Limit restricted to a path

dag 8

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ and let $x_0 \in \mathrm{int}(A)$. Define $P(x_0)$ set of all $\gamma: (0, \alpha) \to \mathbb{R}^n$ such that

$$\lim_{t \to 0^+} \|\gamma(t) - x_0\| = 0$$

we define:

$$\lim_{\gamma} f(x) = \lim_{t \to 0^+} f(\gamma(t))$$

$$\lim f(x) = l \iff \forall \gamma \in P(x_0): \lim_{\gamma} f(x) = l$$

$$\exists \gamma_1, \gamma_2 \in P(x_0): \lim_{\gamma_1} f(x) \neq \lim_{\gamma_2} f(x) \implies \lim_{x \to x_0} f(x) \text{ does not exist}$$

EXAMPLE

Evaluate the $\lim\limits_{(x,y)\to(0,0)} 3x^2(x^2-y^2)$

Solution

consider the path $\gamma(\theta): \begin{cases} x = t\cos\theta \\ y = t\sin\theta \end{cases}$ with $t \to 0^+$

Then

$$\lim_{\gamma(\theta)} f(x,y) = \lim_{\gamma(\theta)} = \frac{x^2-y^2}{x^2+y^2} = \lim_{t\to 0^+} \frac{t^2\cos^2\theta - t^2\sin^2\theta}{t^2\cos^2\theta + t^2\sin^2\theta} = \lim_{t\to 0^+} \frac{t^2(\cos^2\theta - \sin^2\theta)}{t^2(\cos^2\theta + \sin^2\theta)} = \cos^2\theta - \sin^2\theta = \cos 2\theta$$

For $\theta = 0$; $\lim\limits_{\gamma(0)} f(x,y) = \cos(2\cdot 0) = \cos 0 = 1$ (1)

For $\theta = \frac{\pi}{4}$: $\lim\limits_{\gamma(\frac{\pi}{4})} f(x,y) = \cos(2\frac{\pi}{4}) = \cos(\frac{\pi}{2}) = 0$ (2)

From (1) and (2)

$\lim\limits_{\gamma(0)} f(x,y) \neq \lim\limits_{\gamma(\frac{\pi}{4})} f(x,y) \Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist

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b) $f(x,y) = \dfrac{xy^2}{x^2+y^4} \longleftarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$

Solution

Consider the path $\gamma(\theta) = \begin{cases} x = t\cos\theta \\ y = t\sin\theta \end{cases}$, with $t \to 0^+$

$$\lim_{\gamma(\theta)} f(x,y) = \lim_{\gamma(\theta)} \frac{xy^2}{x^2+y^4} = \lim_{t\to0^+} \frac{(t\cos\theta)(t\sin\theta)^2}{(t\cos\theta)^2+(t\sin\theta)^4}$$

$$= \lim_{t\to0^+} \frac{t^3\cos\theta\sin^2\theta}{t^2\cos^2\theta + t^4\sin^4\theta}$$

$$= \lim_{t\to0^+} \frac{t^3\cos\theta\sin^2\theta}{t^2(\cos^2\theta + t^2\sin^4\theta)}$$

$$= \lim_{t\to0^+} \frac{t\cos\theta\sin^2\theta}{\cos^2 + t^2\sin^4\theta} \qquad (1)$$

From (1): for $\cos\theta \neq 0$:

$$\lim_{\gamma(\theta)} f(x,y) = \frac{0\cos\theta\sin^2\theta}{\cos^2\theta + 0^2\sin^4\theta} = \frac{0}{\cos^2\theta + 0} = 0$$

For $\cos\theta = 0 \Rightarrow \sin\theta = \pm1 \Rightarrow$

$$\Rightarrow \lim_{\gamma(\theta)} f(x,y) = \lim_{t\to0^+} \frac{t\cdot 0\cdot(\pm1)^2}{0^2+t^2(\pm1)^4} = \lim_{t\to0^+}\left(\frac{0}{t}\right) = 0$$

Nevertheless, although all linear paths $\gamma(\theta)$ are in agreement, the limit does <u>not</u> in fact exist. To show this, we consider the path γ_0 defined as

$\gamma_0: \begin{cases} x = t^2 \\ y = t \end{cases}$ with $t \to 0^+$

and note that

$$\lim_{\gamma_0} f(x,y) = \lim_{\gamma_0} \frac{xy^2}{x^2+y^4} = \lim_{t\to0^+} \frac{t^2t^2}{(t^2)^2+t^4} = \lim_{t\to0^+} \frac{t^4}{t^4+t^4} = \lim_{t\to0} \frac{t^4}{2t^4} = \frac{1}{2}$$

Since $\lim\limits_{\gamma_0} f(x,y) \neq \lim\limits_{\gamma(\theta)} f(x,y) \Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist

Proposition 4.3

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field, let $g: B \to \mathbb{R}$ with $B \subset \mathbb{R}$ be a function, and let $x_0 \in \mathbb{R}^n$ be an accumulation point of A and l_0 be an accumulation point of B. Then

$$\left.\begin{array}{l} \lim_{x \to x_0} f(x) = l_0 \\ \lim_{t \to l_0} g(t) = l \\ \exists \delta > 0 : \forall x \in N(x_0, \delta) : f(x) \neq l_0 \end{array}\right\} \Rightarrow \lim_{x \to x_0} g(f(x)) = l$$

Alternate Solution of previous problem $\lim_{(x,y) \to (0,0)} \frac{\sin(x^2+y^2)}{3x^2+3y^2}$

we note that

$$\lim_{(x,y) \to (0,0)} (x^2+y^2) = 0 + 0^2 = 0$$

and for $t = x^2+y^2$

$$\lim_{t \to 0} \frac{\sin t}{3t} = \frac{1}{3}$$

we also note that

$$x^2+y^2 \neq 0, \forall (x,y) \in \mathbb{R}^2 - \{(0,0)\}$$

it follows that

$$\lim_{(x,y) \to (0,0)} \frac{\sin(x^2+y^2)}{3x^2+3y^2} = \frac{1}{3}$$

EXAMPLE

$$f(x,y) = \frac{x^4(\sin(xy) + \sin(x+y))}{(x^2+y^2)^2} \longleftarrow \lim_{(x,y)\to(0,0)} f(x,y)$$

Solution

Define $b(x,y) = \dfrac{x^4}{(x^2+y^2)^2}$ $\forall (x,y) \in \mathbb{R}^2 - \{(0,0)\}$

we note that

$$|b(x,y)| = \left|\frac{x^4}{(x^2+y^2)^2}\right| = \frac{x^4}{(x^2+y^2)^2} = \frac{x^4}{x^4+2x^2y^2+y^4} \leq \frac{x^4+2x^2y^2+y^4}{x^4+2x^2y^2+y^4} = 1$$

$\forall (x,y) \in \mathbb{R}^2 - \{(0,0)\} \Rightarrow b$ bounded at $\mathbb{R}^2 - \{(0,0)\}$ (1)

Furthermore:

$$\lim_{(x,y)\to(0,0)} (xy) = 0 \cdot 0 = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \sin(xy) = \sin 0 = 0 \quad (2)$$

$$\lim_{(x,y)\to(0,0)} (x+y) = 0+0 = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \sin(x+y) = \sin 0 = 0 \quad (3)$$

From (2) and (3)

$$\lim_{(x,y)\to(0,0)} [\sin(xy) + \sin(x+y)] = 0 \quad (4)$$

From (1) and (4) $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Directional and Partial Derivatives

- Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $x \in \operatorname{int} A$ and $y \in \mathbb{R}^n$. The directional derivative $f'(x|y)$ is defined as

$$f'(x|y) = \lim_{h \to 0} \frac{f(x+hy) - f(x)}{h}$$

when the limit exists

- $f'(x|y)$ give the rate of chang of f along the line connecting x and $x+y$ at the point x. However, it is important to realize that $f'(x|y)$ depends on Both the direction AND the magnitude of y:

$$f'(x|hy) = hf'(x|y) \quad \forall h \in \mathbb{R}-\{0\}$$

Method. To calculate the directional derivative $f'(x|y)$ we use the following proposition

$$g(t) = f(x+ty),\ \forall t \in (-\varepsilon, \varepsilon) \Rightarrow f'(x|y) = g'(0)$$

EXAMPLE

Evaluate $f'(x,y|e)$ for $f(x,y) = xy(x+y)$ and $e = (1,3)$

Solution

Let $x, y \in \mathbb{R}$ be given. Define

$$g(t) = f((x,y)+t(1,3)) = f(x+t, y+3t) = (x+t)(y+3t)(x+t+y+3t) = (x+t)(y+3t)(x+y+4t) \Rightarrow$$

$$\Rightarrow g'(t) = (y+3t)(x+y+4t) + (x+t)\cdot 3(x+4+4t) + (x+t)(y+3t)\cdot 4 \Rightarrow$$

$$\Rightarrow f'((x,y)|(1,3)) = g'(0) = y(x+y) + 3x(x+y) + 4xy$$
$$= xy + y^2 + 3x^2 + 3xy + 4xy$$
$$= 3x^2 + 8xy + y^2$$

while calculating $g'(t)$ we treat x, y as given constants

Mean Value Theorem

Thm: let $f: A \to \mathbb{R}$ be a scalar field with $A \subseteq \mathbb{R}^n$ Then

$f'(x+ty|y)$ exists. $\forall t \in [0,1] \Rightarrow \exists \xi \in (0,1): f(x+y)-f(x) = f'(x+\xi y|y)$

Partial Derivatives

- Let $f(x) = f(x_1, x_2, \ldots, x_n)$. $\forall x \in A$ with $A \subseteq \mathbb{R}^n$ be a scalar field and consider the unit vectors

$e_1 = (1, 0, 0, \ldots, 0)$
$e_2 = (0, 1, 0, \ldots, 0)$
$e_3 = (0, 0, 1, \ldots, 0)$
$\vdots$
$e_n = (0, 0, 0, \ldots, 1)$

we define the partial derivative of f with respect to x_n as

$$\frac{\partial f}{\partial x_k} = f'(x|e_n)$$

Method: To calculate $\frac{\partial f}{\partial x_n}$ we differentiate f with respect to x_n treating all other variables as constant

Notation: Other notation for partial derivatives for example for 2 variables

$$\frac{\partial f(x,y)}{\partial x} = f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = D_1 f(x,y)$$

$$\frac{\partial f(x,y)}{\partial y} = f_y(x,y) = \frac{\partial}{\partial y} f(x,y) = D_2 f(x,y)$$

Example

For $f(x,y) = xy^2(x^2+y^2)^3$, evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Solution

Recall $([f(x)]^n)' = n[f(x)]^{n-1} f'(x)$

$(x^n)' = nx^{n-1}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[xy^2(x^2+y^2)^3]$$

$$= \frac{\partial}{\partial x}[xy^2]\cdot(x^2+y^2)^3 + xy^2\left[\frac{\partial}{\partial x}(x^2+y^2)^3\right]$$

$$= y^2(x^2+y^2)^3 + xy^2 3(x^2+y^2)^2\left[\frac{\partial}{\partial x}(x^2+y^2)\right]$$

$$= y^2(x^2+y^2)^3 + 3xy^2(x^2+y^2)^2\cdot(2x)$$

$$= y^2(x^2+y^2)^2[(x^2+y^2)+3x(2x)]$$

$$= y^2(x^2+y^2)^2(x^2+y^2+6x^2)$$

$$= y^2(x^2+y^2)^2(7x^2+y^2) \quad \blacksquare$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[xy^2(x^2+y^2)^3 \right]$$

$$= \left[\frac{\partial}{\partial y}(xy^2)\right](x^2+y^2)^3 + xy^2\left[\frac{\partial}{\partial y}(x^2+y^2)^3\right]$$

$$= 2xy(x^2+y^2)^3 + xy^2 \cdot 3(x^2+y^2)^2\left[\frac{\partial}{\partial y}(x^2+y^2)\right]$$

$$= 2xy(x^2+y^2)^3 + 3xy^2(x^2+y^2)^2\,2y$$

$$= 2xy(x^2+y^2)^2\left[(x^2+y^2) + 3y(y)\right]$$

$$= 2xy(x^2+y^2)^2(x^2+y^2+3y^2)$$

$$= 2xy(x^2+y^2)^2(x^2+4y^2) \quad \blacksquare$$

Directional derivatives and Continuity

NOTE: The existence of directional derivatives in all directions does not guarantee that your scalar field is continuous. This is different from single variable Calculus

Counter Example

Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Directional derivatives

For $d = (a,b)$ with $a \neq 0$

$$f'(0,0;d) = \lim_{h\to 0} \frac{f(ha,hb) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(ha,hb)}{h} =$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\,\frac{(ha)(hb)^2}{(ha)^2+(hb)^4}\right] = \lim_{h\to 0}\left[\frac{h^3ab^2}{h^3(a^2+h^2b^4)}\right]$$

$$= \lim_{h\to 0}\left[\frac{ab^2}{a^2+h^2b^4}\right] = \frac{ab^2}{a^2+0} = \frac{b^2}{a}$$

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For $d=(0,b)$

$$f'(0,0|d) = \lim_{h\to 0} \frac{f(0,hb)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

Thus $f'(0,0|d)$ exists for all directions $d \in \mathbb{R}^2 - \{(0,b)\}$

• Continuity

For (γ_1): $\begin{cases} x=0 \\ y=t \end{cases}$ with $t \to 0^+$,

we have

$$\lim_{\gamma_1} f(x,y) = \lim_{t\to 0^+} f(0,t) = 0 \quad (\text{since } f(0,t)=0, \forall t\in\mathbb{R})$$

For (γ_2): $\begin{cases} x=t^2 \\ y=b \end{cases}$ with $t\to 0^+$, we have

$$\lim_{\gamma_2} f(x,y) = \lim_{t\to 0^+} f(t^2,t) = \lim_{t\to 0^+} \frac{t^2 t^2}{(t^2)^2+t^4}$$

$$= \lim_{t\to 0^+} \frac{t^4}{t^4+t^4} = \lim_{t\to 0^+} \frac{t^4}{2t^4} = \frac{1}{2}$$

Since $\lim_{\gamma_1} f(x,y) \neq \lim_{\gamma_2} f(x,y) \Rightarrow f$ not continuous at $(0,0)$.

Mixed partial derivatives

Mixed partial derivatives are defined by succesive partial differentiation, as follows.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = D_1 D_1 f$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy} = D_2 D_2 f$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy} = D_1 D_2 f$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx} = D_2 D_1 f$$

Clairat's theorem (is $f_{xy} = f_{yx}$?)

Thm: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ be a scalar field. Let $(a,b) \in A$ be a point and let $S \subseteq A$ be an open set such that $(a,b) \in \text{int}(S)$

Assume that

a) f_x, f_y, f_{xy} exist in S (i.e for all points in S)

b) f_{xy} continuous in S. Then $f_{xy}(a,b) = f_{yx}(a,b)$

In condition (b) we can replace f_{xy} with f_{yx}.

Example with $f_{xy}(a,b) \neq f_{yx}(a,b)$

Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \in \mathbb{R}^2 - \{(0,0)\} \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Note that $f(x,0) = f(0,y) = 0$

- $f_{xy}(0,0)$ calculation

By definition:

$$f_y(x,0) = \lim_{h\to 0} \frac{f(x,h) - f(x,0)}{h} = \lim_{h\to 0} \frac{f(x,h)}{h}$$

$$= \lim_{h\to 0} \left[\frac{1}{h} \frac{xh(x^2-h^2)}{x^2+h^2} \right] = \lim_{h\to 0} \frac{x(x^2-h)}{x^2+h}$$

$$= \frac{x(x^2-0)}{x^2+0} = x \Rightarrow f_{xy}(x,0) = 1 \Rightarrow f_{x,y}(0,0) = 1$$

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$f_{yx}(0,0)$ calculation

$$f_x(0,y) = \lim_{h \to 0} \frac{f(h,y) - f(0,y)}{h} = \lim_{h \to 0} \frac{f(h,y)}{h}$$

$$= \lim_{h \to 0} \left[\frac{1}{h} \, \frac{hy(h^2 - y^2)}{h^2 + y^2} \right]$$

$$= \lim_{h \to 0} \frac{y(h^2 - y^2)}{h^2 + y^2} = \frac{y(0 - y^2)}{0 + y^2} = -y \Rightarrow$$

$$\Rightarrow f_{yx}(0,y) = -1 \Rightarrow f_{yx}(0,0) = -1$$

Since $\begin{cases} f_{xy}(0,0) = 1 \\ f_{yx}(0,0) = -1 \end{cases} \Rightarrow f_{xy}(0,0) \neq f_{yx}(0,0)$

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Differentiable scalar fields

- Directional derivatives account for the rate of change of the scalar field f accross linear directions. A proper definition of differentiability has to account for curved direction as well

Def: Let $T: \mathbb{R}^n \to \mathbb{R}$ be a scalar field we say that

T linear $\iff \forall \lambda_1, \lambda_2 \in \mathbb{R}: \forall x, y \in \mathbb{R}^n: T(\lambda_1 x, \lambda_2 y) = \lambda_1 T(x) + \lambda_2 T(y)$

Definition of Total Derivative (Young-Fréchet)

Def: Let $f: S \to \mathbb{R}$ with $S \subseteq \mathbb{R}^n$ be a scalar field and let $a \in \text{Int}(S)$ we say that f is differentiable at a if and only if there exist:

a) A linear scalar field $T_a: \mathbb{R}^n \to \mathbb{R}$

b) A scalar field $E_a: \mathbb{R}^n \to \mathbb{R}$

c) A number $\rho \in (0, +\infty)$

$$\begin{cases} \forall x \in B(0, \rho): f(a) + T_a(x) + \|x\| E_a(x) \\ \lim_{x \to 0} E_a(x) = 0 \end{cases}$$

- $T_a(x)$ is the total derivative of f at a. Note that T_a itself is a scalar field, satisfying:

$\forall \lambda_1, \lambda_2 \in \mathbb{R}: \forall x, y \in \mathbb{R}^n: T_a(\lambda_1 x, \lambda_2 y) = \lambda_1 T_a(x) + \lambda_2 T_a(y)$

Gradient of a scalar fied Day 9

- Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scaler field
we define the gradient $\nabla f : A \to \mathbb{R}^n$ as
$\nabla f = (D_1 f, D_2 f, D_3 f, \dots, D_n f)$

provided that the partial derivatives exist

- Note that ∇f is not a scalar fied. It is a vect : fied

Properties of differentiable

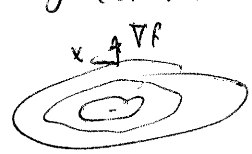
Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $a \in \operatorname{Int}(A)$. it can be shown that:

1) f differentiable at $a \Rightarrow \forall x \in \mathbb{R}^n : T_a(x) = f'(a|x) = \nabla f(a) \cdot x$
2) f differentiable at $a \Rightarrow f$ continuous at a
3) $\left.\begin{array}{l} \exists \rho \in (0, +\infty) : D_1 f, D_2 f, D_n f \text{ exist on } B(a, \rho) \\ D_1 f, D_2 f, \dots, D_n f \text{ continuous} \end{array}\right\} \Rightarrow$
$\Rightarrow f$ differentiable at a

- Properties (1), (2) are consequences of differentiable
Property (3) is a sufficient condition for differentiability

- Property (1) indicates a method for evaluating directiond derivathves via the gradient ∇f

Physical interpretation of the gradient is the directra with fastest chang



Example

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For $f(x,y) = xe^y + x^2y$ evaluate $f'(2,0|a,b)$ for $a,b \in \mathbb{R}-\{0\}$

Solution

$$f_x(x,y) = \frac{\partial}{\partial x}(xe^y + x^2y) = e^y + 2xy \Rightarrow$$

$$\Rightarrow f_x(2,0) = e^0 + 2\cdot 2\cdot 0 = 1+0 = 1$$

$$f_y(x,y) = \left(\frac{\partial}{\partial y}\right)(xe^y + x^2y) = xe^y + x^2 \Rightarrow$$

$$\Rightarrow f_y(2,0) = 2\cdot e^0 + 2^2 = 2+4 = 6$$

It follows that

$$\begin{aligned} f'(2,0|a,b) &= \nabla f(2,0)\cdot(a,b) \\ &= a f_x(2,0) + b f_y(2,0) \\ &= a + 6b \end{aligned}$$

Application: Tangent plane to surface

- Consider a surface $(S): z = f(x,y)$ with $f: A \to \mathbb{R}$ and $A \subseteq \mathbb{R}^2$ a scalar field differentiable in A. Let $p \in A$ be a point. The tangent plane (tp) to the surface (S) at point (S) at point (p) is given by: $(tp): z = f(p) + \nabla f(p) \cdot [(x,y) - p]$

EXAMPLE

Find the tangent plane (t) to the surface $(S): z = x^2y(x^2+y^2-3x)$ at the point $(x,y) = (1,2)$

Solution

Define $f(x,y) = x^2y(x^2+y^2-3xy) = x^4y + x^2y^3 - 3x^3y^2$
and therefore:

$f_x(x,y) = 4x^3y + 2xy^3 - 9x^2y^2$

$f_y(x,y) = x^4 + 3x^2y^2 - 6x^3y$

At $(x,y) = (1,2)$

$f(1,2) = 1^4 \cdot 2 + 1^2 \cdot 2^3 - 3 \cdot 1^3 \cdot 2^2 = 2 + 8 - 12 = -2$

$f_x(1,2) = 4 \cdot 1^3 \cdot 2 + 2 \cdot 1 \cdot 2^3 - 9 \cdot 1^2 \cdot 2^2 = 8 + 16 - 36 = 8 - 20 = -12$

$f_y(1,2) = 1^4 + 3 \cdot 1^2 \cdot 2^2 - 6 \cdot 1^3 \cdot 2 = 1 + 12 - 12 = 1$

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It follows that

$$(t): z = f(1,2) + \nabla f(1,2) \cdot [(x,y) - (1,2)] \iff$$

$$\iff z = -2 + (-12,1) \cdot (x-1, y-2) \iff$$

$$\iff z = -2 - 12(x-1) + (y-2) \iff$$

$$\iff z = -2 - 12x + 12 + y - 2 \iff$$

$$\iff z = -12x + y + 8 \iff$$

$$\iff 12x - y + z = 8$$

thus:

$$(t): 12x - y + z = 8.$$

The chain rule

Thm: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $a: I \to int(A)$ be a vector function with $I \subseteq \mathbb{R}$ Assum that.

a) $a(t)$ differentiable at t
b) f differentiable at $a(t)$

Then

$$\left(\frac{d}{dt}\right) f(a(t)) = \nabla f(a(t)) \cdot a'(t)$$

Application to scalar fields on $\mathbb{R}^2$

1) For $z = f(x,y)$ with $x = x(t)$ and $y = y(t)$

$$\Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

2) For $z = f(x,y)$ with $x = x(t,s)$ and $y = y(t,s)$

$$\Rightarrow \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

The chain rule can be used to derive rules for transforming partial derivatives to other coordinates systems

Example

Consider the two-dimensional polar coordinates system defined via

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Given a scalar field $f(x,y)$ write $\frac{\partial f}{\partial r}$, $\frac{\partial f}{\partial \theta}$ in terms of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

<u>Solution</u>

We note that

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta \qquad \frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(r\cos\theta) = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(r\sin\theta) = \sin\theta \qquad \frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(r\sin\theta) = r\cos\theta$$

and therefore:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = (\cos\theta)\frac{\partial f}{\partial x} + (\sin\theta)\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = (-r\sin\theta)\frac{\partial f}{\partial x} + (r\cos\theta)\frac{\partial f}{\partial y}$$

Using matrix notation:

$$\begin{bmatrix} \partial f/\partial r \\ \partial f/\partial \theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix}$$

Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}$

Then

$$\det A = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = \cos\theta \cdot (r\cos\theta) - \sin\theta(-r\sin\theta) = r(\cos^2\theta + \sin^2\theta) = r$$

and therefore

$$A^{-1} = \frac{1}{r}\begin{bmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -(\sin\theta)/r \\ \sin\theta & \cos\theta/r \end{bmatrix}$$

It follows that

$$\frac{\partial f}{\partial x} = (\cos\theta)\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}$$

$$\frac{\partial f}{\partial y} = (\sin\theta)\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta} \quad \blacksquare$$

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Coordinate Change to polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \longrightarrow \begin{aligned} \frac{\partial f}{\partial r} &= (\cos\theta)\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial \theta} &= (-r\sin\theta)\frac{\partial f}{\partial x} + (r\cos\theta)\frac{\partial f}{\partial y} \end{aligned}$$

$$\begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}$. Then

$$\det A = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = \cos\theta \cdot (r\cos\theta) - \sin\theta(-r\sin\theta) = r(\cos^2\theta + \sin^2\theta) = r$$

and therefore

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \theta} \end{bmatrix} = \frac{1}{r} \begin{bmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \frac{(-\sin\theta)}{r} \\ \sin\theta & \frac{(\cos\theta)}{r} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \theta} \end{bmatrix}$$

$$\Longrightarrow \quad \frac{\partial f}{\partial x} = (\cos\theta)\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}$$

$$\frac{\partial f}{\partial y} = (\sin\theta)\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}$$

Chain Rule and Implicit Differentiation

Implicit Function Theorem

- Let $F: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field let $a \in \text{int}(A)$ and let $\rho \in (0, +\infty)$ be given. Assume that:

a) $F(a) = 0$

b) $D_1F, D_2F, \dots, D_nF$ continuous at $B(a, \rho)$

c) $\forall x \in B(a, \rho)$: $D_nF(x) \neq 0$

Then there is an $f: B \to \mathbb{R}$ with $B \subseteq \mathbb{R}^{n-1}$ such that

$\forall (x_1, x_2, \dots, x_n) \in B(a, \rho)$: $F(x_1, x_2, \dots, x_{n-1}, f(x_1, x_2, \dots, x_{n-1}))$

- It follows that given the condition of the implicit function theorem, an equation $F(x_1, x_2, \dots, x_{n-1}, x_n) = 0$
Implicitly defines a new function f with $x_n = f(x_1, x_2, \dots, x_{n-1})$

Application: The case $F(x, y) = 0$

Consider the function $y = f(x)$ defined implicitly by the equation $F(x, y) = 0$. It follows that $F(x, f(x)) = 0$

We use the chain rule to differentiate with respect x.

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \implies \frac{\partial F}{\partial y}\frac{dy}{dx} = -\frac{\partial f}{\partial x}$$

$$\implies \frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y}$$

Application: The general case

Consider the function $x_n = f(x_1, x_2, \dots, x_{n-1})$
defined implicitly by the equation $F(x_1, x_2, \dots, x_{n-1}, x_n) = \emptyset$

Then, we can show that

$$\frac{\partial x_n}{\partial x_a} = \frac{-\partial F/\partial x_a}{\partial F/\partial x_n}, \quad \text{with } a \in [n-1]$$

Proof

We note that $\partial x_b/\partial x_a = \delta_{ab}$ with

$$\delta_{ab} = \begin{cases} 1 & \text{if } a=b \\ 0 & \text{if } a \neq b \end{cases}$$

Using the chain rule, we differentiate $F(x_1, x_2, \dots, x_{n-1}, x_n) = 0$

with respect to x_a: with $1 \leq a \leq n-1$

$$\sum_{b=1}^{n-1} \frac{\partial F}{\partial x_a} \frac{\partial x_b}{\partial x_a} + \frac{\partial F}{\partial x_n}\frac{\partial x_n}{\partial x_a} = 0 \Rightarrow \frac{\partial F}{\partial x_n}\frac{\partial x_n}{\partial x_a} = -\sum_{b=1}^{n} \frac{\partial F_n}{\partial x_b}\frac{\partial x_b}{\partial x_a}$$

$$= -\sum_{b=1}^{n} \frac{\partial F_n}{\partial x_b} \delta_{ab} = \frac{-\partial F}{\partial x_a} \Rightarrow$$

$$\Rightarrow \frac{\partial x_n}{\partial x_a} = \frac{-\partial F/\partial x_a}{\partial F/\partial x_n}$$

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Example

If x, y, z are constrained via

$$x^3 + g^3 + z^3 + 6xgz = 1$$

evaluate $\partial z/\partial x$ and $\partial z/\partial y$ in terms of x, y, z

Solution

Define $F(x,y,z) = x^3 + g^3 + z^3 + 6xyz - 1$

We note that

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(x^3 + y^3 + z^3 + 6xyz - 1) = 3x^2 + 6yz$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(x^3 + y^3 + z^3 + 6xyz - 1) = 3y^2 + 6xz$$

$$\frac{\partial F}{\partial z} = \frac{\partial}{\partial z}(x^3 + y^3 + z^3 + 6xyz - 1) = 3z^2 + 6xy$$

and therefore

$$\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z} = \frac{-(3x^2 + 6gz)}{3z^2 + 6xg} = \frac{-(x^2 + 2yz)}{z^2 + 2xg}$$

$$\frac{\partial z}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z} = \frac{-(3y^2 + 6xz)}{3z^2 + 6xy} = \frac{-(y^2 + 2xz)}{z^2 + 2xy}$$

Level sets and tangent lines / planes

- Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a differentiable scalar field. we define the level set of at $c \in \mathbb{R}$ as $L(c) = \{x \in A \mid f(x) = c\}$

Level curves and tangent lines

- For $n=2$ (i.e $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$). the level sets $L(c)$ are curves. Note that

a) For every $(x,y) \in A$, there is a unique level curve passing through (x,y)

b) Two distinct level curves never intersect.

Symbolically

$$\forall (x,y) \in A : \exists!\, c \in \mathbb{R} : (x,y) \in L(c)$$

$$c_1 \neq c_2 \Rightarrow L(c_1) \cap L(c_2) = \emptyset$$

- Let $l(x,y)$ be the tangent line to the unique level curve passing through $(x,y) \in A$. Then.

$$\nabla f(x,y) \perp l(x,y)$$

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Proof

Let $a: I \to \mathbb{R}^2$ be a vector field defining a level curve of the scalar field f such that

$\forall t \in I: f(a(t)) = c$

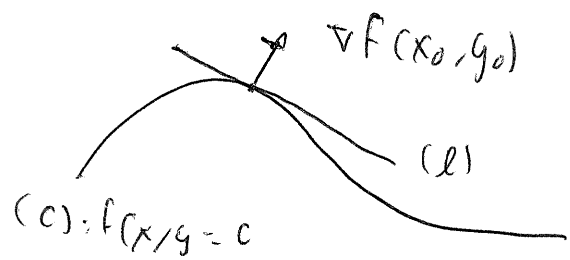
Then $\dot{a}(t)$ is the direction vector of the tangent line $l(t)$ at the point $a(t)$. We note that:

$\forall t \in I: \nabla f(a(t)) \perp \dot{a}(t) = (d/dt) f(a(t)) - (d/dt) c = 0$

$\Rightarrow \forall t \in I. \nabla f(a(t)) \perp \dot{a}(t) \Rightarrow \forall t \in I \; \nabla f(a(t)) \perp l(t)$

* Consider the curve $(c): f(x,y) = c$ be a level curve of the scalar field f. Let $(x_0, y_0) \in (c)$ be a point on the curve (c) and let (l) be the tangent line to (c) at the point (x_0, y_0). Then (l) is given by:

$(l): \nabla f(x_0, y_0) \cdot [(x, y) - (x_0, y_0)] = 0$



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EXAMPLE

Consider the circle (c) $x^2+y^2=a^2$

Let $(x_0,y_0)\in(c)$. Show that the tangent line (ℓ) to (c) at (x_0,y_0) is

$$(\ell): x_0x+y_0y=a^2$$

<u>Solution</u>

Define $f(x,y)=x^2+y^2 \Rightarrow \begin{cases} f_x(x,y)=2x \\ f_y(x,y)=2y \end{cases} \Rightarrow \nabla f(x,y)=(2x,2y)$

Note that since $(x_0,y_0)\in(c) \Rightarrow x_0^2+y_0^2=a^2$ (1)

It follows that

(ℓ): $\nabla f(x_0,y_0)\cdot[(x,y)-(x_0,y_0)]=0 \Leftrightarrow$

$\Leftrightarrow (2x_0,2y_0)\cdot(x-x_0,y-y_0)=0 \Leftrightarrow$

$\Leftrightarrow 2x_0(x-x_0)+2y_0(y-y_0)=0 \Leftrightarrow$

$\Leftrightarrow x_0(x-x_0)+y_0(y-y_0)=0 \Leftrightarrow$

$\Leftrightarrow x_0x-x_0^2+y_0y-y_0^2=0 \Leftrightarrow$

$\Leftrightarrow x_0x+y_0y=x_0^2+y_0^2 \Leftrightarrow$ [use (1)]

$\Leftrightarrow x_0x+y_0y=a^2$

and therefore

(ℓ) $x_0x+y_0y=a^2$

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Level surfaces and tangent planes

- For $n=2$, a level set $L(c)$ represents a <u>surface</u>.

The following properties still hold:

a) $\forall (x,y,z) \in A : \exists!\ c \in \mathbb{R} : (x,y,z) \in L(c)$

b) $c_1 \neq c_2 \Rightarrow L(c_1) \cap L(c_2) = \emptyset$

- Let (s) be a level surface given by:

$$(s): f(x,y,z) = c$$

Let $p(x_0,y_0,z_0) \in (s)$ be a point on (s) and consider a curve $r(t)$ going through the point p such that $\forall t \in I : f(r(t)) = c$ (i.e the curve is constrained on the surface (s)). Then $\nabla f(p)$ is perpendicular to all such curves

<u>Tangent plane to a Surface</u>

Consider the surface (s) given by

$(s): f(x,y,z) = c$

and choose a point $p = (x_0, y_0, z_0) \in (s)$. The tangent plane (t) to the surface (s) at the point p is the plane passing through the point p with normal vector $\nabla f(p)$ and it is given by.

$$(t): \nabla f(x_0,y_0,z_0) \cdot [(x,y,z) - (x_0,y_0,z_0)] = 0$$

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Normal line to surface

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Consider the surface $(S): f(x,y,z)=c$ and let $P=(x_0,y_0,z_0)\in(S)$ be a point on (S). The normal line at P is the line passing through the point P with direction vector $\nabla f(P)$.

- It follows that

$$(\ell): (x,y,z)=(x_0,y_0,z_0)+t\nabla f(x_0,y_0,z_0).\ \forall t\in\mathbb{R}$$

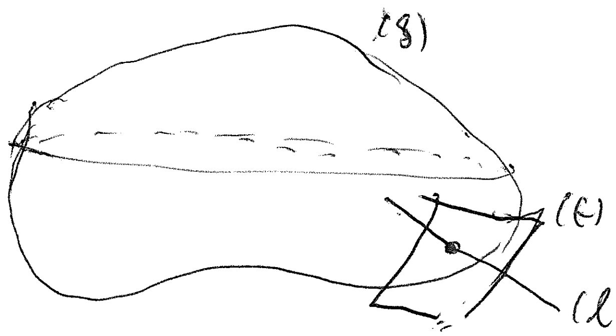
- Equivalently in terms of the symmetric equations the line (ℓ) is given by:

$$(\ell): \frac{x-x_0}{f_x(x_0,y_0,z_0)}=\frac{y-y_0}{f_y(x_0,y_0,z_0)}=\frac{z-z_0}{f_z(x_0,y_0,z_0)}$$

- It is easy to show that if

a) (ℓ) is the normal line at point P

b) (t) is the tangent plane at point P then $(\ell)\perp(t)$



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Maximum - Minimum values

- Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field. Let $p \in A$ be a point in A. We say that

a) p maximum of $f \iff \forall x \in A: f(x) \leq f(p)$

b) p minimum of $f \iff \forall x \in A: f(x) \geq f(p)$

c) p local maximum of $f \iff \exists r > 0: \forall x \in B(p,r) \cap A: f(x) \leq f(p)$

d) p local minimum of $f \iff \exists r > 0: \forall x \in B(p,r) \cap A: f(x) \geq f(p)$

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Generalized Fermat Theorem

- Let $A \subseteq \mathbb{R}^n$ and $f: A \to \mathbb{R}$ and choose a point $p \in A$
Then if:

$$\left.\begin{array}{l} A \text{ open set} \\ f \text{ differentiable at } p \\ p \text{ local extremum of } f \end{array}\right\} \Rightarrow \nabla f(p) = 0$$

- It follows from the generalized Fermat theorem that local extrema $p \in A$ can occur only at points where at least one of the following conditions is satisfied.

a) $\nabla f(p) = 0$

b) f not differentiable at p

c) $p \in \partial A$ (p is on the boundary of A)

- If $p \in A$ satisfies one of these conditions, we say that p is a stationary point (or critical point) of f.
- To find the local min/max of a function, we

a) Find all stationary points

b) Use classification theorems to see if the point is a min or max.

Another restatement

$$\nabla f(p) \neq 0 \Rightarrow (A \text{ not open} \vee f \text{ not differentiable at } p$$
$$\vee\ p \text{ not local extremum})$$

day 11

Failure of 1st derivative test

We will now show that the 1st derivative test cannot be generalized to functions with 2 or more variables

This is bad news, since the 1st derivative test is the best method for functions of 1 variable

Conjecture: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ and let $p \in \text{int } A$ with $\nabla f(p) = 0$. Then:

a) p local max $\iff \forall u \in \mathbb{R}^n : \exists a > 0 : g(t) = f(p + tu) \searrow (0, a)$

b) p local min $\iff \forall u \in \mathbb{R}^n : \exists a > 0 \quad g(t) = f(p + tu) \nearrow (0, a)$

Intuitively we expect his conjecture to be true.

- we now exhibit a counter example

CounterExample

Let $f(x, y) = 3x^4 - 4x^2y + y^2$ on $A = \mathbb{R}^2$

- $(0,0)$ is a stationary point

$$\frac{\partial f}{\partial x} = 12x^3 - 8xy$$

$$\frac{\partial f}{\partial y} = -4x^2 + 2y$$

Thus, for $(x, y) = (0, 0) \Rightarrow \left(\frac{\partial f}{\partial x}\right)(0,0) = 0 \wedge \left(\frac{\partial f}{\partial y}\right)(0,0) = 0$

$\Rightarrow \nabla f(0,0) = 0 \Rightarrow (0,0)$ is a stationary point

- claim: Conditions of conjecture are satisfied.

Let $u=(a,b)\in\mathbb{R}^2$ with $a\neq 0$ and $b\neq 0$ be given

Define

$$g(t) = f((0,0)+tu) = f(ta,tb)$$
$$= 3(at)^4 - 4(at)^2(bt) + (bt)^2$$
$$= 3a^4t^4 - 4a^2bt^3 + b^2t^2 \implies$$

$$\implies g'(t) = 12a^4t^3 - 12a^2bt^2 + 2b^2t = t(12a^4t^2 - 12a^2bt + 2b^2)$$
$$= t\phi(t\,|\,a,b)$$

with $\phi(t|a,b) = 12a^4t^2 - 12a^2bt + 2b^2$

Since $\phi(0|a,b) = 2b^2 > 0$, there is at least some interval $(0,t_0)$ such that $\forall t\in(0,t_0): \phi(t|a,b) > 0$ and therefore.

$\forall t\in(0,t_0): g'(t) = t\phi(t|a,b) > 0 \implies g \nearrow (0,t_0)$

$\implies$ the conjecture predicts that $(0,0)$ is a local minimum

- Claim: $(0,0)$ is Not a local minimum

First we note that $f(0,0)=0$.

Now consider the values of f along the curve $(c): y=2x^2$ we see that

$f(x,2x^2) = 3x^4 - 4x^2(2x^2) + (2x^2)^2 = 3x^4 - 8x^4 + 4x^4 = -x^4 < 0,\ \forall x\in(0,+\infty)$

It follows that

$\forall r>0: \exists x\in B((0,0),r): f(x) < 0 = f(0) \implies 0$ is Not a local min of f!!

We conclude that the stated conjecture (generalized 1st derivative test) is false. More precisely it is:

true in $\Rightarrow$ direction

not always true in $\Leftarrow$ direction

Second derivative test

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ and let $p \in \text{int}(A)$ with $\nabla f(p) = 0$. Define

$$D = f_{xx}(p) f_{yy}(p) - [f_{xy}(p)]^2$$

Then

a) $D > 0 \wedge f_{xx}(p) > 0 \Rightarrow p$ local minimum

b) $D > 0 \wedge f_{xx}(p) < 0 \Rightarrow p$ local maximum

c) $D < 0 \Rightarrow p$ saddle point

d) $D = 0 \Rightarrow$ inconclusive

Examples

a) Clasify all stationary points of $f(x,y) = x^3 + y^3 - 3xy$

Solution

a) clasify all stationary points

Solution

$f_x(x,y) = \left(\frac{\partial}{\partial x}\right)(x^3 + y^3 - 3xy) = 3x^2 - 3y$

$f_y(x,y) = \left(\frac{\partial}{\partial y}\right)(x^3 + y^3 - 3xy) = 3y^2 - 3x$

(x,y) stationary point $\Leftrightarrow \nabla f(x,y) = 0 \Leftrightarrow \begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 - y = 0 \\ y^2 - x = 0 \end{cases} \Leftrightarrow \begin{cases} y = x^2 \\ x^4 - x = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = x^2 \\ x(x-1)(x^2+x+1) = 0 \end{cases} \Leftrightarrow \begin{cases} y = x^2 \\ x = 0 \end{cases} \vee \begin{cases} y = x^2 \\ \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = 1 \\ y = 1 \end{cases}$$

Thus $(0,0)$ and $(1,1)$ are stationary points

Evaluate:

$f_{xx}(x,y) = \left(\frac{\partial}{\partial x}\right)(3x^2 - 3y) = 6x$

$f_{xy}(x,y) = \left(\frac{\partial}{\partial x}\right)(3y^2 - 3x) = -3$

$f_{yy}(x,y) = \left(\frac{\partial}{\partial y}\right)(3y^2 - 3x) = 6y$

$D = f_{xx}(x,y) f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9 = 9(4xy - 1)$

For $(x,y) = (0,0)$: $f_{xx}(0,0) = 0$

$D(0,0) = 9(4 \cdot 0 \cdot 1 - 1) = -9 < 0 \Rightarrow (0,0)$ is a saddle point

For $(x,y) = (1,1)$

$$\left.\begin{array}{l} D(1,1) = 9(4 \cdot 1 \cdot 1 - 1) = 9 \cdot 3 = 27 > 0 \\ f_{xx}(1,1) = 6 \cdot 1 = 6 > 0 \end{array}\right\} \Rightarrow (1,1) \text{ is a local minimum}$$

If $D = 0$ then the 2nd derivative test is inconclusive in that case, if the stationary point is a saddle point, we could be able to identify it as such using the following 1st derivative test. Note however that for local min or max, the 1st derivative test does not work as we have show previously dag 11

1st derivative test for saddle points

Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ and let $p \in \operatorname{int}(A)$ with $\nabla f(p) = 0$. Then

$$\left[\exists a, b \in \mathbb{R}^n, \exists t_1, t_2 \in (0, +\infty): \begin{cases} g_1(t) = f(p + at) \nearrow (0, t_1) \\ g_2(t) = f(p + bt) \searrow (0, t_2) \end{cases}\right] \Rightarrow$$

$\Rightarrow p$ saddle point of f.

EXAMPLE

Show that $f(x,y) = x^4 y^7$ has a saddle point at $(0,0)$

Solution

$$\left.\begin{aligned} f_x(x,y) &= \left(\tfrac{\partial}{\partial x}\right)(x^4 y^7) = 4x^3 y^7 \\ f_y(x,y) &= \left(\tfrac{\partial}{\partial y}\right)(x^4 y^7) = 7x^4 y^6 \end{aligned}\right\} \Rightarrow \nabla f(0,0) = (f_x(0,0), f_y(0,0))$$

$$= (4 \cdot 0 \cdot 0, 7 \cdot 0 \cdot 0) = (0,0) = 0 \Rightarrow$$

$\Rightarrow (0,0)$ is a stationary point

Let $a, b \in \mathbb{R} - \{0\}$ be given. Define: $g(t) = f((0,0) + t(a,b))$:

$f(at, bt) = (at)^4 (bt)^7 = a^4 b^7 t^{11}, \forall t \in (0, +\infty) \Rightarrow$

$\Rightarrow g'(t) = 11 a^4 b^7 t^{10}, \forall t \in (0, +\infty)$

For $a = b = 1$: $g'(t) = 11 t^{10}, \forall t \in (0, +\infty) \Rightarrow g'(t) > 0, \forall t \in (0, +\infty) \Rightarrow g \nearrow (0, +\infty)$

For $a = 1$ and $b = -1$: $g'(t) = 11 \cdot 1^4 \cdot (-1)^7 t^{10} = -11 t^{10}, \forall t \in (0, +\infty)$

$\Rightarrow g'(t) < 0, \forall t \in (0, +\infty) \Rightarrow g \searrow (0, +\infty)$

It follows that $(0,0)$ is a saddle point

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To see why the second derivative test fails, we note that

$f_{xx}(x,y) = \left(\frac{\partial}{\partial x}\right)(4x^3y^7) = 12x^2y^7$

$f_{xy}(x,y) = \left(\frac{\partial}{\partial x}\right)(7x^4y^6) = 28x^3y^6$

$f_{yy}(x,y) = \left(\frac{\partial}{\partial y}\right)(7x^4y^6) = 42x^4y^5$

and therefore

$$D(x,y) = f_{xx}(x,y)\,f_{yy}(x,y) - [f_{xy}(x,y)]^2$$
$$= (12x^2y^7)(42x^4y^5) - (28x^3y^6)^2$$
$$= 504x^6y^{12} - 784x^6y^{12} = -280x^6y^{12}$$
$$\Rightarrow D(0,0) = -280 \cdot 0^6 \cdot 0^{12} = 0$$

$\Rightarrow$ 2nd derivative test inconclusive with respect to $(0,0)$

The Lagrange multipliers method gives the stationary points of the constrained optimization problem but does not establish whether these points are local min or max. Doing so is very difficult, but there are occasional tricks we can try

When constraints yield a finite curve

- Recall from single variable calculus that for any function $f: A \to \mathbb{R}$ with $A \subset \mathbb{R}$ and $a, b \in A$: f continuous in $[a,b] \Rightarrow f$ bounded in $[a,b]$.
- Let S be the set defined by the problem constraints

$$S = \{x \in \mathbb{R}^n \mid \forall k \in [m]: g_k(x) = 0\}$$

Def: we say that S is a finite curve if we can define a parametrization $\gamma: [0,1] \to S$ such that $\gamma([0,1]) = S$ and γ continuous is $[0,1]$

Now we can parameterize the function f on S by defining $g(t) = f(\gamma(t))$. Then, it follows that.

$$\left.\begin{array}{l} f \text{ continuous in } A \Rightarrow f \text{ continuous in } S \\ \gamma \text{ continuous in } [0,1] \end{array}\right\} \Rightarrow$$

$\Rightarrow g$ continuous in $[0,1] \Rightarrow$

$\Rightarrow \exists t_1, t_2 \in [0,1]: \forall t \in [0,1]: g(t_1) \leq g(t) \leq g(t_2)$

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The points $\gamma(t_1)$ $\gamma(t_2)$ will show up as stationary points of the lagrange multiplier method solutions and are correspondingly the minimum and maximum of f under the given constraints.

- <u>To conclude</u>: If S is a finite curve, then the constrained optimization has a maximum and a minimum among the existing stationary points. To identify the minimum and maximum, we simply evaluate the function for all stationary points. and choose the stationary points that give the minimum and maximum values

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Example #1

find the min and max values for

$f(x,y,z) = x+2y+3z$ ① with constraint

$\begin{cases} x^2+y^2=2 & ② \\ y+z=1 & ③ \end{cases}$

Solution:

define $g_1(x,y,z) = 2-x^2-y^2$

$g_2(x,y,z) = y+z-1$

Note that

$\nabla f(x,y,z) = (1,2,3)$

$\nabla g_1(x,y,z) = (-2x,-2y,0)$

$\nabla g_2(x,y,z) = (0,1,1)$

for linear independence.

$\nabla g_1 \times \nabla g_2 = (-2x,-2y,0) \times (0,1,1)$

$= \begin{vmatrix} i & j & k \\ -2x & -2y & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} i & j & -j+k \\ -2x & -2y & 2y \\ 0 & 1 & 0 \end{vmatrix}$

$= 1 \cdot (-1)^{3+2} \cdot \begin{vmatrix} i & -j+k \\ -2x & 2y \end{vmatrix}$

$= -1\left(i \cdot 2y - (-j+k)(-2x)\right)$

$= -(2y, -2x, +2x)$

$= (-2y, +2x, -2x)$

$\therefore \nabla g_1, \nabla g_2$ are linearly independent iff. $\nabla g_1 \times g_2 \neq \vec{0}$

$\therefore x \neq 0$ or $y \neq 0$ ($x \neq 0 \lor y \neq 0$).

$\therefore$ this linear indepence fails at $(x,y,z) = (0,0,z)$.

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by (4) $z = 1 - y$, and since $x = -\frac{1}{2\lambda_1}$, $y = \frac{1}{2\lambda_1}$

$$\therefore \begin{cases} x = \frac{-1}{2\lambda_1} \\ y = \frac{1}{2\lambda_1} \\ \lambda_1 = \frac{1}{2} \vee \lambda_1 = -\frac{1}{2} \\ z = 1 - y \end{cases}$$

$$\therefore \begin{cases} x = -1 \\ y = 1 \\ z = 0 \end{cases} \vee \begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases}$$

$\therefore (x, y, z) = (-1, 1, 0) \vee (x, y, z) = (1, -1, 2)$

Note that both point satisfy the linear independence condition

considering $f(x,y,z) = x + 2y + 3z$, then.

$f(-1, 1, 0) = -1 + 2 = 1$.

$f(1, -1, 2) = 1 - 2 + 6 = 5$

$\therefore (-1, 1, 0)$ is the minimum and

$(1, -1, 2)$ is the maximum

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Example #2. find the stationary point.

$f(x,y,z) = xyz$, with $x,y,z \in (0,+\infty)$.

under constraint

$xy + yz + zx = 1$

$\therefore$ define $g(x,y,z) = xy + yz + xz - 1$.

$\therefore$ $\nabla f = (yz, xz, xy)$. $\because x,y,z \in (0,+\infty)$

$\nabla g = (y+z, x+z, y+x)$. $\therefore \nabla g \neq \vec{0}$ $\therefore \nabla g$ is linearly independent.

$\therefore$ (x,y,z) is stationary iff.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 0 \end{cases} \quad \therefore \begin{cases} (yz, xz, xy) = \lambda (y+z, x+z, y+x) \\ xy + yz + zx - 1 = 0 \end{cases}$$

$$\therefore \begin{cases} yz = \lambda (y+z) \\ xz = \lambda (x+z) \\ xy = \lambda (x+y) \\ xy + yz + zx - 1 = 0 \end{cases} \quad \therefore \begin{cases} xyz = \lambda x (y+z) & (7) \\ xyz = \lambda y (x+z) & (8) \\ xyz = \lambda z (x+y) & (9) \\ xy + yz + zx = 1 & (10). \end{cases}$$

by (7)(8),

$$\lambda x(y+z) - \lambda y(x+z) = 0$$

$$\therefore \lambda (xy + xz - yx - yz) = 0$$

$$\therefore \lambda y(x - z) = 0$$

$$\therefore \lambda = 0 \vee y = 0 \vee x = z$$

if $\lambda = 0$ then

$$\begin{cases} yz = 0 \\ xz = 0 \\ xy = 0 \\ xy + yz + zx = 1. \end{cases} \quad \therefore 0 + 0 + 0 = 1 \text{ inconsistent} \quad \therefore \lambda \neq 0$$

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optimization problem on a bounded set

Def $\forall A \subset \mathbb{R}^n$, A is said to be bounded iff.
$\exists p \in \mathbb{R}^n$, $\exists r \in (0, +\infty)$ s.t. $A \subset B(p, r)$

Thm Let $f: A \to \mathbb{R}$ with $A \subset \mathbb{R}^n$. Assume that
a) f is continuous on A
b) A is closed ($\partial A \subset A$)
c) A is bounded

then $\exists P_1, P_2 \in A$, such that $\forall x \in A$, $f(P_1) \le f(x) \le f(P_2)$

Example. find the min and max of
$f(x,y) = 2 + x^2 + y^2$.
under the constraint
$x^2 + \frac{y^2}{4} \le 1$

solution. Define $S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2/4 \le 1\}$

(interior points)

$\nabla f(x,y) = (2x, 2y)$.
let $\nabla f = 0$, $\therefore (x,y) = (0,0)$

boundary
let $g(x,y) = x^2 + \frac{y^2}{4} - 1$ $\quad \because x^2 + \frac{y^2}{4} = 1 \quad \therefore \nabla g \ne \vec{0}$
$\therefore \nabla g = (2x, \frac{y}{2})$ $\quad \therefore \nabla g$ is linearly independent

$\therefore \begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases} \quad \therefore \begin{cases} 2x = \lambda 2x & (1) \\ 2y = \lambda \cdot \frac{y}{2} & (2) \\ x^2 + \frac{y^2}{4} - 1 = 0 & (3) \end{cases}$

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2nd methord

$\because \ x^2+\frac{y^2}{4}\le 1$

$\therefore$ let $\begin{cases} x(t)=\cos t \\ y(t)=2\sin t \end{cases}$ $t\in[0,2\pi)$

$\therefore$ it follows that

$g(t)=f(\cos t, 2\sin t) = 2+\cos^2 t+4\sin^2 t$

$= 3+3\sin^2 t$

$g'(t)=(3+3\sin^2 t)'=(3\sin^2 t)' = 3\cdot 2\sin t\cdot(\sin t)' = 6\sin t\cos t$.

$= 3\sin(2t)$

let $g'(t)=0$ $\quad\therefore \sin(2t)=0$

$\therefore 2t=n\pi$, $\quad\therefore t=\frac{n\pi}{2}$

$\because t\in[0,2\pi)$ $\quad\therefore t=0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

$\therefore$ for $t=0$ $\quad(x,y)=(\cos t, 2\sin t)=(1,0)$.

$t=\frac{\pi}{2}$ $\quad(x,y)= \dots\dots = (0,2)$.

$t=\pi$ $\quad(x,y)= \dots\dots = (-1,0)$

$t=\frac{3\pi}{2}$ $\quad(x,y)= \dots\dots = (0,-2)$.