Week 6 Monday August 15th

<u>Thm</u>: Let $S = \{a(t,s) \mid t,s \in A\}$ be a surface with $A \subseteq \mathbb{R}^2$ and let $f: B \to \mathbb{R}^3$ be a vector field with $B \subseteq \mathbb{R}^3$ and $S \in B$. Assume that

a) S is a Jordan-bounded surface

b) $a(t,s)$ has continuous 2^{nd} derivative on A

c) f differentiable in B

d) ∇f is continuous in B

Then:

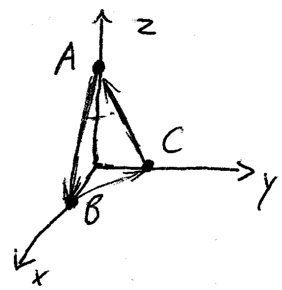
$$\boxed{\iint_S (\nabla \times f) \cdot dS = \oint_{\partial S} f \cdot dl}$$

- The direction of the path ∂S is determined by the intersection that $w(\partial A) = +1$ and the mapping $a: A \to \mathbb{R}^3$. Since $\partial A \in \mathrm{Jord}(\mathbb{R}^3)$, it follows that $\partial S \in \mathrm{Loop}(\mathbb{R}^3)$, which means that the line integral above is circular, hence the $\oint$ notation.

$\Rightarrow$ To formulate the <u>Gauss divergence theorem</u>, we have to define first what we mean by

Example

Let $f(x,y,z) = (2z,\ 8x-3y,\ 3x+y)$
and let C triangular curve $A \to B \to C \to A$
with $A(0,0,2)$, $B(1,0,0)$, $C(0,1,0$
Evaluate $I = \oint_C f \cdot d\ell$



Solution

Let S be the surface defined by ABC. Need the equation for the plane (p) defined by A, B, C

$$(p):\ (\vec{AB} \times \vec{AC}) \cdot ((x,y,z) - (0,0,2)) = 0$$

$$\begin{cases} A(0,0,2) \\ B(1,0,0) \end{cases} \Rightarrow \vec{AB} = (1,0,-2) \quad \text{and} \quad \begin{cases} A(0,0,2) \\ C(0,1,0) \end{cases} \Rightarrow \vec{AC} = 0,1,-2)$$

$$\vec{AB} \times \vec{AC} = \left| \begin{matrix} e_1 & e_2 & e_3 \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{matrix} \right| \begin{matrix} e_1 & e_2 \\ 1 & 0 \\ 0 & 1 \end{matrix} = e_3 - 1(-2)e_1 - (-2)e_2$$

Expansion

$$= (2, 2, 1)$$

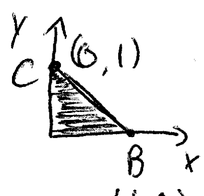
$$(p):\ (2,2,1) \cdot (x, y, z-\overset{2}{2}) = 0$$

$$\Rightarrow$$

$$2x + 2y + z - 2 = 0$$

$$z = -2x - 2y + 2 = g(x,y)$$

To visualize:



this is our plane projected onto x, y plane
$\overset{\Delta ABC}{\wedge}$

$$D = \{(x,y) \in \mathbb{R}^2 \mid x \in [0,1] \wedge y \in [0, 1-x]\}$$

cont.

therefore, ΔABC is given by a set

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid z = -2x - 2y + 2 \wedge x,y \in D\}$$

with $\partial S = C_1$. Therefore,

$$I = \oint_C f \cdot d\ell = \iint_S (\nabla \times f) \cdot dS = \iint_D dx\,dy\, (\nabla \times f)(x, y, 2-2x-2y) \cdot R(x,y \mid S)$$

Find $(\nabla \times f)$ first

$$(\nabla \times f) = \left| \begin{matrix} e_1 & e_2 & e_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z & 8x-3y & 3x+y \end{matrix} \right| \begin{matrix} e_1 & e_2 \\ \partial/\partial x & \partial/\partial y \\ 2z & 8x-3y \end{matrix}$$

$$= e_1(\partial/\partial y)(3x+y) + e_2(\partial/\partial z)(2z) + e_3(\partial/\partial x)(8x-3y) - e_3(\partial/\partial y)(2z) - e_1(\partial/\partial z)(8x-3y) - e_2(\partial/\partial x)(3x+y)$$

$$= e_1 + 2e_2 + 8e_3 - 0e_3 - 0e_1 - 3e_2$$

$$= e_1 - e_2 + 8e_3 = (1,-1,8) = (\nabla \times f)$$

For $R(x,y \mid S)$, we have

$$R(x,y \mid S) = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right) \quad \text{"g defined earlier"}$$

$$= (2, 2, 1)$$

so it follows that

$$I = \iint_D dx\,dy\, (1,-1,8) \cdot (2,2,1)$$

factor out since constant

$$= (1,-1,8) \cdot (2,2,1) \iint dx\,dy = (1,-1,8) \cdot (2,2,1) \int_0^1 dx \int_0^{1-x} dy$$

$$= (1 \cdot 2 + (-1) \cdot 2 + 8 \cdot 1) \int_0^1 dx\,(1-x)$$

$$= 8\left[x - \frac{x^2}{2}\right]\Big|_0^1 = 8 - 4 = 4$$

Ex 2)

Evaluate $I = \oint_C F \cdot d\ell$

with $F(x,y,z) = (-y^2, x, z^2)$

and: C is the curve obtained by intersection of plane

$(\rho): y + z = 2$

with cylinder $(C) = x^2 + y^2 = 1$

Solution

$$C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \wedge y + z = 2\}$$

Take a surface S with $\partial S = C$ such that

$$C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1 \wedge z = 2 - y$$

$$= \{(x,y\,z) \mid z = 2 - y \wedge (x,y) \in A\}$$

with

$$A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

$$= \{(r\cos(\theta), r\sin(\theta)) \in \mathbb{R}^2 \mid r \doteq [0,1] \wedge \theta \in [0, 2\pi]\}$$

Define $B = \{(r,\theta) \mid r \in [0,1] \wedge \theta \in (0, 2\pi)$

$$R(x,y \mid S) = \left(\frac{-\partial(2-y)}{\partial x}, \frac{-\partial(2-y)}{\partial y}, 1\right) = (0,1,1)$$

For $(\nabla \times F)$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix} \begin{matrix} e_1 & e_2 \\ \partial/\partial x & \partial/\partial y \\ -y^2 & x \end{matrix}$$

$$= e_1 \, \partial/\partial y (z^2) + e_2 \, \partial/\partial z (-y^2) + e_3 \, \partial/\partial x (x)$$

$$- e_3 \, \partial/\partial y (-y^2) - e_1 \, \partial/\partial z (x) - e_2 \, \partial/\partial x (z^2)$$

$$= e_1 0 + 0 e_2 + 1 e_3 + 2y e_3 - 0 e_1 - 0 e_2$$

$$= e_3 (1 + 2y)$$

$$= (0, 0, 1 + 2y)$$

cont. ex 2

$$I = \oint_C F \cdot \partial l = \iint_{S'} (\nabla \times F)\, \partial S$$

$$= \iint_A dx\, dy\, (\nabla \times F)(x, y, 2-y) \cdot R(x, y \mid S')$$

$$= \iint dx\, dy\, (0, 0, 1+2y) \cdot (0, 1, 1)$$

$$= \iint_A dx\, dy\, (1+2y)$$

$$= \iint_B dr\, d\theta\, r\, (1 + 2(r \sin\theta))$$

$$= \int_0^1 dr \int_0^{2\pi} d\theta \quad r \quad 1 + 2r \sin\theta$$

$$= \int_0^1 dr \quad r\, [\theta - 2r \cos\theta]\Big|_0^{2\pi}$$

$$= \int_0^1 dr \quad r\, [2\pi - 2r \cos(2\pi) - (0 - 2r \cos\theta)]$$

$$= \cancel{2\pi} \int_0^1 dr \quad r \quad [2\pi - 2r + 2r] = \int_0^1 \partial r\, 2\pi r$$

$$= 2\pi \left[\frac{r^2}{2}\right]\Big|_0^1 = 2\pi \cdot (1/2) = \pi$$

Application

Let $F(x,y,z)$ such that $(\nabla \times F)(x,y,z) = 0, \ \forall (x,y,z) \in \mathbb{R}^3$

Show that F is conservative

Solution

Let $C \in \text{Loop}(\mathbb{R}^3)$ be given

Let $C = \bigcup_{\alpha \in I} C_\alpha$ be a decomposition of C in terms of Jordan curves

Define corresponding Surfaces S_α with $\partial S_\alpha = C_\alpha$

Then:

$$\oint_C F \cdot \partial l = \sum_{\alpha \in I} \oint_{C_\alpha} F \cdot \partial l$$

$$= \sum_{\alpha \in I} \iint_{S_\alpha} (\nabla \times F) \cdot \partial S$$

$$= \sum_{\alpha \in I} \iint_{S_\alpha} 0 \cdot \partial S = 0$$

It follows that:

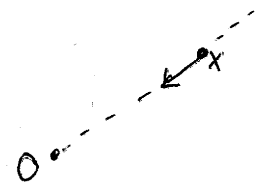
$$\forall C \in \text{Loop}(\mathbb{R}^3) : \oint_C F \cdot \partial l = 0$$

$$\Rightarrow F \text{ conservative on } \mathbb{R}^3$$

Applies to electrostatics, Newtonian Gravity theorem, etc.

Let $F(x) = f(\|x\|)\, x \quad , \quad \forall x \in \mathbb{R}^3$

with $f : \mathbb{R} \to \mathbb{R}$ [redacted]



Then for a Jordan curve C

$$\oint_C F \cdot \partial l = 0$$

as long as $0 \notin C$

Note that

$$\partial_\alpha \|x\| = \partial_\alpha \sqrt{x_1^2 + x_2^2 + x_3^2} = \frac{2 x_\alpha}{2\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$= \frac{x_\alpha}{\|x\|}$$

$$\partial_\alpha x_\beta = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases} = \delta_{\alpha\beta}$$

Therefore:

$$(\nabla \times F)_\alpha = [\nabla \times (f(\|x\|)\, x)]_\alpha$$

$$= \varepsilon_{\alpha\beta c}\, \partial_\beta [f(\|x\|) x]_c$$

$$= \varepsilon_{\alpha\beta c}\, \partial_\beta [f(\|x\|)\, x_c]$$

$$= \varepsilon_{\alpha\beta c}\, x_c\, \partial_\beta (f(\|x\|)) + \varepsilon_{\alpha\beta c}\, f(\|x\|)\, \partial_\beta (x_c)$$

$$= \varepsilon_{\alpha\beta c} \cdot x_c \cdot f'(\|x\|)\, \partial_\beta \|x\| + \varepsilon_{\alpha\beta c}\, f\|x\|\, \delta_{\beta c}$$

$$= \varepsilon_{\alpha\beta c} \cdot x_c \cdot f'(\|x\|) \cdot \frac{x_\beta}{\|x\|} + f(\|x\|)\, \varepsilon_{\alpha\beta c}\, \delta_{\beta c}$$

$$= \frac{f'(\|x\|)}{\|x\|} [\varepsilon_{\alpha\beta c}\, x_\beta x_c] + f(\|x\|)\, \varepsilon_{\alpha\beta\beta}$$

$$= \frac{f'(\|x\|)}{\|x\|}\, 0 + f(\|x\|)\, 0$$

$$= 0 \quad \Rightarrow (\nabla \times F) = \mathbf{0} \Rightarrow \oint_C F \cdot \partial l = \iint (\nabla \times F) \cdot \partial S = \iint \mathbf{0} \cdot \partial S = 0$$