day 3

16a) Given $u, v, w \in \mathbb{R}^3$

Show that $u \perp v \wedge u \perp w \Rightarrow [\forall \alpha, b \in \mathbb{R} : u \perp (\alpha v + bw)]$

Solution

Assume that $u \perp v$ and $u \perp w$

Then: $\begin{cases} u \perp v \\ u \perp w \end{cases} \Rightarrow \begin{cases} u \cdot v = 0 \\ u \cdot w = 0 \end{cases}$

Let $\alpha, b \in \mathbb{R}$ be given, then

$$\begin{aligned} u \cdot (\alpha v + bw) &= u \cdot (\alpha v) + u \cdot (bw) \\ &= \alpha (u \cdot v) + b (u \cdot w) \\ &= \alpha \cdot \emptyset + b \cdot \emptyset \\ &= \emptyset \end{aligned}$$

Hence $u \perp (\alpha v + bw)$. It follows that $\forall \alpha, b \in \mathbb{R} : u \perp (\alpha v + bw)$.

18a) Show that

$\text{Proj}_u(v+w) = \text{Proj}_u(v) + \text{Proj}_u(w)$

Solution

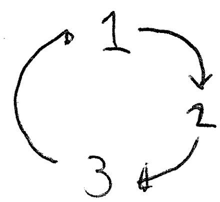
$$\begin{aligned} \text{Proj}_u(v+w) &= \left(\frac{u \cdot (v+w)}{\|u\|^2}\right) u \\ &= \frac{u \cdot v + u \cdot w}{\|u\|^2} u \\ &= \left[\frac{u \cdot v}{\|u\|^2} + \frac{u \cdot w}{\|u\|^2}\right] u \\ &= \frac{u \cdot v}{\|u\|^2} u + \frac{u \cdot w}{\|u\|^2} u \\ &= \text{Proj}_u(v) + \text{Proj}_u(w) \end{aligned}$$

④ Cross Product

Let $u=(u_1, u_2, u_3)$ and $v=(v_1, v_2, v_3)$

The cross product $w=u\times v$ is defined as

$$w=u\times v=(w_1, w_2, w_3) \Longleftrightarrow \begin{cases} w_1 = u_2v_3 - u_3v_2 \\ w_2 = u_3v_1 - u_1v_3 \\ w_3 = u_1v_2 - u_2v_1 \end{cases}$$



To calculate the cross product, given the unit vectors

$e_1=(1,0,0) \quad e_2=(0,1,0) \quad e_3=(0,0,1)$

$$u\times v = \left|\begin{array}{ccc|cc} e_1 & e_2 & e_3 & e_1 & e_2 \\ u_1 & u_2 & u_3 & u_1 & u_2 \\ v_1 & v_2 & v_3 & v_1 & v_2 \end{array}\right.$$

$$= u_2v_3e_1 + u_3v_1e_2 + u_1v_2e_3 - u_2v_1e_3 - u_3v_2e_1 - u_1v_3e_2$$

EXAMPLE

$u=(1,3,4)$

$v=(2,1,1)$

$u\times v = (1,3,4)\times(2,1,1)$

$$= \left|\begin{array}{ccc|cc} e_1 & e_2 & e_3 & e_1 & e_2 \\ 1 & 3 & 4 & 1 & 3 \\ 2 & 1 & 1 & 2 & 1 \end{array}\right.$$

$$= 3\cdot 1e_1 + 4\cdot 2e_2 + 1\cdot 1e_3 - 3\cdot 2e_3 - 4\cdot 1e_1 - 1\cdot 1e_2$$

$$= 3e_1 + 8e_2 + 1e_3 - 6e_3 - 4e_1 - 1e_2$$

$$= (3-4)e_1 + (8-1)e_2 + (1-6)e_3$$

day 3

EXAMPLE Cont. from last page

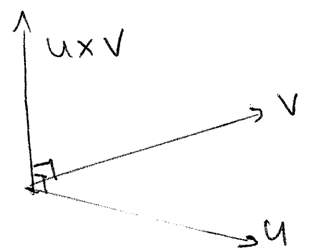
$= (3-4)e_1 + (8-1)e_2 + (1-6)e_3$

$= -e_1 + 7e_2 - 5e_3$

$= (-1, 7, -5)$

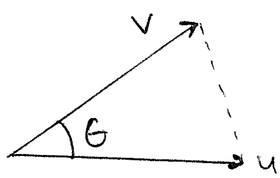
Geometric properties of cross product

1) $\forall u, v \in \mathbb{R}^3 : u \perp u \times v \wedge v \perp u \times v$



2) Let θ be the interior angle between $u, v \in \mathbb{R}^3$

$\|u \times v\| = \|u\| \|v\| \sin\theta$



The proof uses the lagrange identity

$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$

$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2$

Proof

$|u \times v|^2 = |(u_2 v_3 - u_3 v_2), (u_3 v_1 - u_1 v_3), (u_1 v_2 - u_2 v_1)|^2$

$= (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2$

$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2$

continuation from last page

$= \|u\|^2\|v\|^2 - (u\cdot v)^2$

$= \|u\|^2\|v\|^2 - \|u\|^2\|v\|^2\cos^2\theta$

$= \|u\|^2\|v\|^2(1-\cos^2\theta)$

$= \|u\|^2\|v\|^2\sin^2\theta \implies \theta \in [0,\pi] \implies \sin\theta$

$\implies \|u\times v\| = \|u\|\,\|v\|\sin\theta$

3) ABCD parallelogram $\implies (ABCD) = \|\vec{AB}\times\vec{AD}\|$

Proof in web notes

Algebraic Properties of cross product

1) $\forall u \in \mathbb{R}^3$: $u\times u = 0$

2) $\forall u, v \in \mathbb{R}^3$: $u\times v = -v\times u$

3) $\forall u, v \in \mathbb{R}^3$: $\forall c \in \mathbb{R}$: $(cu)\times v = u\times(cv) = c(u\times v)$

4) $\forall u, v, w \in \mathbb{R}^3$: $u\times(v+w) = u\times v + u\times w$

: $(v+w)\times u = v\times u + w\times u$

5) $\forall u, v, w \in \mathbb{R}^3$: $u\cdot(v\times w) = v\cdot(w\times u) = w\cdot(u\times v)$ ← Volume of Parallelepiped defined by u, v, w

6) $\forall u, v, w \in \mathbb{R}^3$: $u\times(v\times w) = (u\cdot w)v - (u\cdot v)w$

Parallel Vectors

Def $\vec{u} \parallel \vec{v} \Longleftrightarrow \exists c \in \mathbb{R} : \vec{u} = c\vec{v}$

Thm $\vec{u} \parallel \vec{v} \Longleftrightarrow u \times v = 0$

Proof

$\Rightarrow$: Assume $\vec{u} \parallel \vec{v} \Rightarrow \exists c \in \mathbb{R} \Rightarrow u \times v = u \times (cu) = c(u \times u) = c \cdot \emptyset = \emptyset$

$\Leftarrow$: Assum that $u \times v = \emptyset$ with $u \neq \emptyset$ and $v \neq \emptyset$ then

$$\begin{cases} u_1 v_2 - u_2 v_1 = 0 \\ u_2 v_3 - u_3 v_2 = 0 \\ u_3 v_1 - u_1 v_3 = 0 \end{cases} \Rightarrow \begin{cases} u_1 v_2 = u_2 v_1 \\ u_2 v_3 = u_3 v_2 \\ u_3 v_1 = u_1 v_3 \end{cases} \Rightarrow \frac{u_1}{v_1} = \frac{u_2}{v_2} = \frac{u_3}{v_3} = c$$

$$\Rightarrow \begin{cases} u_1 = c v_1 \\ u_2 = c v_2 \\ u_3 = c v_3 \end{cases} \Rightarrow u = cv \Rightarrow \vec{u} \parallel \vec{v}$$

Scalar Triple product

$$u \cdot (v \times w) = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

Note that $u \cdot (v \times w) = v \cdot (w \times u) = w \cdot (u \times v)$

Application $\begin{cases} u \perp w \\ v \perp w \end{cases} \Rightarrow w \parallel u \times v \quad \forall u, v, w \in \mathbb{R}^3$

(24a) Let $u, v, w \in \mathbb{R}^3$

Recall: $u \parallel v \Leftrightarrow \exists c \in \mathbb{R} : u = cv$

$u \parallel v \wedge v \parallel w \to u \parallel w$

Solution

Assume that $u \parallel v$ and $v \parallel w$

Then $u \parallel v \Rightarrow \exists c_1 \in \mathbb{R} : u = c_1 v$

$v \parallel w \Rightarrow \exists c_2 \in \mathbb{R} : v = c_2 w$

It follows that

$$u = c_1 v = c_1 (c_2 w) = (c_1 c_2) w$$

$\Rightarrow \exists c \in \mathbb{R} : u = cw$ (for $c = c_1 c_2$)

$\Rightarrow u \parallel w$

(24d) $u \perp v \Rightarrow u \times (u \times v) = -\|u\|^2 v$

Solution

Assume that $u \perp v$. Then $u \cdot v = 0$

It follows that

$$\begin{aligned} u \times (u \times v) &= (u \cdot v) u - (u \cdot u) v \\ &= 0 \cdot u - \|u\|^2 v \\ &= 0 - \|u\|^2 v \\ &= -\|u\|^2 v \end{aligned}$$

(23) Let $u, v, w \in \mathbb{R}^3$ with

day 4

$\|u\| = 2$. $\|v\| = 3$ $\|w\| = 1$

$\varphi =$ (interior angle from u to v) $= \frac{\pi}{4}$

$\theta =$ (interior angle from w to $u \times v$) $= \frac{\pi}{3}$

Evaluate $\|u+v\|$ and $|u \cdot (v \times w)|$

Solution

$$\begin{aligned}
\|u+v\|^2 &= (u+v)\cdot(u+v) \\
&= (u+v)\cdot u + (u+v)\cdot v \\
&= u\cdot u + v\cdot u + u\cdot v + v\cdot v \\
&= \|u\|^2 + 2u\cdot v + \|v\|^2 \\
&= \|u\|^2 + 2\|u\|\|v\|\cos\varphi + \|v\|^2 \\
&= 2^2 + 2\cdot 2\cdot 3\cos(\tfrac{\pi}{4}) + 3^2 \\
&= 4 + 12\left(\tfrac{\sqrt{2}}{2}\right) + 9 \\
&= 13 + 6\sqrt{2}
\end{aligned}$$

$$\Longrightarrow \|u+v\| = \sqrt{13+6\sqrt{2}}$$

Recall that $u\cdot(v\times w) = v\cdot(w\times u) = w\cdot(u\times v)$

$$\begin{aligned}
|u\cdot(v\times w)| &= |v\cdot(w\times u)| \\
&= |w\cdot(u\times v)| \\
&= \|w\|\|u\times v\||\cos\theta| \\
&= \|w\|\left[\|u\|\|v\|\sin\varphi\right]|\cos\theta| \\
&= 1\left[2\cdot 3\sin(\tfrac{\pi}{4})\right]\cos(\tfrac{\pi}{3}) \\
&= 6\left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{\sqrt{3}}{2}\right. \\
&= \frac{6}{2\cdot 2}\sqrt{6} \\
&= \frac{3\sqrt{6}}{2}
\end{aligned}$$

dag 4

Lines in $\mathbb{R}^3$

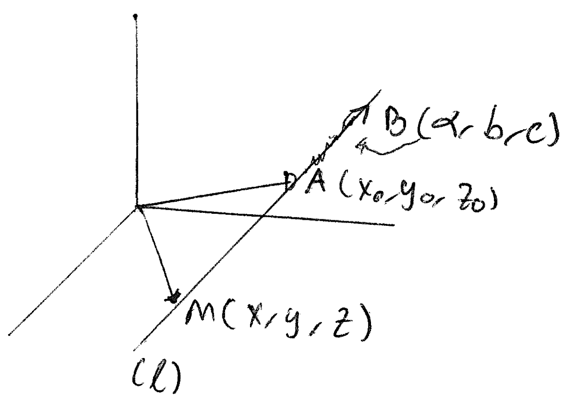
Let $\emptyset$ be the origin of our coordinate system

<u>Def</u>: The parametric equation for a line (ℓ) going through the points A, B is:

$$(\ell): \quad \vec{r} = \overrightarrow{\emptyset A} + t\overrightarrow{AB}, \quad \forall t \in \mathbb{R}$$

The above statement is equivalent to:

$$M \in (\ell) \iff \exists t \in \mathbb{R} : \overrightarrow{OM} = \overrightarrow{\emptyset A} + t\overrightarrow{AB}$$



- $\overrightarrow{AB}$ = direction Vector of (ℓ)
- For $\vec{r} = (x, y, z)$, $\overrightarrow{\emptyset A} = (x_0, y_0, z_0)$, and $\overrightarrow{AB} = (a, b, c)$ the parametric equation is equivalent to

$$(\ell) : \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}, \quad \forall t \in \mathbb{R}$$

- Eliminating t from the above equations gives the <u>Symetric equations</u> representation of the line (ℓ):

day 4

$$(\ell): \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

The symmetric equations can be reduced to a system of the form

$$(\ell): \begin{cases} A_1x + B_1y + C_1 = 0 \\ A_2y + B_2z + C_2 = 0 \end{cases}$$

which essentially defines the line (ℓ) as an intersection of two planes.

Relative position of two lines

Consider the lines

(ℓ_1): $\vec{r} = \vec{a_1} + t\vec{b_1}$, $\forall t \in \mathbb{R}$

(ℓ_2): $\vec{r} = \vec{a_2} + t\vec{b_2}$, $\forall t \in \mathbb{R}$

Then

1) $(\ell_1) \parallel (\ell_2) \Leftrightarrow \vec{b_1} \parallel \vec{b_2} \Leftrightarrow \vec{b_1} \times \vec{b_2} = 0$

day 4

EXAMPLE

a) write the symmetric equations for the line (AB) with $A(1,2,-1)$ and $B(5,4,1)$

Solution

$$\left.\begin{array}{l} A(1,2,-1) \\ B(5,4,1) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \overrightarrow{OA} = (1,2,-1) \\ \overrightarrow{AB} = (5-1,\ 4-2,\ 1-(-1)) = (4,2,2) \end{array}\right.$$

therefore

$$(l): (x,y,z) = \overrightarrow{OA} + t\overrightarrow{AB} = (1,2,-1) + t(4,2,2)$$
$$= (1+4t,\ 2+2t,\ -1+2t) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 1+4t \\ y = 2+2t \\ z = -1+2t \end{cases} \Leftrightarrow \frac{x-1}{4} = \frac{y-2}{2} = \frac{z+1}{2}$$

$$\Leftrightarrow \begin{cases} 2(x-1) = 4(y-2) \\ 2(y-2) = 2(z+1) \end{cases} \Leftrightarrow \begin{cases} 2x-2 = 4y-8 \\ 2y-4 = 2z+2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2x-4y+6=0 \\ 2y-2z-6=0 \end{cases} \Leftrightarrow \boxed{\begin{cases} x-4y+3=0 \\ y-z-3=0 \end{cases}}$$

day 4

c) Given the points $A(1,1,0)$, $B(3,1,5)$, $C(a+b, b+c, c+a)$ and $D(a+1, b+1, c+1)$, find all $a, b, c \in \mathbb{R}$ such that $(AB) \parallel (CD)$

Solution

we calculate $\vec{AB} \times \vec{CD}$

$$\left.\begin{array}{l} A(1,1,0) \\ B(3,1,5) \end{array}\right\} \Rightarrow \vec{AB} = (3-1, 1-1, 5-0) = (2, 0, 5) \qquad (1)$$

$$\left.\begin{array}{l} C(a+b, b+c, c+a) \\ D(a+1, b+1, c+1) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \vec{CD} = ((a+1)-(a+b), (b+1)-(b+c), (c+1)-(c+a))$$

$$= (a+1-a-b, b+1-b-c, c+1-c-a)$$

$$= (1-b, 1-c, 1-a). \qquad (2)$$

From (1) and (2):

$$\vec{AB} \times \vec{CD} = \left|\begin{array}{ccc} i & j & K \\ 2 & 0 & 5 \\ 1-b & 1-c & 1-a \end{array}\right| \begin{array}{cc} i & j \\ 2 & 0 \\ 1-b & 1-c \end{array}$$

Note: it is better to use e_1, e_2, e_3 notation.

$$= 0(1-a)i + 5(1-b)j + 2(1-c)K - 0(1-b)K - 5(1-c)i - 2(1-a)j$$

$$= -5(1-c)i + [5(1-b) - 2(1-a)]j + 2(1-c)K$$

$$= (-5+5c, 5-5b-2+2a, 2-2c)$$

$$= (-5+5c, 2a-5b+3, 2-2c$$

It follows that

From last Page

$\vec{AB} \times \vec{CD} = (-5+5c, 2a-5b+3, 2-2c)$

$(AB) \parallel (CD) \Leftrightarrow \vec{AB} \parallel \vec{CD} \Leftrightarrow \vec{AB} \times \vec{CD} = \vec{0} \Leftrightarrow$

$\Leftrightarrow (-5+5c, 2a-5b+3, 2-2c) = (0,0,0) \Leftrightarrow$

$\Leftrightarrow \begin{cases} -5+5c=0 \\ 2a-5b+3=0 \\ 2-2c=0 \end{cases} \Leftrightarrow \begin{cases} 5c=5 \\ 2a-5b=-3 \\ 2c=2 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} c=1 \\ 2a-5b=-3 \\ c=1 \end{cases} \Leftrightarrow \begin{cases} 2a-5b=-3 \\ c=1 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} a = \frac{5b-3}{2} \\ c=1 \end{cases} \Leftrightarrow$

$\Leftrightarrow (a,b,c) = \left(\frac{5b-3}{2}, b, 1\right)$

$= \left(-\frac{3}{2}, 0, 1\right) + \left(\frac{5}{2}b, b, 0\right)$

$= \left(-\frac{3}{2}, 0, 1\right) + b\left(\frac{5}{2}, 1, 0\right)$

$\Leftrightarrow (a,b,c) \in \left\{\left(-\frac{3}{2}, 0, 1\right) + b\left(\frac{5}{2}, 1, 0\right) \middle| b \in \mathbb{R}\right\}$

day 4

d) write the parametric equation for the line (ℓ) defined by

$$(\ell): \begin{cases} x - 3y + 2 = 0 \\ 2y - z + 5 = 0 \end{cases}$$

Solution

$$(\ell): \begin{cases} x - 3y + 2 = 0 \\ 2y - z + 5 = 0 \end{cases} \iff \begin{cases} x = 3y - 2 \\ z = 2y + 5 \end{cases} \iff$$

$$\iff (x, y, z) = (3y - 2, y, 2y + 5)$$
$$= (3y, y, 2y) + (-2, 0, 5)$$
$$= y(3, 1, 2) + (-2, 0, 5)$$

It follows that

$$(\ell): \; r = (-2, 0, 5) + t(3, 1, 2)$$

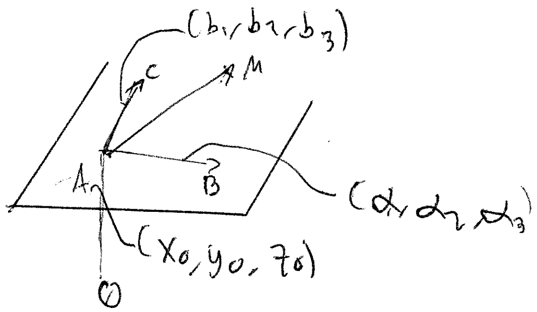
Note: Homework problem (27) remove it

Also problem (30)

Planes in $\mathbb{R}^3$ day 4

Let A, B, C be three non-collinear points (i.e. A, B, C are not on the same line). Then these three points define a unique plane with equation

$$(P): \vec{r} = \vec{OA} + t\vec{AB} + s\vec{AC} \quad \forall t, s \in \mathbb{R}$$



Equivalently if we let

$\vec{OA} = (x_0, y_0, z_0)$, $\vec{AB} = (d_1, d_2, d_3)$ $\vec{AC} = (b_1, b_2, b_3)$

then

$$(P): \begin{cases} x = x_0 + d_1 t + b_1 s \\ y = y_0 + d_2 t + b_2 s \\ z = z_0 + d_3 t + b_3 s \end{cases} \quad \forall t, s \in \mathbb{R}$$

Similarly the belonging condition for (P) is.

$$M(x, y, z) \in (P) \Leftrightarrow \exists t, s \in \mathbb{R}: \begin{cases} x = x_0 + d_1 t + b_1 s \\ y = y_0 + d_2 t + b_2 s \\ z = z_0 + d_3 t + b_3 s \end{cases}$$

or Equivalently in the notes

$$M \in (P) \Leftrightarrow \exists t, s \in \mathbb{R}: \vec{OM} = \vec{OA} + t\vec{AB} + s\vec{AC}$$

Eliminating t, s gives an equivalent equation of the form

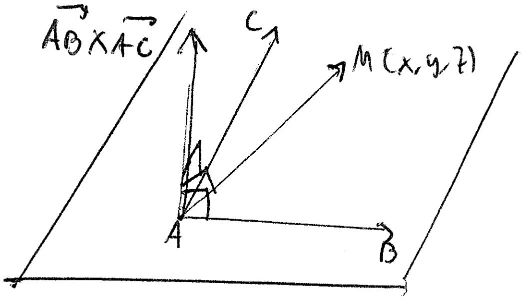
$(P): Ax + By + Cz + D = 0$

which is called the <u>scalar equation</u> of (P)

Scalar equation for plane from 3 points

day 4

Let A, B, C be three points with $\vec{AB} \times \vec{AC} \neq 0$. The plane (p) defined by A, B, C has scalar equation:



$$(P): (\vec{AB} \times \vec{AC}) \cdot [\underbrace{(x,y,z) - \vec{OA}}_{\vec{AM}}] = 0$$

EXAMPLE

Find the equation of the plane (P) containing the points A(3,1,2), B(4,3,1), and C(2,5,6)

Solution

$$\left.\begin{array}{l} A(3,1,2) \\ B(4,3,1) \end{array}\right\} \Rightarrow \vec{AB} = (4-3, 3-1, 1-2) = (1,2,-1) \quad (1)$$

$$\left.\begin{array}{l} A(3,1,2) \\ C(2,5,6) \end{array}\right\} \Rightarrow \vec{AC} = (2-3, 5-1, 6-2) = (-1,4,4) \quad (2)$$

From (1) and (2)

$$\vec{AB} \times \vec{AC} = (1,2,-1) \times (-1,4,4)$$

$$= \left|\begin{array}{ccc} i & j & k \\ 1 & 2 & -1 \\ -1 & 4 & 4 \end{array}\right| \begin{array}{cc} i & j \\ 1 & 2 \\ -1 & 4 \end{array}$$

$$= 2\cdot 4 i + (-1)(-1) j + 4 \cdot 1 k - (-1)\cdot 2 k - 4(-1) i - 4 \cdot 1 j$$

$$= 8i + j + 4k + 2k + 4i - 4j$$

$$= 12i - 3j + 6k$$

$$= (12, -3, 6)$$

therefore

$(P): (\vec{AB} \times \vec{AC}) \cdot (\vec{r} - \vec{OA}) = 0 \iff$

$\iff (12, -3, 6) \cdot (x-3, y-1, z-2) = 0 \iff$

$\iff 12(x-3) - 3(y-1) + 6(z-2) = 0 \iff$

$\iff 4(x-3) - (y-1) + 2(z-2) = 0 \iff$

$\iff 4x - 12 - y + 1 + 2z - 4 = 0 \iff$

$\iff 4x - y + 2z - 15 = 0 \iff$

$\iff 4x - y + 2z = 15$

Thus $(P): 4x - y + 2z = 15$

Note that the plane $(P): Ax + By + Cz + D = 0$ has normal vector $\vec{n} = (A, B, C)$

Plane equation to parametric equation

The first step is to solve the plane equation for one of the 3 variables and use the result to rewrite (x, y, z) as in the following

b) write the parametric equation for the plane $(P): 2x + y + 3z = 7$

Solution

day 4

$2x+y+3z=7 \Longleftrightarrow y=7-2x-3z \Longleftrightarrow$

$\Longleftrightarrow (x,y,z)=(x,7-2x-3z,z)$

$=(0,7,0)+(x,-2x,0)+(0,-3z,z)$

$=(0,7,0)+x(1,-2,0)+z(0,-3,1)$

therefore

$$(p):\begin{cases} x=0+1t+0s \\ y=7+(-2)t+(-3)s \\ z=0+0t+1s \end{cases} \quad \forall t,s\in\mathbb{R}$$

day 5

Relative position of two planes

Consider two planes

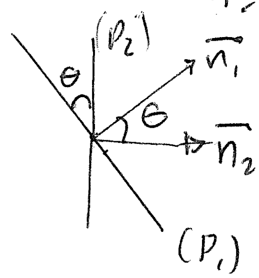
$(P_1) : \vec{n_1} \cdot (\vec{r} - \vec{r_1}) = 0$
$(P_2) : \vec{n_2} \cdot (\vec{r} - \vec{r_2}) = 0$

with $\vec{n_1}, \vec{n_2}, \vec{r_1}, \vec{r_2} \in \mathbb{R}^3$. Then:

- $(P_1) // (P_2) \Leftrightarrow \vec{n_1} // \vec{n_2} \Leftrightarrow \vec{n_1} \times \vec{n_2} = 0$
  $(P_1) \perp (P_2) \Leftrightarrow \vec{n_1} \perp \vec{n_2} \Leftrightarrow \vec{n_1} \cdot \vec{n_2} = 0$

- The angle θ between (P_1) and (P_2) is equal to the angle between the normal vectors $\vec{n_1}, \vec{n_2}$:

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{\|\vec{n_1}\| \|\vec{n_2}\|}$$



- If the planes $(P_1), (P_2)$ intersect, then $(\ell) = (P_1) \cap (P_2)$ is a line.
  To find the equation of the line (ℓ)

a) Find a point $A \in (\ell)$

b) Find the direction vector $u \in \mathbb{R}^3$ of the line (ℓ) by noting that

$$(\ell) = (P_1) \cap (P_2) \Rightarrow \vec{n_1} \perp (\ell) \wedge \vec{n_2} \perp (\ell) \Rightarrow$$
$$\Rightarrow \vec{n_1} \times \vec{n_2} // (\ell)$$
$$\Rightarrow \vec{n_1} \times \vec{n_2} // \vec{u}.$$

EXAMPLES

a) Consider the planes $(P_1): x+y+z=1$ and $(P_2): x-y+z=1$

Find the angle θ between (P_1) and (P_2) and the line $(\ell) = (P_1) \cap (P_2)$

Solution

(P_1) has normal vector $n_1 = (1,1,1)$

(P_2) has normal vector $n_2 = (1,-1,1)$

Since:

$$\|\vec{n_1}\| = \|(1,1,1)\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\|\vec{n_2}\| = \|(1,-1,1)\| = \sqrt{1^2+(-1)^2+1^2} = \sqrt{3}$$

$$\vec{n_1} \cdot \vec{n_2} = (1,1,1) \cdot (1,-1,1) = 1 \cdot 1 + 1 \cdot (-1) + 1 \cdot 1 = 1-1+1 = 1$$

It follows that:

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{\|\vec{n_1}\|\|\vec{n_2}\|} = \frac{1}{\sqrt{3}\sqrt{3}} = \frac{1}{3}$$

• The line $(\ell) = (P_1) \cap (P_2)$

For $z=0$:

Recall: $(P_1): x+y+z=1$
$(P_2): x-y+z=1$

$$(x,y,0) \in (\ell) \Leftrightarrow (x,y,0) \in (P_1) \cap (P_2) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x+y+0=1 \\ x-y+0=1 \end{cases} \Leftrightarrow \begin{cases} x+y=1 \\ x-y=1 \end{cases} \Leftrightarrow \begin{cases} 2x=2 \\ x-y=1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=1 \\ x-y=1 \end{cases} \Leftrightarrow \begin{cases} x=1 \\ 1-y=1 \end{cases} \Leftrightarrow \begin{cases} x=1 \\ y=0 \end{cases}$$

Therefore $A(1,0,0) \in (\ell)$

Since $(\ell) \perp \vec{n_1} \wedge (\ell) \perp \vec{n_2} \Rightarrow (\ell) \parallel \vec{n_1} \times \vec{n_2}$

$\Rightarrow (\ell)$ has direction vector $\vec{u} = \vec{n_1} \times \vec{n_2}$

we note that

$$\vec{n_1} \times \vec{n_2} = (1,1,1) \times (1,-1,1) = \left| \begin{array}{ccc|cc} i & j & k & i & j \\ 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 \end{array} \right.$$

$$= i + j - k - k - (-1)i - j$$

$$= i + j - k - k + i - j$$

$$= 2i - 2k$$

$$= (2, 0, -2)$$

and therefore

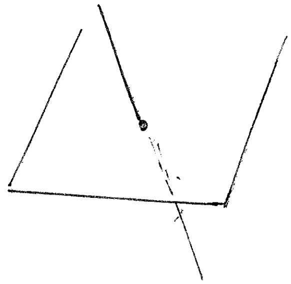
$$(\ell): (x, y, z) = (1, 0, 0) + t(2, 0, -2)$$

$$= (1 + 2t, 0, -2t) \Rightarrow$$

$$\Rightarrow (\ell): \begin{cases} x = 1 + 2t \\ y = 0 \\ z = -2t \end{cases}$$

day 5

Intersection of line and plane



Consider the line

$$(\ell): (x, y, z) = (a_1, a_2, a_3) + t(b_1, b_2, b_3)$$

and the plane $(P): Ax + By + Cz + D = 0$

To find the intersection $(\ell) \cap (P)$, we note that

$$(x, y, z) \in (\ell) \cap (P) \Leftrightarrow \begin{cases} (x, y, z) \in (\ell) \\ (x, y, z) \in (P) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \exists t \in \mathbb{R}: \begin{cases} (x, y, z) = (a_1, a_2, a_3) + t(b_1, b_2, b_3) \\ Ax + By + Cz + D = 0 \end{cases}$$

It follows that

$$Ax + By + Cz + D = 0 \Leftrightarrow$$

$$\Leftrightarrow A(a_1, b_1 t) + B(a_2, b_2 t) + C(a_3 + b_3 t) + D = 0$$

$$\Leftrightarrow \ldots \Leftrightarrow t = t_0$$

Given t, we find $(x, y, z) = (a_1, a_2, a_3) + t_0 (b_1, b_2, b_3) = \ldots$

Possibilities: Unique solution (a point): $pt = q \rightarrow t = \frac{q}{p}$ if $p \neq 0$

No solution (line is parallel to the plane): $0t = q$ with $q \neq 0$

Infinite solutions (line is in the plane): $0t = 0 \rightarrow$ all $t \in \mathbb{R}$ are solutions.

dag 5

EXAMPLE

Find the intersection between the line (ℓ) going through the points $A(2,1,0)$ and $B(3,1,5)$ and the plane $(P): x+y-2z=1$

Solution

we not that

$$\left.\begin{array}{l} A(2,1,0) \\ B(3,1,5) \end{array}\right\} \Rightarrow \vec{AB} = (3-2, 1-1, 5-0) = (1,0,5) \Rightarrow$$

$$\Rightarrow (\ell): (x,y,z) = (2,1,0) + (1,0,5)t \Rightarrow$$

$$\Rightarrow (\ell): \begin{cases} x = 2+t \\ y = 1 \\ z = 5t \end{cases}$$

Since

$$(x,y,z) \in (\ell) \cap (P) \Leftrightarrow \begin{cases} (x,y,z) \in (\ell) \\ (x,y,z) \in (P) \end{cases} \Leftrightarrow$$

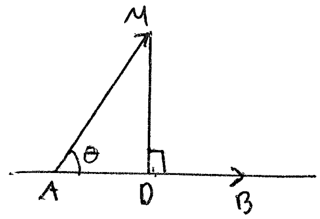
$$\Leftrightarrow \exists\, t \in \mathbb{R}: \begin{cases} (x,y,z) = (2+t, 1, 5t) \\ x+y-2z = 1 \end{cases}$$

It follows that

$$\begin{aligned} x+y-2z=1 &\Leftrightarrow (2+t)+1-2(5t) = 1 \Leftrightarrow \\ &\Leftrightarrow 2+t+1-10t = 1 \Leftrightarrow \\ &\Leftrightarrow 2-9t = 0 \Leftrightarrow \\ &\Leftrightarrow t = \frac{9}{2} \\ &\Leftrightarrow (x,y,z) = \left(2+\tfrac{2}{9}, 1, 5\cdot\tfrac{2}{9}\right) \\ &\qquad = \left(\tfrac{20}{9}, 1, \tfrac{10}{9}\right) \end{aligned}$$

Distances between points, lines and planes

Distance of point M from line (AB)



$$d(M,(AB)) = MD = \frac{\|\vec{AB} \times \vec{AM}\|}{\|\vec{AB}\|}$$

Proof

Let $D \in (AB)$ such that $MD \perp AB$

Let $\theta = \hat{MAB} \Rightarrow \|\vec{AB} \times \vec{AM}\| = \|\vec{AB}\| \|\vec{AM}\| \sin\theta$

$$\Rightarrow \sin\theta = \frac{\|\vec{AB} \times \vec{AM}\|}{\|\vec{AB}\|\|\vec{AM}\|}$$

$$\Rightarrow d(M,AB) = MD$$
$$= AM \sin\theta$$
$$= \|\vec{AM}\| \frac{\|\vec{AB} \times \vec{AM}\|}{\|\vec{AB}\|\|\vec{AN}\|}$$
$$= \frac{\|\vec{AB} \times \vec{AM}\|}{\|\vec{AB}\|}$$

In $\mathbb{R}^2$

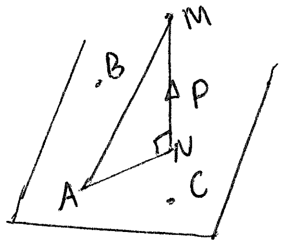
$(\ell): Ax + By + C = 0$

$M(x_1, y_1)$

$$d(M,(\ell)) = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2+B^2}}$$

Distance of point from Plane

Case 1: The distance $d(M,(P))$ between the point M and the plane (P) defined by the point A, B, C is given by



$$d(M,(P)) = \frac{|(\vec{AB} \times \vec{AC}) \cdot \vec{MA}|}{\|\vec{AB} \times \vec{AC}\|}$$

Proof

Let $N \in (P)$ be the projection of M on (P) such that $MN \perp (P)$.
Let $D \in (MN)$ such that $\vec{ND} = \vec{AB} \times \vec{AC}$:

$$d(M,(P)) = MN = \|\vec{MN}\| = \|Proj_{\vec{ND}}(\vec{AM})\|$$

$$= |Comp_{\vec{ND}}(\vec{AM})|$$

$$= \left| \frac{\vec{AM} \cdot \vec{ND}}{\|\vec{ND}\|} \right|$$

$$= \frac{|\vec{AM} \cdot \vec{ND}|}{\|\vec{ND}\|}$$

$$= \frac{|\vec{AM} \cdot (\vec{AB} \times \vec{AC})|}{\|\vec{AB} \times \vec{AC}\|}$$

$$= \frac{|(\vec{AB} \times \vec{AC}) \cdot \vec{MA}|}{\|\vec{AB} \times \vec{AC}\|}$$

Case 2:

The distance between $M(x_0, y_0, z_0)$ and the plane $(P): Ax + By + Cz + D = 0$ is given by:

$$d(M, (P)) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Proof

Let $N, P, Q \in (P)$ be three non-collinear points on the plane (P) with $N(x_1, y_1, z_1)$. Since $N(x_1, y_1, z_1) \in (P) \Rightarrow Ax_1 + By_1 + Cz_1 + D = 0 \Rightarrow$
$\Rightarrow Ax_1 + By_1 + Cz_1 = -D$ (1)

We also note that:

$$\left.\begin{array}{l} (P): Ax + By + Cz + D = 0 \Rightarrow \vec{u} = (A, B, C) \perp (P) \\ \overrightarrow{NP} \times \overrightarrow{NQ} \perp (P) \end{array}\right\} \Rightarrow$$

$\Rightarrow \vec{u} \,//\, \overrightarrow{NP} \times \overrightarrow{NQ} \Rightarrow \exists \lambda \in \mathbb{R}: \underline{\overrightarrow{NP} \times \overrightarrow{NQ} = \lambda \vec{u}}$ (2)

and

$$\left.\begin{array}{l} M(x_0, y_0, z_0) \\ N(x_1, y_1, z_1) \end{array}\right\} \Rightarrow \underline{\overrightarrow{NM} = (x_0 - x_1, y_0 - y_1, z_0 - z_1)} \quad (3)$$

From (1), (2), (3) using the previous result it follows that

(on next page)

(from previous page)

$$d(M,(p)) = \frac{|\vec{NM} \cdot (\vec{NP} \times \vec{NQ})|}{\|\vec{NP} \times \vec{NQ}\|}$$

$$= \frac{|\vec{NM} \cdot (\lambda \vec{u})|}{\|\lambda \vec{u}\|}$$

$$= \frac{|\lambda (\vec{NM} \cdot \vec{u})|}{\|\lambda \vec{u}\|}$$

$$= \frac{|\lambda| \cdot |\vec{NM} \cdot \vec{u}|}{|\lambda| \|\vec{u}\|}$$

$$= \frac{|\vec{NM} \cdot \vec{u}|}{\|\vec{u}\|}$$

$$= \frac{|(x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (A, B, C)|}{\|(A, B, C)\|}$$

$$= \frac{|A(x_0 - x_1) + B(y_0 - y_1) + C(z_0 - z_1)|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|(Ax_0 + By_0 + Cz_0) - (Ax_1 + By_1 + Cz_1)|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|(Ax_0 + By_0 + Cz_0) - (-D)|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Note Summer 2 2016: Until here is test 1

Vector - Valued Functions

Definitions

- A vector valued function is a mapping $f: A \to \mathbb{R}^3$ with $A \subseteq \mathbb{R}$.

Limit of a vector-valued function

Def: Let $f: A \to \mathbb{R}^3$ be a vector-valued function and let $a \in A$ be a limit point of A. Then

$$\lim_{t \to a} f(t) = L \iff$$

$$\iff \forall \varepsilon > 0 : \exists \delta > 0 : \forall t \in A : (0 < |t-a| < \delta \Rightarrow \|f(t) - L\| < \varepsilon)$$

Note that:

$|t-a|$ = absolute value of $t-a$

$\|f(t)-L\|$ = vector norm of $f(t)-L$

(23) Let $u, v, w \in \mathbb{R}^3$ with

day 4

$\|u\| = 2$. $\|v\| = 3$ $\|w\| = 1$

$\varphi = (\text{interior angle from } u \text{ to } v) = \frac{\pi}{4}$

$\theta = (\text{interior angle from } w \text{ to } u \times v) = \frac{\pi}{3}$

Evaluate $\|u+v\|$ and $|u \cdot (v \times w)|$

Solution

$$\begin{aligned}
\|u+v\|^2 &= (u+v)\cdot(u+v) \\
&= (u+v)\cdot u + (u+v)\cdot v \\
&= u\cdot u + v\cdot u + u\cdot v + v\cdot v \\
&= \|u\|^2 + 2u\cdot v + \|v\|^2 \\
&= \|u\|^2 + 2\|u\|\|v\|\cos\varphi + \|v\|^2 \\
&= 2^2 + 2\cdot 2\cdot 3\cos(\tfrac{\pi}{4}) + 3^2 \\
&= 4 + 12(\tfrac{\sqrt{2}}{2}) + 9 \\
&= 13 + 6\sqrt{2}
\end{aligned}$$

$$\Longrightarrow \|u+v\| = \sqrt{13+6\sqrt{2}}$$

$$\begin{aligned}
|u\cdot(v\times w)| &= |v\cdot(w\times u)| \\
&= |w\cdot(u\times v)| \\
&= \|w\|\|u\times v\||\cos\theta| \\
&= \|w\|\left[\|u\|\|v\|\sin\varphi\right]|\cos\theta| \\
&= 1\left[2\cdot 3\sin(\tfrac{\pi}{4})\right]\cos(\tfrac{\pi}{3}) \\
&= 6(\tfrac{\sqrt{2}}{2})(\tfrac{\sqrt{3}}{2} \\
&= \frac{6}{2\cdot 2}\sqrt{6} \\
&= \frac{3\sqrt{6}}{2}
\end{aligned}$$

Recall that $u\cdot(v\times w) = v\cdot(w\times u) = w\cdot(u\times v)$

dag 4

Lines in $\mathbb{R}^3$

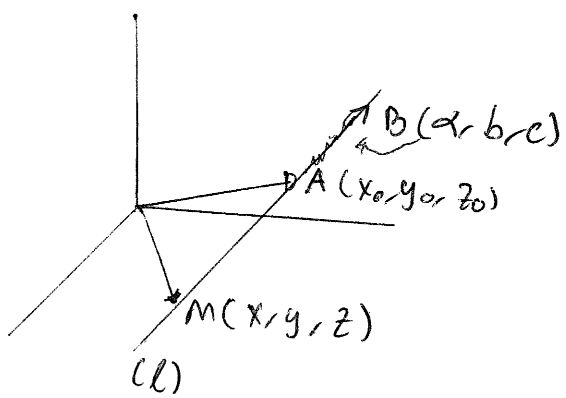
Let $\emptyset$ be the origin of our coordinate system

<u>Def</u>: The parametric equation for a line (ℓ) going through the points A, B is:

$$(\ell): \quad \vec{r} = \overrightarrow{\emptyset A} + t\overrightarrow{AB}, \quad \forall t \in \mathbb{R}$$

The above statement is equivalent to:

$$M \in (\ell) \iff \exists t \in \mathbb{R} : \overrightarrow{OM} = \overrightarrow{\emptyset A} + t\overrightarrow{AB}$$



- $\overrightarrow{AB}$ = direction Vector of (ℓ)
- For $\vec{r} = (x, y, z)$, $\overrightarrow{\emptyset A} = (x_0, y_0, z_0)$, and $\overrightarrow{AB} = (a, b, c)$ the parametric equation is equivalent to

$$(\ell) : \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}, \quad \forall t \in \mathbb{R}$$

- Eliminating t from the above equations gives the <u>Symetric equations</u> representation of the line (ℓ):

$$(\ell): \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

The symmetric equations can be reduced to a system of the form

$$(\ell): \begin{cases} A_1 x + B_1 y + C_1 = 0 \\ A_2 y + B_2 z + C_2 = 0 \end{cases}$$

which essentially defines the line (ℓ) as an intersection of two planes.

Relative position of two lines

Consider the lines

$(\ell_1): \vec{r} = \vec{a_1} + t\vec{b_1}, \quad \forall t \in \mathbb{R}$
$(\ell_2): \vec{r} = \vec{a_2} + t\vec{b_2}, \quad \forall t \in \mathbb{R}$

Then

1) $(\ell_1) \parallel (\ell_2) \Leftrightarrow \vec{b_1} \parallel \vec{b_2} \Leftrightarrow \vec{b_1} \times \vec{b_2} = 0$

EXAMPLE

a) write the symmetric equations for the line (AB) with $A(1,2,-1)$ and $B(5,4,1)$

Solution

$$\left.\begin{array}{l} A(1,2,-1) \\ B(5,4,1) \end{array}\right\} \Rightarrow \begin{cases} \overrightarrow{OA} = (1,2,-1) \\ \overrightarrow{AB} = (5-1,\ 4-2,\ 1-(-1)) = (4,2,2) \end{cases}$$

therefore

$$(l): (x,y,z) = \overrightarrow{OA} + t\overrightarrow{AB} = (1,2,-1) + t(4,2,2)$$
$$= (1+4t,\ 2+2t,\ -1+2t) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 1+4t \\ y = 2+2t \\ z = -1+2t \end{cases} \Leftrightarrow \frac{x-1}{4} = \frac{y-2}{2} = \frac{z+1}{2}$$

$$\Leftrightarrow \begin{cases} 2(x-1) = 4(y-2) \\ 2(y-2) = 2(z+1) \end{cases} \Leftrightarrow \begin{cases} 2x-2 = 4y-8 \\ 2y-4 = 2z+2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2x-4y+6=0 \\ 2y-2z-6=0 \end{cases} \Leftrightarrow \boxed{\begin{cases} x-4y+3=0 \\ y-z-3=0 \end{cases}}$$

day 4

c) Given the points $A(1,1,0)$, $B(3,1,5)$, $C(a+b, b+c, c+a)$ and $D(a+1, b+1, c+1)$, find all $a,b,c \in \mathbb{R}$ such that $(AB) \parallel (CD)$

<u>Solution</u>

we calculate $\vec{AB} \times \vec{CD}$

$$\left.\begin{array}{l} A(1,1,0) \\ B(3,1,5) \end{array}\right\} \Longrightarrow \vec{AB} = (3-1, 1-1, 5-0) = (2, 0, 5) \qquad (1)$$

$$\left.\begin{array}{l} C(a+b, b+c, c+a) \\ D(a+1, b+1, c+1) \end{array}\right\} \Longrightarrow$$

$$\Rightarrow \vec{CD} = ((a+1)-(a+b), (b+1)-(b+c), (c+1)-(c+a))$$
$$= (a+1-a-b, b+1-b-c, c+1-c-a)$$
$$= (1-b, 1-c, 1-a). \qquad (2)$$

From (1) and (2):

$$\vec{AB} \times \vec{CD} = \left|\begin{array}{ccc|cc} i & j & K & i & j \\ 2 & 0 & 5 & 2 & 0 \\ 1-b & 1-c & 1-a & 1-b & 1-c \end{array}\right|$$

Note: It is better to use e_1, e_2, e_3 notation.

$$= 0(1-a)i + 5(1-b)j + 2(1-c)K - 0(1-b)K - 5(1-c)i - 2(1-a)j$$
$$= -5(1-c)i + [5(1-b) - 2(1-a)]j + 2(1-c)K$$
$$= (-5+5c, 5-5b-2+2a, 2-2c)$$
$$= (-5+5c, 2a-5b+3, 2-2c$$

It follows that

day 4

From last Page

$\vec{AB} \times \vec{CD} = (-5+5c, 2a-5b+3, 2-2c)$

$(AB) \parallel (CD) \Leftrightarrow \vec{AB} \parallel \vec{CD} \Leftrightarrow \vec{AB} \times \vec{CD} = \vec{0} \Leftrightarrow$

$\Leftrightarrow (-5+5c, 2a-5b+3, 2-2c) = (0,0,0) \Leftrightarrow$

$$\Leftrightarrow \begin{cases} -5+5c=0 \\ 2a-5b+3=0 \\ 2-2c=0 \end{cases} \Leftrightarrow \begin{cases} 5c=5 \\ 2a-5b=-3 \\ 2c=2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} c=1 \\ 2a-5b=-3 \\ c=1 \end{cases} \Leftrightarrow \begin{cases} 2a-5b=-3 \\ c=1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a = \frac{5b-3}{2} \\ c=1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow (a,b,c) = \left(\frac{5b-3}{2}, b, 1\right)$$

$$= \left(-\frac{3}{2}, 0, 1\right) + \left(\frac{5}{2}b, b, 0\right)$$

$$= \left(-\frac{3}{2}, 0, 1\right) + b\left(\frac{5}{2}, 1, 0\right)$$

$$\Leftrightarrow (a,b,c) \in \left\{ \left(-\frac{3}{2}, 0, 1\right) + b\left(\frac{5}{2}, 1, 0\right) \mid b \in \mathbb{R} \right\}$$

day 4

d) write the parametric equation for the line (ℓ) defined by

$$(\ell): \begin{cases} x - 3y + 2 = 0 \\ 2y - z + 5 = 0 \end{cases}$$

Solution

$$(\ell): \begin{cases} x - 3y + 2 = 0 \\ 2y - z + 5 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 3y - 2 \\ z = 2y + 5 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow (x, y, z) = (3y - 2, y, 2y + 5)$$
$$= (3y, y, 2y) + (-2, 0, 5)$$
$$= y(3, 1, 2) + (-2, 0, 5)$$

It follows that

$$(\ell): r = (-2, 0, 5) + t(3, 1, 2)$$

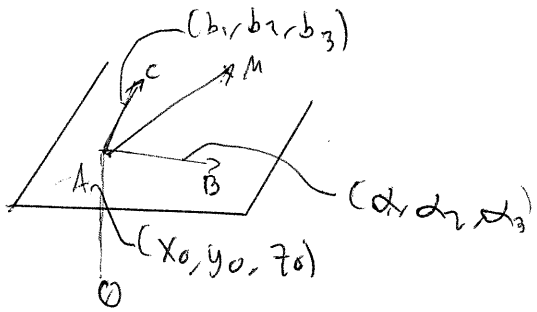
Note: Homework problem (27) remove it

Also problem (30)

Planes in $\mathbb{R}^3$ day 4

Let A, B, C be three non-collinear points (i.e. A, B, C are not on the same line). Then these three points define a unique plane with equation

$$(P): \vec{r} = \vec{OA} + t\vec{AB} + s\vec{AC} \quad \forall t, s \in \mathbb{R}$$



Equivalently if we let

$\vec{OA} = (x_0, y_0, z_0)$, $\vec{AB} = (\alpha_1, \alpha_2, \alpha_3)$ $\vec{AC} = (b_1, b_2, b_3)$

then

$$(P): \begin{cases} x = x_0 + \alpha_1 t + b_1 s \\ y = y_0 + \alpha_2 t + b_2 s \\ z = z_0 + \alpha_3 t + b_3 s \end{cases} \quad \forall t, s \in \mathbb{R}$$

Similarly the belonging condition for (P) is.

$$M(x, y, z) \in (P) \Leftrightarrow \exists t, s \in \mathbb{R}: \begin{cases} x = x_0 + \alpha_1 t + b_1 s \\ y = y_0 + \alpha_2 t + b_2 s \\ z = z_0 + \alpha_3 t + b_3 s \end{cases}$$

or Equivalently in the notes

$$M \in (P) \Leftrightarrow \exists t, s \in \mathbb{R}: \vec{OM} = \vec{OA} + t\vec{AB} + s\vec{AC}$$

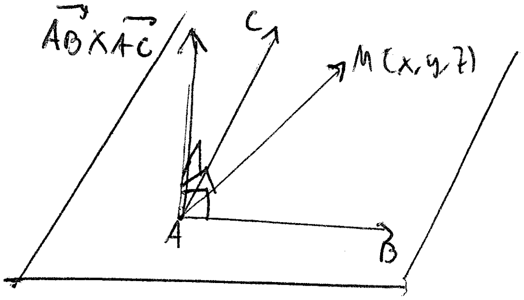
Eliminating t, s gives an equivalent equation of the form

$$(P): Ax + By + Cz + D = 0$$

which is called the <u>scalar equation</u> of (P)

Scalar equation for plane from 3 points $\qquad$ day 4

Let A, B, C be three points with $\vec{AB} \times \vec{AC} \neq 0$. The plane (p) defined by A, B, C has scalar equation:



$$(P): (\vec{AB} \times \vec{AC}) \cdot [\underbrace{(x,y,z) - \vec{OA}}_{\vec{AM}}] = 0$$

EXAMPLE

Find the equation of the plane (P) containing the points A(3,1,2), B(4,3,1), and C(2,5,6)

Solution

$$\left.\begin{array}{l} A(3,1,2) \\ B(4,3,1) \end{array}\right\} \Rightarrow \vec{AB} = (4-3,\ 3-1,\ 1-2) = (1,2,-1) \quad (1)$$

$$\left.\begin{array}{l} A(3,1,2) \\ C(2,5,6) \end{array}\right\} \Rightarrow \vec{AC} = (2-3,\ 5-1,\ 6-2) = (-1,4,4) \quad (2)$$

From (1) and (2)

$$\vec{AB} \times \vec{AC} = (1,2,-1) \times (-1,4,4)$$

$$= \left|\begin{array}{ccc} i & j & K \\ 1 & 2 & -1 \\ -1 & 4 & 4 \end{array}\right| \begin{array}{cc} i & j \\ 1 & 2 \\ -1 & 4 \end{array}$$

$$= 2 \cdot 4 i + (-1)(-1) j + 4 \cdot 1 K - (-1) \cdot 2 K - 4(-1) i - 4 \cdot 1 j$$

$$= 8i + j + 4K + 2K + 4i - 4j$$

$$= 12i - 3j + 6K$$

$$= (12, -3, 6)$$

day 4

therefore

$(P): (\vec{AB} \times \vec{AC}) \cdot (\vec{r} - \vec{OA}) = 0 \iff$

$\iff (12, -3, 6) \cdot (x-3, y-1, z-2) = 0 \iff$

$\iff 12(x-3) - 3(y-1) + 6(z-2) = 0 \iff$

$\iff 4(x-3) - (y-1) + 2(z-2) = 0 \iff$

$\iff 4x - 12 - y + 1 + 2z - 4 = 0 \iff$

$\iff 4x - y + 2z - 15 = 0 \iff$

$\iff 4x - y + 2z = 15$

Thus $(P): 4x - y + 2z = 15$

Note that the plane $(P): Ax + By + Cz + D = 0$ has normal vector $\vec{n} = (A, B, C)$

Plane equation to parametric equation

The first step is to solve the plane equation for one of the 3 variables and use the result to rewrite (x, y, z) as in the following

b) write the parametric equation for the plane $(P): 2x + y + 3z = 7$

Solution

day 4

$2x + y + 3z = 7 \iff y = 7 - 2x - 3z \iff$

$\iff (x, y, z) = (x, 7 - 2x - 3z, z)$

$= (0, 7, 0) + (x, -2x, 0) + (0, -3z, z)$

$= (0, 7, 0) + x(1, -2, 0) + z(0, -3, 1)$

there fore

$$(p): \begin{cases} x = 0 + 1t + 0s \\ y = 7 + (-2)t + (-3)s \\ z = 0 + 0t + 1s \end{cases} \quad \forall t, s \in \mathbb{R}$$

dag 6

Quantifier notation:

$\forall x \in A: P(x) \rightarrow$ "For all $x \in A$, $P(x)$ is true"

$\exists x \in A: P(x) \rightarrow$ "There is at least one $x \in A$ such that $P(x)$ is true"

$\sum_{\alpha \in \{1,2,3\}} f(\alpha) = f(1) + f(2) + f(3)$

Equivalence:

$[\forall x \in \{a, b, c\}: P(x)] \Leftrightarrow P(a) \wedge P(b) \wedge P(c)$

$[\exists x \in \{a, b, c\}: P(x)] \Leftrightarrow P(a) \vee P(b) \vee P(c)$

Combination of quantifiers.

$\forall x \in A: \exists y \in B: P(x, y)$

Refer: notes for intro to Math Proof website

Limit Definition

Let $f: A \rightarrow \mathbb{R}^3$ let $a \in A$ be a limit point of A

$\lim_{t \to a} f(t) = L \Leftrightarrow$

$\Leftrightarrow \forall \varepsilon \in (0, +\infty): \exists \delta \in (0, +\infty): \forall t \in A: (0 < |t - a| < \delta \Rightarrow \|f(t) - L\| < \varepsilon)$

dag 6

Thm: Let $f: A \to \mathbb{R}^3$ be a vector-valued function and assume that

a) $f(t) = (x(t), y(t), z(t)), \ \forall t \in A$

b) a limit point of A

Then:

$$\lim_{t \to a} f(t) = \left(\lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right)$$

Continuity of vector valued function

Def. Let $f: A \to \mathbb{R}^2$ be a vector-valued function, and let $J \subseteq A$ and $a \in A$

a) f continuous at $t = a \Leftrightarrow \lim_{t \to a} f(t) = f(a)$

b) f continuous at $J \Leftrightarrow \forall a \in J: f$ continuous at $t = a$.

Derivatives of vector valued functions

Def. Let $f: A \to \mathbb{R}^3$ be a vector-valued function we say

a) f differentiable at $a \in A \Leftrightarrow \exists v \in \mathbb{R}^3: \lim_{\Delta t \to 0} \dfrac{f(t+\Delta t) - f(t)}{\Delta t} = v$

b) f differentiable at $J \subseteq A \Leftrightarrow \forall a \in J: f$ differentiable at $t = a$

Def: Let $f: A \to \mathbb{R}^3$ be a vector-valued function that is differentiable in $J \subseteq A$. We define the derivative function $\dot{f}: J \to \mathbb{R}^3$ as:

$$\forall t \in J: \dot{f}(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Notation: $\dot{f}(t) = f'(t) = \frac{d f(t)}{dt}$

For higher-order derivatives

$$\ddot{f}(t) = f''(t) = \frac{d \dot{f}(t)}{dt} = \frac{d^2 f(t)}{dt^2}$$

$$\dddot{f}(t) = f'''(t) = \frac{d \ddot{f}(t)}{dt} = \frac{d^3 f(t)}{dt^3}$$

Thm: Let $f: A \to \mathbb{R}^3$ with $f(t) = (x(t), y(t), z(t))$ $\forall t \in A$ differentiable in $B \subseteq A$. Then

$$f'(t) = (x'(t), y'(t), z'(t))$$

Interpretation

If a vector-valued function $r: [0, +\infty) \to \mathbb{R}^3$ represents the position vector of an object in motion, then:

$u(t) = \dot{r}(t) =$ velocity of object

$a(t) = \dot{u}(t) = \ddot{r}(t) =$ acceleration of object

Examples: web notes CAL 3.2 pages: 4–6

day 6

Properties of differentiation

Thm: Let $u: A \to \mathbb{R}^3$ and $v: A \to \mathbb{R}^3$ vector-valued functions. Then.

a) $\frac{d}{dt}[u(t) + v(t)] = \dot{u}(t) + \dot{v}(t)$

b) $\frac{d}{dt}[\lambda u(t)] = \lambda \dot{u}(t)$

c) $\frac{d}{dt}[f(t) u(t)] = f'(t) u(t) + f(t) \dot{u}(t)$

d) $\frac{d}{dt}[u(t) \cdot v(t)] = \dot{u}(t) \cdot v(t) + u(t) \cdot \dot{v}(t)$

e) $\frac{d}{dt}[u(t) \times v(t)] = \dot{u}(t) \times v(t) + u(t) \times \dot{v}(t)$

Application

Show that if $\forall t \in \mathbb{R} : \|u(t)\| = c$, then $\forall t \in \mathbb{R} : \dot{u}(t) \perp u(t)$

Solution

we noth that

$$\frac{d}{dt}\|u(t)\|^2 = \frac{d}{dt}[u(t) \cdot u(t)] = \dot{u}(t) \cdot u(t) + u(t) \cdot \dot{u}(t) = 2u(t) \cdot \dot{u}(t)$$

It follows that

$$\|u(t)\| = c, \forall t \in \mathbb{R} \Rightarrow \frac{d}{dt}\|u(t)\|^2 = 0, \forall t \in \mathbb{R} \Rightarrow$$

$$\Rightarrow 2u(t) \cdot \dot{u}(t) = 0, \quad \forall t \in \mathbb{R}$$

$$\Rightarrow u(t) \cdot \dot{u}(t) = 0, \quad \forall t \in \mathbb{R}$$

$$\Rightarrow \dot{u}(t) \perp u(t), \quad \forall t \in \mathbb{R}$$

day 6

Arclenght

- Let $C(C): r(t),\ \forall t \in [a,b]$ be a finite curve

The lenght of the curve is given by:

$$l = \int_a^b |\dot{r}(t)|\, dt$$

- For the more general case of an infinite curve $C(C): r(t),\ \forall t \in \mathbb{R}$

we define the <u>arclength function</u> $s(t)$ as:

$$s(t) = \int_{t_0}^{t} |\dot{r}(\alpha)|\, d\alpha, \quad \forall t \in \mathbb{R}$$

Here $t_0 \in \mathbb{R}$ represents an initial time, usually chosen by default as $t_0 = 0$. The <u>arclength function</u> $s(t)$ gives the distance travelled during the interval $[t_0, t]$ for $t > t_0$

- We note, from the fundamental theorem of calculus, that

$$\frac{ds(t)}{dt} = \|\dot{r}(t)\|$$

EXAMPLES

Find the arclength function from $t_0=0$ for curve

$C(t) = r(t) = (e^{2t}\cos 2t, 2, e^{2t}\sin 2t)$

Solution

For $x(t) = e^{2t}\cos 2t$, $y(t)=2$, and $z(t) = e^{2t}\sin 2t$ we find that

$\dot{x}(t) = (e^{2t})'\cos 2t + e^{2t}(\cos 2t)' = 2e^{2t}\cos 2t - 2e^{2t}\sin 2t = 2e^{2t}(\cos 2t - \sin 2t)$

$\dot{y}(t) = 0$

$\dot{z}(t) = (e^{2t})'\sin 2t + e^{2t}(\sin 2t)' = 2e^{2t}\sin 2t + 2e^{2t}\cos 2t = 2e^{2t}(\sin 2t + \cos 2t)$

It follows that

$$\begin{aligned}\|\dot{r}(t)\|^2 &= [\dot{x}(t)]^2 + [\dot{y}(t)]^2 + [\dot{z}(t)]^2 \\ &= [2e^{2t}(\cos 2t - \sin 2t)]^2 + [2e^{2t}(\sin 2t + \cos 2t)]^2 \\ &= 4e^{4t}[(\cos 2t - \sin 2t)^2 + (\cos 2t + \sin 2t)^2] \\ &= 4e^{4t}[\cos^2 2t - 2\cos 2t\sin 2t + \sin^2 2t + \cos^2 2t + 2\cos 2t\sin 2t + \sin^2 2t] \\ &= 4e^{4t}[2\cos^2 2t + 2\sin^2 2t] \\ &= 4e^{4t}\cdot 2 \\ &= 8^{4t}\end{aligned}$$

$\Rightarrow \|\dot{r}(t)\| = 2\sqrt{2}\, e^{2t}$

$\Rightarrow l = \int_0^t \|\dot{r}(t)\| dt$

$$\begin{aligned}\Rightarrow s(t) = \int_0^t \|\dot{r}(\tau)\| d\tau &= \int_0^t 2\sqrt{2}e^{2\tau} d\tau \\ &= 2\sqrt{2}\int_0^t e^{2\tau} d\tau = 2\sqrt{2}\left[\frac{e^{2\tau}}{2}\right]_0^t \\ &= 2\sqrt{2}\,\frac{e^{2t}-1}{2} = \sqrt{2}(e^{2t}-1)\end{aligned}$$

Curvature of a curve (C) day 6

$T(t)$ tangent to trajectory follows direction of motion

$\|T(t)\| = 1$

Assuming continuous movement

$\|\dot{r}(t)\| \neq \quad \forall t \in A$

Then the tangent vector is defined as

$$T(t) = \frac{\dot{r}(t)}{\|\dot{r}(t)\|}$$

Curvature definition

$$K(t) = \left\| \frac{dT}{ds} \right\|$$

Noting that $s(t) \to t = f(s)$ (inverting the function $\to T(t) = T(f(s))$)

Thm : $K(t) = \dfrac{\|\dot{T}(t)\|}{\|\dot{r}(t)\|}$

Proof

From the chain rule

$$\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} = \frac{dT}{ds}\|\dot{r}(t)\| \Rightarrow$$

$$\Rightarrow \frac{dT}{ds} = \frac{1}{\|\dot{r}(t)\|}\frac{dT}{dt} = \frac{\dot{T}(t)}{\|\dot{r}(t)\|} \Rightarrow$$

$$\Rightarrow K(t) = \left\|\frac{dT}{ds}\right\| = \left\|\frac{\dot{T}(t)}{\|\dot{r}(t)\|}\right\| = \frac{\|\dot{T}(t)\|}{\|\dot{r}(t)\|} \blacksquare$$

day 7

Given $r(t)$ with $\|r(t)\| \neq 0, \ \forall t \in \mathbb{R}$

$$T(t) = \frac{1}{\|\dot{r}(t)\|} \dot{r}(t)$$

$$K(t) = \left\| \frac{dT}{ds} \right\|$$

$$K(t) = \frac{\|\dot{T}(t)\|}{\|\dot{r}(t)\|}$$

$$K(t) = \frac{\|\dot{r}(t) \times \ddot{r}(t)\|}{\|\dot{r}(t)\|^3}$$

$\downarrow$

Proof

Since $\dot{r}(t) = \|\dot{r}(t)\| T(t) = s'(t) T(t)$

and $\ddot{r}(t) = s''(t) T(t) + s'(t) \dot{T}(t)$

therefore

$$\begin{aligned} \dot{r}(t) \times \ddot{r}(t) &= [s'(t) T(t)] \times [s'' T(t) + s'(t) \dot{T}(t)] \\ &= [s'(t) T(t)] \times [s'(t) \dot{T}(t)] \\ &= [s'(t)]^2 [T(t) \times \dot{T}(t)] \end{aligned}$$

note $[s'(t) T(t)] \times [s'' T(t)] = 0$

day 7

Since $\|T(t)\| = 1, \ \forall t \in \mathbb{R}$

$\Rightarrow T(t) \perp \dot{T}(t), \ \forall t \in \mathbb{R}$

$\Rightarrow \|T(t) \times \dot{T}(t)\| = \|C(t)\| \|\dot{T}(t)\| \sin(\frac{\pi}{2}) = \|\dot{T}(t)\| \Rightarrow$

$$\Rightarrow \|\dot{r}(t) \times \ddot{r}(t)\| = \|[s'(t)]^2 [C(t) \times \dot{T}(t)]\|$$
$$= |[s'(t)]^2| \|T(t) \times \dot{T}(t)\|$$
$$= \|\dot{r}(t)\|^2 \|\dot{T}(t)\|$$
$$= \|\dot{r}(t)\|^2 K(t) \|\dot{r}(t)\| \Rightarrow$$

$$\Rightarrow K(t) = \frac{\|\dot{r}(t) \times \ddot{r}(t)\|}{\|\dot{r}(t)\|^3}$$

APPLICATION

Show that a circle (c) with radius R has constant curvature $K(t) = \frac{1}{R}$

Proof

Consider the circle (c), $r(t) = (R\cos t, R\sin t, 0)$.
It follows that

$\dot{r}(t) = (-R\sin t, R\cos t, \emptyset)$
$\ddot{r}(t) = (-R\cos t, -R\sin t, \emptyset)$

and therefore

$$\dot{r}(t) \times \ddot{r}(t) = \begin{vmatrix} i & j & K \\ -R\sin t & R\cos t & \emptyset \\ -R\cos t & -R\sin t & \emptyset \end{vmatrix} = \begin{vmatrix} -R\sin t & R\cos t \\ -R\cos t & -R\sin t \end{vmatrix} K$$

dag 7

$= [(-R\sin t)^2 - (-R\cos t)(R\cos t)]k$

$= [R^2\sin^2 t + R^2\cos^2 t]k$

$= R^2(\sin^2 t + \cos^2 t)k$

$= R^2 k$

$\Rightarrow \|\dot{r}(t) \times \ddot{r}(t)\| = R^2$

Also

$\|\dot{r}(t)\| = \|(-R\sin t, R\cos t, 0)\| = \sqrt{(-R\sin t)^2 + (R\cos t)^2}$

$= \sqrt{R^2(\sin^2 t + \cos^2 t)}$

$= \sqrt{R^2}$

$= |R|$ since $R > 0$

$= R$

and therefore

$$\kappa(t) = \frac{\|\dot{r}(t) \times \ddot{r}(t)\|}{\|\dot{r}(t)\|^3} = \frac{R^2}{R^3} = \frac{1}{R} \quad \blacksquare$$

Scalar fields

Definitions

- we define the n-dimensional space $\mathbb{R}^n$ as follows:
  $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid \forall k \in [n]: x_k \in \mathbb{R}\}$ (n-tuple)
  with $[n]$ defined as
  $[n] = \{1, 2, 3, \dots, n\}$
- The elements $x \in \mathbb{R}^n$ are <u>n-dimensional vectors</u> with $x = (x_1, x_2, \dots, x_n)$
  The numbers $x_1, x_2, \dots, x_n \in \mathbb{R}$ are the <u>components</u> of x

Algebra on $\mathbb{R}^n$

Let $x, y, z \in \mathbb{R}^n$ with $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$ and $z = (z_1, z_2, \dots, z_n)$. We define:

$x = y \Leftrightarrow \forall k \in [n]: x_k = y_k$

$z = x + y \Leftrightarrow \forall k \in [n]: z_k = x_k + y_k$

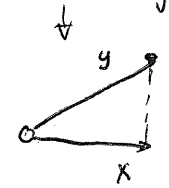
$z = ax \Leftrightarrow \forall k \in [n]: z_k = a x_k$ (with $a \in \mathbb{R}$)

$x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_{k=1}^{n} x_k y_k$ (Inner product)

$\|x\| = (x \cdot x)^{1/2} = \sqrt{\sum_{k=1}^{n} x_k^2}$ (norm)

Balls on $\mathbb{R}^n$

- Let $x, y \in \mathbb{R}^n$ be two vectors. Assume that x, y represent two points in an n-dimensional space. Then $\|x-y\|$ is the distance between x and y



- we therefore define:

$$B(x,\rho) = \{ u \in \mathbb{R}^n \mid \|x-u\| < \rho \}$$

with $x \in \mathbb{R}^n$ and $\rho \in (0,+\infty)$

- The set $B(x,\rho)$ contains all the points in $\mathbb{R}^n$ whose distance from x is less then ρ

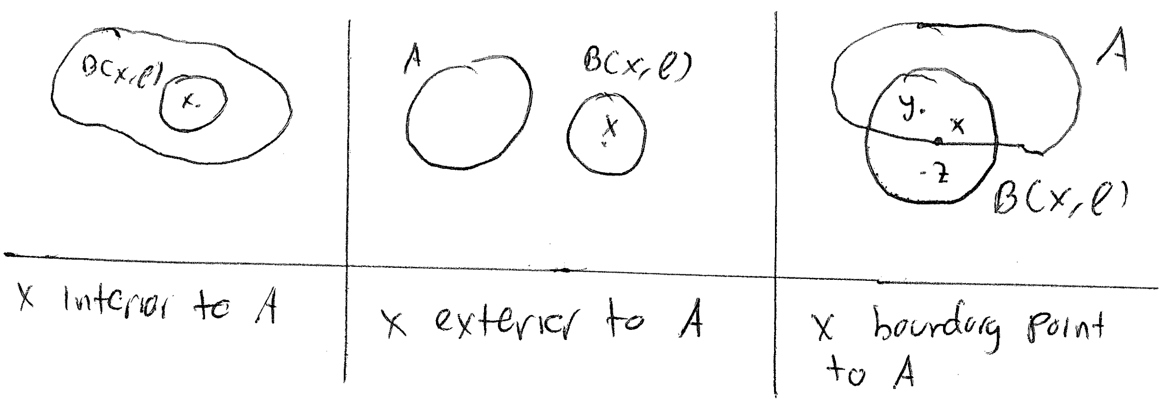
Open and closed sets

- Let $A \subseteq \mathbb{R}^n$ and $x \in A$. We say that

x interior to $A \Leftrightarrow \exists \rho \in (0,+\infty) : B(x,\rho) \subseteq A$

x exterior to $A \Leftrightarrow \exists \rho \in (0,+\infty) : B(x,\rho) \cap A = \emptyset$

x boundary point to $A \Leftrightarrow \forall \rho \in (0,+\infty) : \exists y, z \in B(x,\rho) : (y \in A \land z \notin A)$.



- We may therefore define:

$\mathrm{Int}(A) = \{ x \in \mathbb{R}^n \mid x \text{ interior to } A \}$

$\mathrm{ext}(A) = \{ x \in \mathbb{R}^n \mid x \text{ exterior to } A \}$

$\partial A = \{ x \in \mathbb{R}^n \mid x \text{ boundary point to } A \}$

day 7

- We say that

A is an open set $\Longleftrightarrow A \cap \partial A \neq \emptyset$

A is a closed set $\Longleftrightarrow \partial A \subseteq A$

- It follows that

a) An open set does not contain any of the points in its boundary

b) A closed set includes all of the points in its boundary

Scalar fields

- A scalar fields is a mapping $f: A \longrightarrow \mathbb{R}$ with $A \subseteq \mathbb{R}^n$.
- A is the domain of f, and we write $dom(f) = A$.
- The range $f(A)$ of A is defined as:

$f(A) = \{f(x) \mid x \subseteq A\}$

Limits of scalar fields

- Let $f: A \longrightarrow \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ be a scalar field and let $x_0 \in int(A$

we define:

$$\lim_{x \to x_0} f(x) = l \Longleftrightarrow \forall \varepsilon \in (0, +\infty): \exists \delta \in (0, +\infty): \forall x \in A: (x \in B(x_0, \delta) - \{x_0\} \Rightarrow |f(x) - l| < \varepsilon)$$

This definition is similar to the Weierstrass definition of $\lim_{x \to a} f(x) = l$

for functions of one variable

Properties of limits

Thm: Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$, and let $a \in \text{Int}(A)$. Assume that $\lim_{x \to a} f(x) = l_1 \wedge \lim_{x \to a} g(x) = l_2$. Then:

a) $\lim_{x \to a} [f(x) + g(x)] = l_1 + l_2$

b) $\lim_{x \to a} [f(x) g(x)] = l_1 l_2$

c) $\forall \lambda \in \mathbb{R}: \lim_{x \to a} [\lambda f(x)] = \lambda \lim_{x \to a} f(x)$

d) $l_2 \neq 0 \Rightarrow \lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{l_1}{l_2}$

Limit of polynomials

Def: A polynomial $f \in \mathbb{R}^n[x]$ is a scalar field $f: \mathbb{R}^n \to \mathbb{R}$ with

$$f(x) = \sum_{(k_1, \dots, k_n) \in ([m] \cup \{0\})^n} A_{k_1 k_2 \dots k_n} x_1^{k_1} x_2^{x_2} \dots x_n^{k_n}$$

with $x = (x_1, x_2, \dots, x_n)$ and $[m] = \{1, 2, 3, \dots, m\}$

$\mathbb{R}^n[x]$ the set of al polynomials on $\mathbb{R}^n$

Thm: $f \in \mathbb{R}[x] \Rightarrow \lim_{x \to x_0} f(x) = f(x_0), \forall x_0 \in \mathbb{R}^n$

EXAMPLE

Evaluate. $\lim_{(x,y) \to (1,3)} [3 + x^2 + y^2 + 3xy]$

Solution

$$\lim_{(x,y) \to (1,3)} [3 + x^2 + y^2 + 3xy] = 3 + 1^2 + 3^2 + 3 \cdot 1 \cdot 3 = 3 + 1 + 9 + 9 = 22$$

dag 7

Zero-bounded theorem

Def: Let $f: A \to \mathbb{R}$ be a scalar field with $A \subseteq \mathbb{R}^n$

Let $B \subseteq A$. we say that

f bounded in $B \Leftrightarrow \exists a \in (0,+\infty): \forall x \in B: |f(x)| \leq a$

(refer to the paper in the web page)

Thm: Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ be scalar fields with $A \subseteq \mathbb{R}^n$, and let $x_0 \in \text{int}(A)$. Then:

$$\left.\begin{array}{r} \exists \delta \in (0,+\infty): f \text{ bounded at } B(x_0,\delta) \\ \lim\limits_{x \to x_0} g(x) = 0 \end{array}\right\} \Rightarrow \lim_{x \to x_0} [f(x) g(x)] = 0$$