Tuesday Week 6

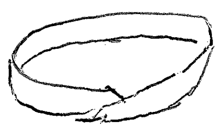
Gauss divergence theorem

Thm: Let $S = \{a(t,s) \mid (t,s) \in A\}$ be a surface and let $f: B \to \mathbb{R}^3$ with $B \subseteq \mathbb{R}^3$. Assume that:

$$\begin{cases} S \text{ smooth and closed} \\ a(t,s) \text{ positive oriented} \\ f \text{ differentiable on } S \cup \text{int}(S) \\ \nabla f \text{ continuous on } S \cup \text{int}(S) \end{cases}$$

then $$\iint_S f \cdot ds = \iiint_{\text{int}(S)} (\nabla \cdot f)\, dx\, dy\, dz$$

Oriented surface



Def. The Mobius strip is the surface

$$m = \{(x(a,\theta), y(a,\theta), z(a,\theta)) \mid a \in [-1,1] \land \theta \in [0, 2\pi]\}$$

with

$$\begin{cases} x(a,\theta) = (1 + (a/2)\cos(\theta/2))\cos\theta \\ y(a,\theta) = (1 + (a/2)\cos(\theta/2))\cos\theta \\ z(a,\theta) = (a/2)\sin(\theta/2) \end{cases}$$

Def: Let $S \subseteq \mathbb{R}^3$ be a smooth surface $S_0 \in \text{Sub}(S) \iff S_0 \subseteq S \land S_0$ smooth surface

Def: Let $S_1 \subseteq \mathbb{R}^3$ and $S_2 \subseteq \mathbb{R}^3$ we say that S_1 homeomorphic to S_2 ($S_1 \cong S_2$) if and only if there is $\phi: S_1 \to S_2$ such that

- ϕ one-to-one
- $\phi(S_1) = S_2$
- ϕ continuous on S_1
- ϕ^{-1} continuous on S_2

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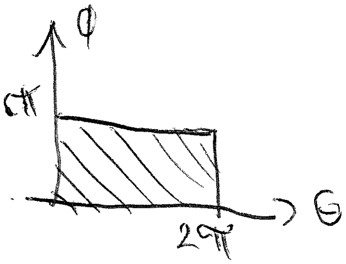
Def: S orientable $\Leftrightarrow \forall S_0 \in Sub(S): S_0 \not\cong m$

S not orientable $\Leftrightarrow \exists S_0 \in Sub(S): S_0 \cong m$

If S orientable we can define a consistent normal vector $\eta(t,s|a)$

closed Surfaces



Def: Let $S = \{a(t,s) \mid (t,s) \in A\}$

Define $P = \{(t,s) \in \partial A \mid \exists (t_0,s_0) \in \partial A - \{(t,s)\} : a(t,s) = a(t_0,s_0)\}$

$\partial S = \{a(t,s) \mid (t,s) \in \partial A - P\}$

S closed $\Leftrightarrow \partial S = \emptyset$

▶ $int(S)$ ← set of points in $\mathbb{R}^3$ that are "inside" the surface S

Positive-Oriented Surface

Define $S = \{a(t,s) \mid (t,s) \in A\}$ and let

$$\eta(t,s|a) = \frac{1}{\|R(t,s|a)\|} R(t,s|a)$$ ← normal vector

$a(t,s)$ positively oriented $\Leftrightarrow$

$\Leftrightarrow \forall (t,s) \in A : \exists \tau_0 \in (0,+\infty) : \{a(t,s) + \tau\eta(t,s|a) \mid \tau \in (0,\tau_0]\} \subseteq ext(S)$

$ext(S) = \mathbb{R}^3 - (S \cup int(S))$

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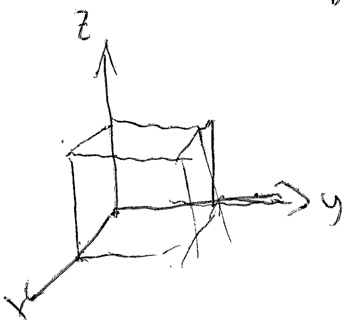
EXAMPLE

For $F(x,y,z) = (x^2y, xz, yz^3)$, $\forall (x,y,z) \in \mathbb{R}^3$ and $S = \partial A$

with $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \in [0,1] \wedge y \in [0,2] \wedge z \in [0,3]$

evaluate $I := \iint_S F \cdot ds$



<u>Solution</u>

$$\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(y z^3) = 2xy + 3yz^2 \Rightarrow$$

$$I = \iint_S F \cdot dS = \iiint_A dx\,dy\,dz\ \nabla \cdot F = \iiint_A dx\,dy\,dz\,(2xy + 3yz^2)$$

$$= \int_0^1 dx \int_0^2 dy \int_0^3 (2xy + 3yz^2)$$

$$= \int_0^1 dx \int_0^2 dy \left[2xyz + \frac{3yz^3}{3}\right]_{z=0}^{z=3}$$

$$= \int_0^1 dx \int_0^2 dy\,(2x \cdot y \cdot 3 + y3^3)$$

$$= \int_0^1 \int_0^2 dy\,(6xy + 27y)$$

$$= \int_0^1 dx \left[3xy^2 + \frac{27y^2}{2}\right]_{y=0}^{y=2}$$

$$= \int_0^1 dx \left[3x2^2 + \frac{27 \cdot 2^2}{2}\right]$$

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$$= \int_0^1 dx\,(12x+54)$$

$$= \left[6x^2+54x\right]_0^1$$

$$= 6+54$$

$$= 60$$

EXAMPLE

Let $F(x,y,z) = (x^3+\tan(yz),\ y^3 - e^{xz},\ 3z+x^3)$

and let $A = \{(x,y,z)\in\mathbb{R}^3 \mid z\in[0,3] \wedge x^2+y^2\leq 4\}$

Evaluate $I = \iint F\cdot dS$

Solution

Note that

$$A = \{(x,y,z)\in\mathbb{R}^3 \mid z\in[0,3] \wedge x^2+y^2\leq 4\}$$

$$= \{(r\cos\theta, r\sin\theta, z) \mid z\in[0,3] \wedge r\in[0,2] \wedge \theta\in[0,2\pi]\}$$

Define $B = \{(r,\theta,z) \mid r\in[0,2] \wedge \theta\in[0,2\pi] \wedge z\in[0,3]\}$

$$\nabla\cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(x^3+\tan(yz)) + \frac{\partial}{\partial y}(y^3-e^{xz}) + \frac{\partial}{\partial z}(3z+x^3)$$

$$= 3x^2+3y^2+3$$

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It follows that

$$F = \iint_{\partial A} F \cdot dS = \iiint_A dx\,dy\,dz\, \nabla \cdot F = \iiint_A dx\,dy\,dz\,(3x^2 + 3y^2 + 3)$$

$$= 3\iiint_A dx\,dy\,dz\,[(x^2+y^2)+1]$$

$$= 3\iiint_B d\cancel{x}\,d\theta\,dz\; r[r^2+1]$$

$$= 3\iiint_B dr\,d\theta\,dz\,(r^3+r)$$

$$= 3\int_0^2 dr \int_0^{2\pi} d\theta \int_0^3 dz\,(r^3+r)$$

$$= 3\left[\int_0^2 dr\,(r^3+r)\right]\left[\int_0^{2\pi} d\theta\right]\left[\int_0^3 dz\right]$$

$$= 3\left[\frac{r^4}{4} + \frac{r^2}{2}\right]_0^{2\pi} 2\pi \cdot 3$$

$$= 18\pi\left[\frac{2^4}{4} + \frac{2^2}{2}\right]$$

$$= 18\pi(4+2)$$

$$= 6 \cdot 18\pi$$

$$= 108\pi$$