Calculus III

Vectors in $\mathbb{R}^2$

Cartesian Product of sets

Let A, B be sets

$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$

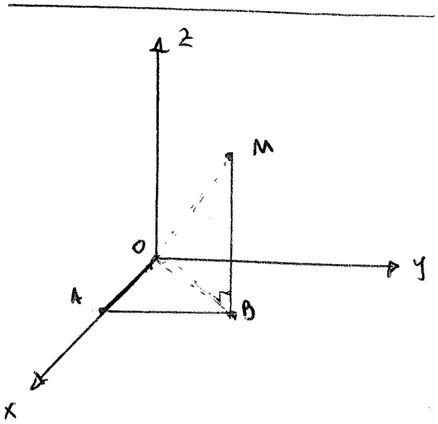
$A \times B \times C = \{(a,b,c) \mid a \in A \wedge b \in B \wedge c \in C\}$

For $A = B = C = \mathbb{R}$

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a,b) \mid a \in \mathbb{R} \wedge b \in \mathbb{R}\}$ ← set of 2d vectors

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(a,b,c) \mid a,b,c \in \mathbb{R}\}$ ← set of 3d vectors

Cartesian Coordinates



$\overline{OA} = x$

$\overline{AB} = y$

$\overline{BM} = z$

↓

$\boxed{M(x,y,z)}$

Distance Formula

$\left.\begin{matrix} M(x,y,z) \\ O(0,0,0) \end{matrix}\right\} \Rightarrow OM = \sqrt{x^2+y^2+z^2}$

$\left.\begin{matrix} A(x_1, y_1, z_1) \\ B(x_2, y_2, z_2) \end{matrix}\right\} \Rightarrow AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

Sphere (A,r) = Center A and radius r

Sphere $(A,r) = \{M \in \mathbb{R}^3 \mid AM = r\}$

(A,r) $(x-x_A)^2+(y-y_A)^2+(z-z_A)^2 = r^2$

EXAMPLE

Find all $\lambda \in \mathbb{R}$ such that $A(\lambda, \lambda+1, \lambda+2)$ and $B(\lambda-1, 2\lambda, 2)$ satisfy $AB = 1$

Solution

Since

$$\begin{cases} A(\lambda, \lambda+1, \lambda+2) \\ B(\lambda-1, 2\lambda, 2) \end{cases} \Rightarrow \quad \begin{aligned} AB &= \sqrt{(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2} \\ &= \sqrt{((\lambda-1)-\lambda)^2+(2\lambda-(\lambda+1))^2+(2-(\lambda+2))^2} \\ &= \sqrt{(-1)^2+(2\lambda-\lambda-1)^2+(2-\lambda-2)^2} \\ &= \sqrt{1+(\lambda-1)^2+(-\lambda)^2} \\ &= \sqrt{1+\lambda^2-2\lambda+\lambda^2} \\ &= \sqrt{2\lambda^2-2\lambda+2} \qquad (1) \end{aligned}$$

It follows that

$$AB = 1 \overset{\text{Eq. (1)}}{\iff} \sqrt{2\lambda^2-2\lambda+2} = 1$$

$$\iff 2\lambda^2-2\lambda+2 = 1^2$$

$$\iff 2\lambda^2-2\lambda+1 = 0$$

For $(a, b, c) = (2, -2, 1)$

$$\Delta = b^2-4ac = (-2)^2-4(+2)1$$

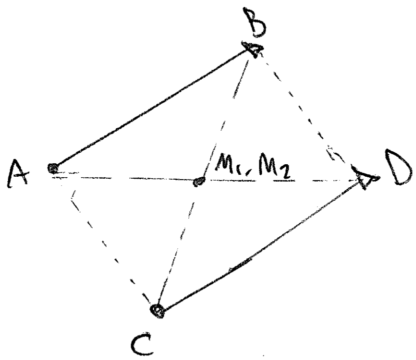
$$= 4-8 < 0 \Rightarrow \text{no real solutions}$$

Therefore for all $\lambda \in \mathbb{R}$, $AB \neq 1$

Geometric Vectors

Let A, B be two points

$\overrightarrow{AB} = (A, B)$ — A initial point
B terminal point



Def: Given geometric vectors $\overrightarrow{AB}, \overrightarrow{CD}$, let
M_1 be midpoint of AD
M_2 be midpoint of BC

$$\boxed{\overrightarrow{AB} = \overrightarrow{CD} \Longleftrightarrow M_1 = M_2}$$

Let $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ the cartesian representation of $\overrightarrow{AB}$ is

$$\overrightarrow{AB} = (x_B - x_A, y_B - y_A, z_B - z_A)$$

$$\downarrow$$

$$\overrightarrow{AA} = (0, 0, 0) = \mathbf{0} \leftarrow \text{zero vector}$$

Vector Operations

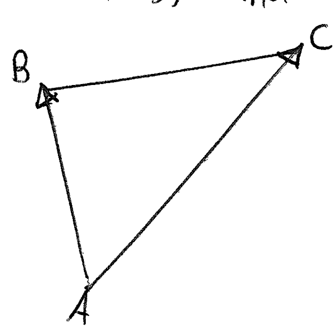
① Addition and subtraction

Let $u \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ with $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$

we define

$u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

$u - v = (u_1 - v_1, u_2 - v_2, u_3 - v_3)$



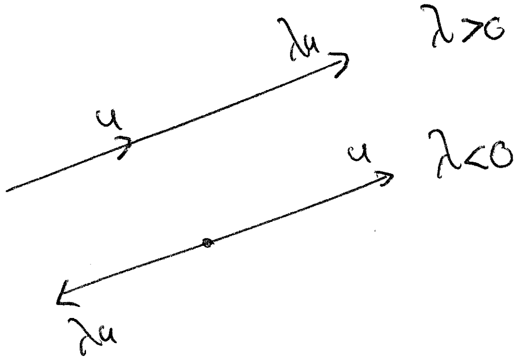
__Prop__ Given points A, B, C

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

② Scalar multiplication

Let $\lambda \in \mathbb{R}$ and let $u \in \mathbb{R}^3$ with $u = (u_1, u_2, u_3)$ we define

$\lambda u = (\lambda u_1, \lambda u_2, \lambda u_3)$



③ Scalar product (Dot Product)

Let $u, v \in \mathbb{R}^3$ with $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$

we define

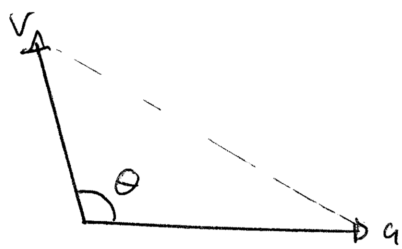
$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$\|u\| = \sqrt{u \cdot u} \quad \longleftarrow \text{norm of } u \text{ (geometric lenght)}$$

$$= \sqrt{u_1^2 + u_2^2 + u_3^2}$$

we will show that

$u \cdot v = \|u\| \|v\| \cos\theta$



day 2

Properties of addition

1) $\forall v, w \in \mathbb{R}^3$: $v+w=w+v$
2) $\forall v, w, u \in \mathbb{R}^3$: $u+(v+w)=(u+v)+w$
3) $\forall u \in \mathbb{R}^3$: $u+\mathbf{0}=u$
4) $\forall u \in \mathbb{R}^3$: $u+(-u)=\mathbf{0}$
5) $\forall c \in \mathbb{R}$: $\forall u, v \in \mathbb{R}^3$: $c(u+v)=cu+cv$
6) $\forall c_1, c_2 \in \mathbb{R}$: $\forall u \in \mathbb{R}^3$: $(c_1+c_2)u=c_1u+c_2u$
7) $\forall c_1, c_2 \in \mathbb{R}$: $\forall u \in \mathbb{R}^3$: $(c_1c_2)u=c_1(c_2u)$
8) $\forall u \in \mathbb{R}^3$: $1u=u$

The lenght of the vector is given by

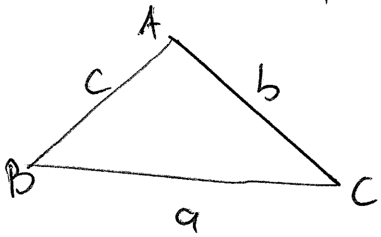
$\|v\| = \sqrt{v \cdot v}$

Properties of Dot Product

1) $\forall u, v \in \mathbb{R}^3$: $u \cdot v = v \cdot u$
2) $\forall u, v, w \in \mathbb{R}^3$: $u \cdot (v+w) = u \cdot v + u \cdot w$
3) $\forall c \in \mathbb{R}$: $\forall u, v \in \mathbb{R}^3$: $(cu) \cdot v = u \cdot (cv) = c(u \cdot v)$
4) $\forall u \in \mathbb{R}^3$: $\mathbf{0} \cdot u = 0$

day 2

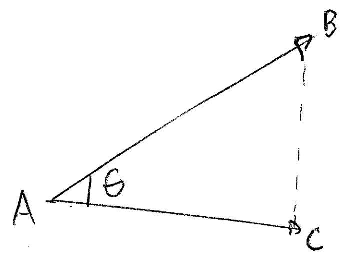
Show that $u \cdot v = \|u\| \|v\| \cos\theta$



From law of cosines

$a^2 = b^2 + c^2 - 2bc \cdot \cos A$

Proof



Let $u = \overrightarrow{AB}$ and $v = \overrightarrow{AC}$

Then $\overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{AB} = v - u$

and

$a = \|\overrightarrow{BC}\| = \|v - u\| = \|u - v\|$

$b = \|\overrightarrow{AC}\| = \|v\|$

$c = \|\overrightarrow{AB}\| = \|u\|$

$\theta = BAC = A$ consequently

$$\cos\theta = \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{\|v\|^2 + \|u\|^2 - \|u - v\|^2}{2\|v\|\|u\|}$$

$$= \frac{\|u\|^2 + \|v\|^2 - (u-v) \cdot (u-v)}{2\|u\|\|v\|}$$

$$= \frac{\|u\|^2 + \|v\|^2 - u \cdot u + u \cdot v + v \cdot u - v \cdot v}{2\|u\|\|v\|}$$

$$= \frac{\|u\|^2 + \|v\|^2 - \|u\|^2 + 2u \cdot v - \|v\|^2}{2\|u\|\|v\|}$$

$$= \frac{2u \cdot v}{2\|u\|\|v\|}$$

$$= \frac{u \cdot v}{\|u\|\|v\|} \Longrightarrow u \cdot v = \|u\|\|v\| \cos\theta \quad ■$$

(7) $u = (1+\sqrt{2},\ 1-\sqrt{2},\ \sqrt{2})$ and

$v = (\sqrt{2}+2,\ 2+3\sqrt{2},\ 1+\sqrt{2})$

Evaluate $(2u-v)\cdot(u+3v)$

Solution

$$(2u-v)\cdot(u+3v) = (2u-v)\cdot u + (2u-v)\cdot(3v)$$
$$= (2u)\cdot u - v\cdot u + (2u)\cdot(3v) - v\cdot(3v)$$
$$= 2\|u\|^2 - u\cdot v + 6u\cdot v - 3\|v\|^2$$
$$= 2\|u\|^2 + 5u\cdot v - 3\|v\|^2 \qquad (1)$$

Calculating the terms

$$\|u\|^2 = \|(1+\sqrt{2},\ 1-\sqrt{2},\ \sqrt{2})\|^2$$
$$= (1+\sqrt{2})^2 + (1-\sqrt{2})^2 + (\sqrt{2})^2$$
$$= 1 + 2\sqrt{2} + (\sqrt{2})^2\ 1 - 2\sqrt{2} + (\sqrt{2})^2 + 2$$
$$= 1+2+1+2+2$$
$$= 8$$

$$u\cdot v = (1+\sqrt{2},\ 1-\sqrt{2},\ \sqrt{2})\cdot(\sqrt{2}+2,\ 2+3\sqrt{2},\ 1+\sqrt{2})$$
$$= (1+\sqrt{2})(\sqrt{2}+2) + (1-\sqrt{2})(2+3\sqrt{2}) + \sqrt{2}(1+\sqrt{2})$$
$$= \sqrt{2}+2+2+2\sqrt{2} + 2+3\sqrt{2}-2\sqrt{2}-3\cdot 2 + \sqrt{2}+2$$
$$= 2+5\sqrt{2}$$

$$\|v\|^2 = \|(\sqrt{2}+2,\ 2+3\sqrt{2},\ 1+\sqrt{2}\|^2$$
$$= (\sqrt{2}+2)^2 + (2+3\sqrt{2})^2 + (1+\sqrt{2})^2$$
$$= 2+4\sqrt{2}+4 + 4+12\sqrt{2}+18 + 1+2\sqrt{2}+2$$
$$= 31+18\sqrt{2}$$

From Eq. (1)

$$(2u-v)\cdot(u+3v) = 2\|u\|^2 + 5u\cdot v - 3\|v\|^2$$
$$= 2\cdot 8 + 5(2+5\sqrt{2}) - 3(31+18\sqrt{2})$$
$$= 16+10+25\sqrt{2} - 93 - 54\sqrt{2}$$
$$= -67 - 29\sqrt{2}$$

day 2

Orthogonal Vectors

$u \perp v \Leftrightarrow u \cdot v = 0 \Leftrightarrow \theta = \frac{\pi}{2}$ "$\perp$" means orthogonal

EXAMPLE

Let $u = (0, x, x+1)$ and $v = (x, x+1, x-2)$ with $x \in \mathbb{R}$. Find all $x \in \mathbb{R}$ such that $u \perp v$

Solution

Since

$$
\begin{aligned}
u \cdot v &= (0, x, x+1) \cdot (x, x+1, x-2) \\
&= 0x + x(x+1) + (x+1)(x-2) \\
&= (x+1)[x + (x-2)] \\
&= (x+1)(2x-2) \\
&= 2(x+1)(x-1) \qquad (1)
\end{aligned}
$$

Thus

$$
\begin{aligned}
u \perp v &\Leftrightarrow u \cdot v = 0 \\
&\Leftrightarrow 2(x+1)(x-1) = 0 \quad [\text{Via Eq. } (1)] \\
&\Leftrightarrow x+1 = 0 \vee x-1 = 0 \\
&\Leftrightarrow x = -1 \vee x = 1 \\
&\Leftrightarrow \underline{x \in \{-1, 1\}}
\end{aligned}
$$

$\vee$ "or"

$\wedge$ "and"

Application: Triangle Inequality

day 2

$\forall u, v \in \mathbb{R}^3: |u \cdot v| \leq \|u\|\|v\|$

Proof: $|u \cdot v| = \big|\|u\|\|v\|\cos\theta\big|$
$= \|u\|\|v\||\cos\theta|$
$\leq \|u\|\|v\| \cdot 1$
$= \|u\|\|v\|$

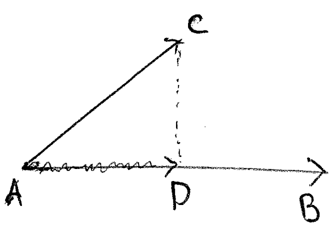
since $\forall x \in \mathbb{R} \begin{cases} |\sin x| \leq 1 \\ |\cos x| \leq 1 \end{cases}$

$\forall u, v \in \mathbb{R}^3: \|u+v\| \leq \|u\| + \|v\|$

Proof: $\|u+v\|^2 = (u+v)\cdot(u+v)$
$= u \cdot u + u \cdot v + v \cdot u + v \cdot v$
$= \|u\|^2 + 2u \cdot v + \|v\|^2$
$= \|u\|^2 + 2\|u\|\|v\|\cos\theta + \|v\|^2$
$\leq \|u\|^2 + 2\|u\|\|v\|$
$= (\|u\| + \|v\|)^2 \implies \|u+v\| \leq \|u\| + \|v\|$

Since $\|u+v\| > 0$

Projections



$\left.\begin{array}{l} v = \overrightarrow{AC} \\ u = \overrightarrow{AB} \end{array}\right\}$ Projection

Projection of u in v "$Proj_u(v)$"

$Proj_u(v) = \dfrac{u \cdot v}{\|u\|^2} u$

$Proj_u(v) = Comp_u(v) \dfrac{1}{\|u\|} u$

$Comp_u(v) = \dfrac{u \cdot v}{\|u\|}$