb{R}^2 \mid y \in [z,1] \wedge f_1(y,z) \leq x \leq 1-z\}$

Wednesdag week 4

To determine $f_1(y,z)$ and $f_2(y,z)$, let $D \in AC$ and $E \in BC$ such that $y_D = y_E = y$. Then we see that $f_1(y,z) = x_D$ and $f_2(y,z) = x_E$.

We note that

$$D \in AC \Leftrightarrow \frac{y_D - y_A}{x_D - x_A} = \frac{y_C - y_A}{x_C - x_A} \Leftrightarrow$$

$$\Leftrightarrow \frac{y-z}{f(y,z)-0} = \frac{1-z}{(1-z)-0} \Leftrightarrow$$

$$\Leftrightarrow \frac{y-z}{f_1(y,z)} = 1 \Leftrightarrow f_1(y,z) = y - z$$

Since BC is vertical (i.e. $x_B = x_C = 1-z$) it follows that

$f_2(y,z) = x_E = x_B = 1-z$

and therefore

$S_{xy}(z) = \{(x,y) \in \mathbb{R}^2 \mid z \leq y \leq 1 \wedge y - z \leq x \leq 1 - z\}$

The trapezoid itself is represented by

$A = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in S_{xy}(z) \wedge z \in [0,1]\}$

We may now evaluate the integral

$$I = \iiint_A xy\, dx\,dy\,dz = \int_0^1 dz \iint_{S_{xy}(z)} dx\,dy\; xy = \int_0^1 dz \int_z^1 dy \int_{y-z}^{1-z} dx\; xy = \int_0^1 dz \int_z^1 dy\; y\left[\int_{y-z}^{1-z} x\,dx\right]$$

$$= \int_0^1 dz \int_z^1 dy\, y \left[\frac{x^2}{2}\right]_{y-z}^{1-z} = \int_0^1 dz \int_z^1 dy\, y\left[\frac{(1-z)^2 - (y-z)^2}{2}\right] = \frac{1}{2}\int_0^1 dz \int_z^1 dy\; y\left[(1-2z+z^2) - (y^2 - 2yz + z^2)\right]$$

$$= \frac{1}{2}\int_0^1 dz \int_z^1 dy\; y(1 - 2z - y^2 + 2yz) = \frac{1}{2}\int_0^1 dz \int_z^1 dy\,(y - 2yz - y^3 + 2y^2 z) =$$

$$= \frac{1}{2}\int_0^1 dz \left[\frac{y^2}{2} - y^2 z - \frac{y^4}{4} + \frac{2y^3 z}{3}\right]_z^1 = \frac{1}{2}\cdot\frac{1}{12}\int_0^1 dz \left[6y^2 - 12y^2 z - 3y^4 + 8y^3 z\right]_z^1 =$$

Wednesday week 4

$$= \frac{1}{24}\int_0^1 dz\left[6(1-z^2) - 12(1-z^2)z - 3(1-z^4) + 8(1-z^3)z\right] =$$

$$= \frac{1}{24}\int_0^1 dz\left[6 - 6z^2 - 12z + 12z^3 - 3 + 3z^4 + 8z - 8z^4\right] =$$

$$= \frac{1}{24}\int_0^1 dz\left[(3-8)z^4 + 12z^3 - 6z^2 + (-12+8)z + (6-3)\right] =$$

$$= \frac{1}{24}\int_0^1 dz\left(-5z^4 + 12z^3 - 6z^2 - 4z + 3\right) =$$

$$= \frac{1}{24}\left[-\frac{5z^5}{5} + 12\frac{z^4}{4} - 6\frac{z^3}{3} - 4\frac{z^2}{2} + 3z\right]_0^1 =$$

$$= \frac{1}{24}\left[-z^5 + 3z^4 - 2z^3 - 2z^2 + 3z\right]_0^1 =$$

$$= \frac{1}{24}\left[-1 + 3 - 2 - 2 + 3\right]$$

$$= \frac{1}{24}$$

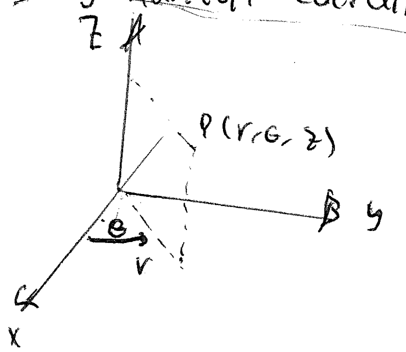
Evaluation of a 3x3 determinant

Wednesday week 4

3x3 determinant can be evaluated as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix} = (a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3) - (a_3 b_2 c_1 + b_3 c_2 a_1 + c_3 a_2 b_1)$$

Cylindrical coordinates



$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

Jacobian: $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$ Differential: $dx\,dy\,dz = r\,dr\,d\theta\,dz$

Proof

For $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, we have

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \cos\theta(r\cos\theta) - \sin\theta(-r\sin\theta) = r(\cos^2\theta + \sin^2\theta) = r$$

Let $A = \{(r\cos\theta, r\sin\theta, z) \mid (r, \theta, z) \in S\} = g(S)$ Wednesday week 4

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^3$ be a scalar field,

Let $g(r, \theta, z) = (r\cos\theta, r\sin\theta, z) = (x, y, z)$ be the transformation

$g: \mathbb{R}^3 \to \mathbb{R}^3$ to cylindrical coordinates.

Let $S \subseteq \mathbb{R}^2$ with $g(S) \subseteq A$. Then

$$\iiint_{g(S)} f(x,y,z)\,dx\,dy\,dz = \iiint_S f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

Example

Evaluate the integral $I = \iiint_A (x^2+y^2)\,dx\,dy\,dz$

with A given by

$A = \{(x,y,z) \in \mathbb{R}^2 \mid x \in [-2, 2] \wedge y \in [-\sqrt{4-x^2}, \sqrt{4-x^2}] \wedge z \in [\sqrt{x^2+y^2}, 2]\}$

by converting to cylindrical coordinates

Solution

- First we determine the domain B of the cylindrical integral

Let $x = r\cos\theta \wedge y = r\sin\theta \wedge z = z$. Then $x^2+y^2 = r^2$

we note that

$y \in [-\sqrt{4-x^2}, \sqrt{4-x^2}] \Leftrightarrow -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2} \Leftrightarrow |y| \leq \sqrt{4-x^2} = \sqrt{4-x^2}$

$\Leftrightarrow |y|^2 \leq (\sqrt{4-x^2})^2 \Leftrightarrow y^2 \leq 4 - x^2 \Leftrightarrow x^2+y^2 \leq 4 \Leftrightarrow 0 \leq r^2 \leq 4$

$\Leftrightarrow 0 \leq r \leq 2$. (since $x^2+y^2 \geq 0 \ \forall x, y \in \mathbb{R}$)

and

$z \in [\sqrt{x^2+y^2}, 2] \iff \sqrt{x^2+y^2} \leq z \leq 2 \iff r \leq z \leq 2$

The condition $x \in [-2,2]$ is implied by the condition on y via the domain restriction $4-x^2 \geq 0$, so it is redundant and does not introduce further restrictions.

Indeed: $|x| = |r\cos\theta| = |r||\cos\theta| \leq |r| = r \leq 2 \Rightarrow |x| \leq 2 \Rightarrow x \in [-2,2]$

Since there are no restriction on the angle θ we have $\theta \in [0, 2\pi]$

It follows that

$A = \{(r\cos\theta, r\sin\theta, z) \mid \theta \in [0,2\pi] \wedge r \in [0,2] \wedge z \in [r,2]\}$

Define

$B = \{(r,\theta,z) \mid \theta \in [0,2\pi] \wedge r \in [0,2] \wedge z \in [r,2]\}$

Now we change variables and evaluate the integral

$$I = \iiint_A (x^2+y^2)\,dx\,dy\,dz = \iiint_B r^2\, r\,dr\,d\theta\,dz = \iiint_B r^3\,dr\,d\theta\,dz = \int_0^{2\pi} d\theta \int_0^2 dr \int_r^2 dz\, r^3$$

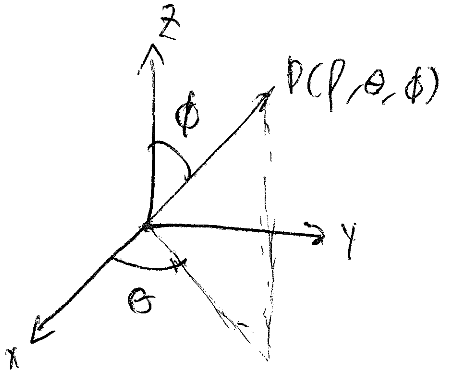
$$= \int_0^{2\pi} d\theta \int_0^2 dr\, r^3 \left[\int_r^2 dz\right] = \int_0^{2\pi} d\theta \int_0^2 dr\, r^3(2-r) = \int_0^{2\pi} d\theta \int_0^2 dr\,(2r^3 - r^4) = \int_0^{2\pi} d\theta \left[\frac{2r^4}{4} - \frac{r^5}{5}\right]_0^2$$

$$= \left[\frac{r^4}{2} - \frac{r^5}{5}\right]_0^2 \int_0^{2\pi} d\theta = 2\pi\left[\frac{2^4-0^4}{2} - \frac{2^5}{5}\right] = 2\pi\left[8 - \frac{32}{5}\right] = 2\pi\,\frac{40-32}{5} = 2\pi\,\frac{8}{5}$$

$$= \frac{16\pi}{5}$$

Thursday week 4

Spherical coordinates



- we define

ρ = distance from origin to P

θ = angle from x-axis to projection of OP onto the xy plane

ϕ = angle fro z-axis to OP.

Note that:

$$\rho \in (0,+\infty) \quad \theta \in [0,2\pi), \quad \phi \in [0,\pi]$$

we define the transformation from spherical to cartesian coordinates

$$g:(\rho,\theta,\phi)\in(0,+\infty)\times[0,2\pi)\times[0,\pi]\to(x,y,z)\in\mathbb{R}^3$$

as follows

$x = [\rho\sin\phi]\cos\theta$

$y = [\rho\sin\phi]\sin\theta$

$z = \rho\cos\phi$

Differential: $dxdydz = \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$

Thursday week 4

- Let $g: [0,+\infty) \times [0, 2\pi) \times [0, \pi] \to \mathbb{R}^3$ be the transformation from spherical to cartesian coordinates with

$$g(\rho, \theta, \phi) = (\rho \sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$$

Let $B \subseteq \mathbb{R}^3$ such that $g(B) \subseteq A$. Then, it follows that:

$$I = \iiint_{g(B)} f(x,y,z)\,dx\,dy\,dz = \iiint_B f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

Proof

Consider the coordinate transformation

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$ with $\rho \in [0,+\infty)$, $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$.

it follows that

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \partial x/\partial\rho & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial\rho & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial\rho & \partial z/\partial\theta & \partial z/\partial\phi \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi(\rho\sin\phi)(\rho\cos\phi) \begin{vmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{vmatrix} - \rho\sin\phi(\sin\phi)(\rho\sin\phi) \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin & \cos\theta \end{vmatrix}$$

Thursday week 4

$$= \ell^2 \sin\phi \cos^2\theta (-\sin^2\theta - \cos^2\theta) - \ell^2 \sin^3\phi(\cos^2\theta + \sin^2\theta) =$$

$$= -\ell^2 \sin\phi \cos^2\phi - \ell^2 \sin^3\phi = -\ell^2 \sin\phi(\cos^2\phi + \sin^2\phi) = -\ell \sin\phi$$

<u>Alternative calculation for 3x3 matrices</u>

$$\frac{\partial(x,y,z)}{\partial(\ell,\theta,\phi)} = \left|\begin{array}{ccc|cc} \sin\phi\cos\theta & -\ell\sin\phi\sin\theta & \ell\cos\phi\cos\theta & \sin\phi\cos\theta & -\ell\sin\phi\sin\theta \\ \sin\phi\sin\theta & \ell\sin\phi\cos\theta & \ell\cos\phi\sin\theta & \sin\phi\sin\theta & \ell\sin\phi\cos\theta \\ \cos\phi & 0 & -\ell\sin\phi & \cos\phi & 0 \end{array}\right.$$

$$= (\sin\phi\cos\theta)(\ell\sin\phi\cos\theta)(-\ell\sin\phi) + (-\ell\sin\phi\sin\theta)(\ell\cos\phi\sin\theta)\cos\phi + 0$$

$$- (\cos\phi)(\ell\sin\phi\cos\theta)(\ell\cos\phi\cos\theta) - 0 - (-\ell\sin\phi)(\sin\phi\sin\theta)(-\ell\sin\phi\sin\theta)$$

$$= -\ell^2 \sin^3\phi\cos^2\theta - \ell^2\sin\phi\cos^2\phi\sin^2\theta - \ell^2\cos^2\phi\sin\phi\cos^2\theta - \ell^2\sin^3\sin^2\theta$$

$$= -\ell^2 \sin\phi[\sin^2\phi\cos^2\theta + \cos^2\phi\sin^2\theta + \cos^2\phi \;\; {}^2\theta + \sin^2\phi\sin^2\theta]$$

$$= -\ell^2\sin\phi[\sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi(\sin^2\theta + \cos^2\theta)]]$$

$$= -\ell^2\sin\phi[\sin^2\phi + \cos^2\phi]$$

$$= -\ell^2\sin\phi$$

$$\Rightarrow dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(\ell,\theta,\phi)}\right| d\ell\, d\theta\, d\phi = |-\ell^2\sin\phi|\, d\ell\, d\theta\, d\phi = \ell^2|\sin\phi|\, d\ell\, d\theta\, d\phi$$

$$= \ell^2\sin\phi\, d\ell\, d\theta\, d\phi$$

Note that $\phi \in [0,\pi] \Rightarrow \sin\phi \geq 0 \Rightarrow |\sin\phi| = \sin\phi$

Thursday week 4

Example

Use spherical coordinates to evaluate the integral

$$I = \iiint_A e^{(x^2+y^2+z^2)^{3/2}} \, dx\,dy\,dz$$

over the region $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\}$

Solution

Let $\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho\cos\phi \end{cases}$ with $\begin{matrix} \rho \in [0, +\infty) \\ \theta \in [0, 2\pi) \\ \phi \in [0, \pi] \end{matrix}$

We note that

$x^2+y^2+z^2 \leq 1 \iff \rho^2 \leq 1 \iff 0 \leq \rho \leq 1$

There are no constraints on θ and ϕ

Then let us define

$$B = \{(\rho, \theta, \phi) \mid \rho \in [0,1] \wedge \theta \in [0, 2\pi) \wedge \phi \in [0, \pi]\} = [0,1] \times [0, 2\pi) \times [0, \pi]$$

It follows that

$$I = \iiint_A e^{(x^2+y^2+z^2)^{3/2}} \, dx\,dy\,dz = \iiint_B e^{(\rho^2)^{3/2}} \rho^2 \sin\phi \, d\rho\,d\theta\,d\phi = \iiint_B \rho^2 e^{\rho^3} \sin\phi \, d\rho\,d\theta\,d\phi$$

$$= \int_0^1 d\rho \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \, \rho^2 e^{\rho^3} \sin\phi = \left[\int_0^1 d\rho \, \rho^2 e^{\rho^3}\right]\left[\int_0^{\pi} d\phi \sin\phi\right]\left[\int_0^{2\pi} d\theta\right] = 2\pi I_1 I_2$$

with $I_1 = \int_0^1 d\rho \, \rho^2 e^{\rho^3}$ and $I_2 = \int_0^{\pi} d\phi \sin\phi$

Thursday week 4

This comes from the trick

$A = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$

$$\iiint_A f(x)\, g(y)\, h(z)\, dx\, dy\, dz = \left[\int_{a_1}^{a_2} dx\, f(x)\right]\left[\int_{b_1}^{b_2} dy\, g(y)\right]\left[\int_{c_1}^{c_2} dz\, h(z)\right]$$

Returning to the problem

To evaluate I_1:

Let $u = e^{\rho^3} = g(\rho) \Rightarrow \begin{cases} du = 3\rho^2 e^{\rho^3} d\rho \\ g(0) = e^0 = 1 \\ g(1) = e^1 = e \end{cases}$

$$\Rightarrow I_1 = \int_1^e \left(\frac{1}{3}\right) du = \frac{1}{3}\int_1^e du = \frac{e-1}{3}$$

To evaluate I_2:

$$I_2 = \int_0^{\pi} d\phi \sin\phi = \left[-\cos\phi\right]_0^{\pi} = (-\cos\pi) - (-\cos 0) = (-(-1)) - (-1) = 1+1 = 2$$

It follows that:

$$I = 2\pi I_1 I_2 = 2\pi\left(\frac{e-1}{3}\right)\cdot 2 = \frac{4\pi(e-1)}{3}$$

Thursday week 4

Vector Fields

Definition

- A three-dimensional vector field f is a mapping $f: A \to \mathbb{R}^3$ with $A \subseteq \mathbb{R}^3$
- If f is a vector field, we write:
  $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$
- The scalar fields f_1, f_2, f_3 are the components of the vector field f.

Derivative of a vector field

- Let $f: A \to \mathbb{R}^3$ be a vector with components f_1, f_2, f_3 that are assumed to be partially differentiable. Then we define

a) The divergence of f: $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$

$$\nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \operatorname{div}(f) \leftarrow \text{Divergence}$$

b) The curl of f

$$\nabla \times f = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix} \to \begin{cases} (\nabla \times f)_1 = \dfrac{\partial f_3}{\partial y} - \dfrac{\partial f_2}{\partial x} \\ (\Delta \times f)_2 = \dfrac{\partial f_1}{\partial z} - \dfrac{\partial f_3}{\partial x} \\ (\Delta \times f)_3 = \dfrac{\partial f_2}{\partial x} - \dfrac{\partial f_1}{\partial y} \end{cases} = \operatorname{curl}(f)$$

- Note that

$\nabla \cdot f$ is a scalar field

$\nabla \times f$ is a vector field

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- Let $\phi: A \to \mathbb{R}$ be a scaler field with $A \subseteq \mathbb{R}^3$ then we define:

a) The gradient of ϕ

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

Laplacian

$$\nabla^2\phi = \nabla\cdot\nabla\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \partial^2\phi$$

$$\nabla^2 f = (\nabla^2 f_1, \nabla^2 f_2, \nabla^2 f_3)$$

Tensor notation

Tensor notation makes it easier to work with the derivatives of vector fields. we follow the following guidelines,

a) we write the vector field "f" as "f_a", representing the "a" component of f. with tensor notation we always work in terms of the components of the ivolved vector fields

b) Repeating indices are automatically summed over all components. when associated with a product.

Example: $f_a g_a = f_1 g_1 + f_2 g_2 + f_3 g_3 = f \cdot g$

represents the dot product However, for the vector sum

$(f+g)_a = f_a + g_a$

no summation is implied

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c) we abreviate the partial derivative $\partial/\partial x_a$ as ∂_a
we may thus write:

$$\nabla \cdot f = \partial_a f_a \ ; \ (\nabla\phi)_a) = \partial_a \phi \ ; \ \nabla^2 \phi = \partial_a \partial_a \phi$$

Notice a scalar - No index
vector - 1 index
Matrix - 2 indxes
⋮
etc

d) To define the curl, we introduce the Levi-Civita tensor ε_{abc} as

$$\varepsilon_{abc} = \frac{(a-b)(b-c)(c-a)}{2}$$

$$= \begin{cases} +1, & \text{if } (a,b,c) \in \{(1,2,3), (2,3,1), (3,1,2)\} \\ -1, & \text{if } (a,b,c) \in \{(3,2,1), (1,3,2), (2,1,3)\} \\ 0, & \text{if } a=b \vee b=c \vee c=a \end{cases}$$

Then the curl reads

$$(\nabla \times f)_a = \varepsilon_{abc} \partial_b$$

Likewise for two vector fields f, g, the cross-product reads.

$$(f \times g)_a = \varepsilon_{abc} f_b g_c$$

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To sumarize: Given the vector fields f, g and the scalar field ϕ:

$(f+g)_a = f_a + g_a$ $\qquad$ $\nabla\phi = \partial_a \phi$ $\qquad$ $\nabla \cdot f = \partial_a f_a$

$f \cdot g = f_a g_a$ $\qquad$ $\nabla^2 \phi = \partial_a \partial_a \phi$ $\qquad$ $\nabla \times f = \varepsilon_{abc} \partial_b f_c$

$f \times g = \varepsilon_{abc} f_b g_c$

Kronecker delta

we define

$$\delta_{ab} = \begin{cases} 1 & \text{if } a=b \\ 0 & \text{if } a \neq b \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows that $\delta_{ab} f_a = \delta_{a1} f_1 + \delta_{a2} f_2 + \delta_{a3} f_3 = f_a$

(because in the implied summation $b = 1, 2, 3$ the only non-zero contribution occurs when $a=b$)

Similar contractions are possible. For example

$\delta_{ab} \varepsilon_{acd} = \varepsilon_{bcd}$

(because in the implied summation $a = 1, 2, 3$, the only non-zero contribution occurs when $a=b$).

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$(f+g)_\alpha = f_\alpha + g_\alpha$

$f \cdot g = f_\alpha g_\alpha$

$(f \times g) = \varepsilon_{\alpha bc} f_b g_c$

$\nabla \cdot f = \partial_\alpha f_\alpha$

$(\nabla \times f)_\alpha = \varepsilon_{abc} \partial_b f_c$

$(\nabla \phi)_\alpha = \partial_\alpha \phi$

$\nabla^2 \phi = \partial_\alpha \partial_\alpha \phi$

$$\delta_{\alpha b} = \begin{cases} 1 & \text{if } \alpha = b \\ 0 & \text{if } \alpha \neq b \end{cases}$$

$\delta_{\alpha b} f_b = f_\alpha$

Properties $\varepsilon_{\alpha bc}$ (Levi-Civita tensor)

1) $\varepsilon_{\alpha bc} f_b f_c = 0_\alpha$

$\varepsilon_{\alpha bc} \partial_b \partial_c = 0_\alpha$

2) $\varepsilon_{\alpha bc} = \varepsilon_{bc\alpha} = \varepsilon_{c\alpha b}$

$\varepsilon_{abc} = -\varepsilon_{cb\alpha}$

3) $$\varepsilon_{\alpha bc} \varepsilon_{pqr} = \begin{vmatrix} \delta_{\alpha p} & \delta_{aq} & \delta_{ar} \\ \delta_{bp} & \delta_{bq} & \delta_{br} \\ \delta_{cp} & \delta_{cq} & \delta_{cr} \end{vmatrix}$$

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4) $\varepsilon_{\alpha bc}\varepsilon_{\alpha pq} = \delta_{bp}\delta_{cq} - \delta_{bq}\delta_{cp}$

$\varepsilon_{\alpha bp}\varepsilon_{abq} = 2\delta_{pq}$

$\varepsilon_{\alpha bc}\varepsilon_{\alpha bc} = 6$

1) $\nabla \times (\nabla\phi) = 0$

Proof

$$[\nabla \times (\nabla\phi)]_\alpha = \varepsilon_{\alpha bc}\partial_b(\nabla\phi)_c = \varepsilon_{abc}\partial_b(\partial_c\phi)$$
$$= [\varepsilon_{abc}\partial_b\partial_c]\phi$$
$$= 0_\alpha\phi$$
$$= 0_\alpha$$

$\Rightarrow \nabla \times (\nabla\phi) = 0$

2) $\nabla \cdot (\nabla \times f) = 0$

Proof

$$\nabla \cdot (\nabla \times f) = \partial_\alpha(\nabla \times f)_\alpha = \partial_\alpha[\varepsilon_{\alpha bc}\partial_b f_c]$$
$$= \varepsilon_{\alpha bc}\partial_\alpha\partial_b f_c$$
$$= [\varepsilon_{cab}\partial_\alpha\partial_b]f_c$$
$$= 0_c f_c$$
$$= 0$$

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3) $\nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \nabla^2 f$

$\hookrightarrow$ $\mathrm{curl}(\mathrm{curl}(f)) = \mathrm{grad}(\mathrm{div}(f)) - \Delta f$

Proof

$$[\nabla \times (\nabla \times f)]_\alpha = \varepsilon_{abc} \partial_b (\nabla_\times f)_c$$
$$= \varepsilon_{abc} \partial_b [\varepsilon_{cpq} \partial_p f_q]$$
$$= \varepsilon_{abc} \varepsilon_{cpq} \partial_b \partial_p f_q$$
$$= \varepsilon_{cab} \varepsilon_{cpq} \partial_b \partial_p f_q$$
$$= [\delta_{\alpha p} \delta_{bq} - \delta_{\alpha q} \delta_{bp}] \partial_b \partial_p f_q$$
$$= \delta_{\alpha p} \delta_{bq} \partial_b \partial_p f_q - \delta_{\alpha q} \delta_{bp} \partial_b \partial_p f_q$$
$$= \partial_b \partial_\alpha f_b - \partial_b \partial_b f_\alpha$$
$$= \partial_\alpha (\partial_b f_b) - \nabla^2 f_\alpha$$
$$= \partial_\alpha (\nabla \cdot f) - (\nabla^2 f)_\alpha$$
$$= [\nabla(\nabla \cdot f)]_\alpha - (\nabla^2 f)_\alpha$$
$$= [\nabla(\nabla \cdot f) - \nabla^2 f]_\alpha$$

Another property

$$a \cdot (b \times c) = b \cdot (c \times a)$$

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$\nabla \cdot (\phi f) = f \cdot \nabla \phi + \phi \nabla \cdot f \rightarrow \operatorname{div}(\phi f) = f \cdot \operatorname{grad}(\phi) + \phi \operatorname{div}(f)$

Proof

$$\nabla \cdot (\phi f) = \partial_\alpha (\phi f)_\alpha = \partial_\alpha (\phi f_\alpha)$$
$$= f_\alpha (\partial_\alpha \phi) + \phi \partial_\alpha f_\alpha$$
$$= f_\alpha (\nabla \phi)_\alpha + \phi \nabla \cdot f$$
$$= f \cdot (\nabla \phi) + \phi \nabla \cdot f$$

Line Integrals

- Line integrals are integrals of vector fields over a path

Definition concerning paths

- Let $a : I \rightarrow \mathbb{R}^n$ be a vector-valued function

a is a path $\Longleftrightarrow \begin{cases} \exists t_1, t_2 \in \mathbb{R} : I = [t_1, t_2] \\ a \text{ Continuous at } [t_1, t_2] \end{cases}$

Furthermore, if a is a path we say that

a is a closed path $\Longleftrightarrow a(t_1) = a(t_2)$

a is an open path $\Longleftrightarrow a(t_1) \neq a(t_2)$

we say that: $a(t_1)$ = initial point

$a(t_2)$ = final point

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- Consider a path $a : [t_1, t_2] \to \mathbb{R}^n$ we say that

$$a \text{ smooth path} \iff \begin{cases} a \text{ continuous at } [t_1, t_2] \\ a \text{ differentiable at } (t_1, t_2) \\ a' \text{ continuous at } (t_1, t_2) \end{cases}$$

- Let $a : [t_1, t_2] \to \mathbb{R}^n$ be a path and let $B \subseteq [t_1, t_2]$. The restriction of "a" to B is denoted as:

$$b = a \restriction B \iff \begin{cases} b : B \to \mathbb{R}^n \\ \forall t \in B : b(t) = a(t) \end{cases}$$

- Let $a : [t_1, t_2] \to \mathbb{R}^n$ be a path we say that a piecewise smooth path $\iff$

$$\iff \begin{cases} \exists \tau_0, \tau_1, \dots, \tau_n \in [t_1, t_2] : t_1 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_n = t_2 \\ \forall n \in [n] : a \restriction [\tau_{k-1}, \tau_k] \text{ is a smooth path} \end{cases}$$

Example

A piecewise smooth path may look like this

Connected Sets

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- Let $x, y \in \mathbb{R}^n$ be given points. we define
  $\mathcal{P}_A(x, y)$ = The set of all piecewise smooth paths in A with initial point x and final point y

- Let $A \subseteq \mathbb{R}^n$ be a given region. We say that

  A path - connected $\iff \forall x, y \in A : \mathcal{P}_A(x, y) \neq \emptyset$

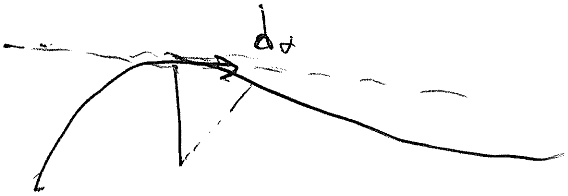
  A path - disconnected $\iff \exists x, y \in A : \mathcal{P}_A(x, y) \; \emptyset$

Interpretation: In a path-connected set A. every two points $x, y \in A$ are connected by at least one.

Definition of line integral

Let $\alpha : [t_1, t_2] \to \mathbb{R}^n$ be piecewise smooth path, and let $f : A \to \mathbb{R}^n$ such that $\alpha([t_1, t_2]) \subseteq A$
we define

$$\int f \cdot d\alpha = \int_{t_1}^{t_2} dt \left[f(\alpha(t)) \cdot \dot{\alpha}(t) \right]$$



$f \cdot dl \leftrightarrow dt \, \dot{\alpha}(t)$

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Notation

$C: \alpha(t) = (\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t))$

$I = \int f \cdot d\alpha = \int_C f \cdot dl$

$f(x) = (f_1(x), f_2(x), \dots, f_n(x))$

$I = \int f_1 d\alpha_1 + f_2 d\alpha_2 + \dots + f_n d\alpha_n = \int_C f_1 dx_1 + f_2 dx_2 + \dots + f_n dx_n$

For a two dimensional fiel $f(x,y)$ and an path

$C: \alpha(t) = (\alpha_1(t), \alpha_2(t)) \ \forall t \in [t_1, t_2]$

$f(x,y) = (f_1(x,y), f_2(x,y))$

$I = \int f \cdot d\alpha = \int_C f_1(x,y)\, dx + f_2(x,y)\, dy$

$= \int_{t_1}^{t_2} dt \left[f_1(\alpha_1(t), \alpha_2(t))\alpha_1'(t) + f_2(\alpha_1(t), \alpha_2(t))\alpha_2'(t) \right]$