Math 1450

Common Final Exam

Fall 2014

(1) Determine the exact value of the other trigonometric functions given that $\sec\theta = -13/5$ and $\sin\theta < 0$.
(i.e. $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\csc\theta$)

Solution

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{-13/5} = \frac{-5}{13}$$

$$\sin^2\theta = 1-\cos^2\theta = 1-\left(\frac{-5}{13}\right)^2 = 1-\frac{5^2}{13^2} = \frac{13^2-5^2}{13^2} =$$

$$= \frac{(13-5)(13+5)}{13^2} = \frac{8\cdot 18}{13^2} = \frac{16\cdot 9}{13^2} = \left(\frac{4\cdot 3}{13}\right)^2 =$$

$$= \left(\frac{12}{13}\right)^2 \Rightarrow \sin\theta = \frac{-12}{13} \quad [\text{via } \sin\theta < 0]$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-12/13}{-5/13} = \frac{12}{5}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{12/5} = \frac{5}{12}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-12/13} = \frac{-13}{12}$$

We conclude that

$\sin\theta = -12/13$

$\cos\theta = -5/13$

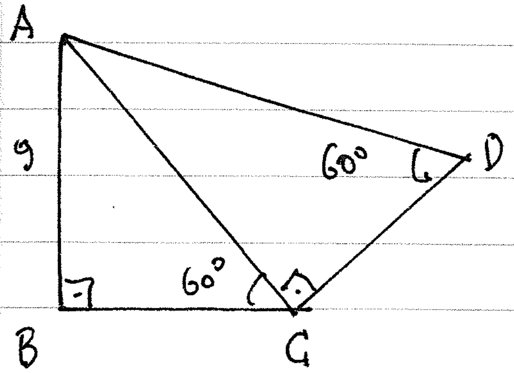
$\tan\theta = 12/5$

$\cot\theta = 5/12$

$\csc\theta = -13/12$

(2) In the figure $A\hat{B}C$ and $A\hat{C}D$ are right angles, $B\hat{C}A = C\hat{D}A = 60°$, and $AB = 9$. Find the length AD.

<u>Solution</u>



From $A\hat{B}C$:

$AB = AC \sin 60° \Rightarrow$

$$\Rightarrow AC = \frac{AB}{\sin 60°} \qquad (1)$$

From $A\hat{C}D$:

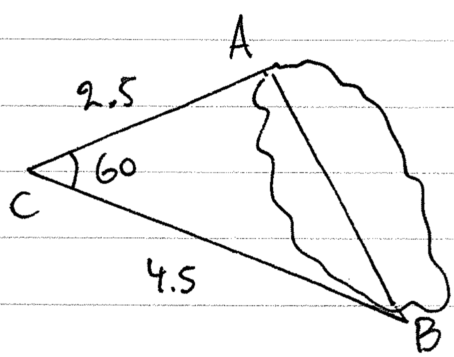
$$AC = AD \sin 60° \Rightarrow AD = \frac{AC}{\sin 60°} \qquad (2)$$

From Eq.(1) and Eq.(2):

$$AD = \frac{AC}{\sin 60°} = \frac{1}{\sin 60°}\left(\frac{AB}{\sin 60°}\right) = \frac{AB}{\sin^2 60°} =$$

$$= \frac{9}{(\sqrt{3}/2)^2} = \frac{9}{3/4} = \frac{9 \cdot 4}{3} = \frac{36}{3} = 12$$

(3) Points A and B are on the edge of a lake. To estimate the distance between them a surveyor picks another point C and finds distances CA and CB to be 2.5 miles and 4.5 miles respectively. If the angle $A\hat{C}B = 60°$, find the distance AB.



$$AB^2 = AC^2 + BC^2 - 2\,AC \cdot BC \cos 60° =$$
$$= 2.5^2 + 4.5^2 - 2 \cdot 2.5 \cdot 4.5 \cdot (1/2) =$$
$$= 6.25 + 20.25 - 11.25 = 15.25 \Rightarrow$$
$$\Rightarrow AB = \sqrt{15.25} \approx 3.905$$

④ Show that $\sin^2x(1+\cot^2x)+\cos^2x(1+\tan^2x)=2$

<u>Solution</u>

$$A=\sin^2x\,(1+\cot^2x)+\cos^2(1+\cot^2x)=$$
$$=\sin^2x+\sin^2x\cot^2x+\cos^2x+\cancel{\cos^2x\cot^2x}\,\cos^2x\tan^2x$$
$$=(\sin^2x+\cos^2x)+(\sin^2x)\frac{\cos^2x}{\sin^2x}+\cos^2x\,\frac{\sin^2x}{\cos^2x}=$$
$$=1+\cos^2x+\sin^2x=1+1=2.$$

⑤ Show that $\cos x+\cos(x+2\pi/3)+\cos(x+4\pi/3)=0$

<u>Solution</u>

Since

$$\cos(x+2\pi/3)=\cos x\cos(2\pi/3)-\sin x\sin(2\pi/3)=$$
$$=\cos x\cos(\pi-\pi/3)-\sin x\sin(\pi-\pi/3)=$$
$$=-\cos x\cos(-\pi/3)+\sin x\sin(-\pi/3)=$$
$$=-\cos x\cos(\pi/3)-\sin x\sin(\pi/3).$$
$$\cos(x+4\pi/3)=\cos(x+\pi+\pi/3)=-\cos(x+\pi/3)=$$
$$=-[\cos x\cos(\pi/3)-\sin x\sin(\pi/3)]$$
$$=-\cos x\cos(\pi/3)+\sin x\sin(\pi/3)$$

it follows that

$$A=\cos x+\cos(x+2\pi/3)+\cos(x+4\pi/3)=$$
$$=\cos x+[-\cos x\cos(\pi/3)-\sin x\sin(\pi/3)]+[-\cos x\cos(\pi/3)+$$
$$+\sin x\sin(\pi/3)]$$
$$=[1-\cos(\pi/3)-\cos(\pi/3)]\cos x=[1-1/2-1/2]\cos x=0.$$

⑥ Find all solutions to the following equation

$\sin x = \sin^3 x$.

<u>Solution</u>

$$\sin x = \sin^3 x \Leftrightarrow \sin^3 x - \sin x = 0 \Leftrightarrow \sin x(\sin^2 x - 1) = 0$$
$$\Leftrightarrow \sin x(\sin x - 1)(\sin x + 1) = 0 \Leftrightarrow$$
$$\Leftrightarrow \sin x = 0 \vee \sin x - 1 = 0 \vee \sin x + 1 = 0 \Leftrightarrow$$
$$\Leftrightarrow \sin x = 0 \vee \sin x = 1 \vee \sin x = -1$$
$$\Leftrightarrow x = k\pi \vee x = 2k\pi + \pi/2 \vee x = 2k\pi - \pi/2$$
$$\Leftrightarrow x = k\pi/2$$

(7) Let $u=(2,7)$ and $v=(3,1)$ be two vectors, with an angle θ between them. Find the following:
(a) $|u+v|$ (b) $3u-2v$ (c) $u\cdot v$ (d) $\cos\theta$

Solution

$$|u+v| = |(2,7)+(3,1)| = |(2+3,\ 7+1)| = |(5,8)| =$$
$$= \sqrt{5^2+8^2} = \sqrt{25+64} = \sqrt{89}$$
$$3u-2v = 3(2,7)-2(3,1) = (6,21)-(6,2) = (6-6,\ 21-2)$$
$$= (0,19)$$
$$u\cdot v = (2,7)\cdot(3,1) = 2\cdot 3+7\cdot 1 = 6+7 = 13$$
$$\cos\theta = \frac{u\cdot v}{|u||v|} = \frac{13}{|(2,7)||(3,1)|} = \frac{13}{\sqrt{2^2+7^2}\ \sqrt{3^2+1^2}} =$$
$$= \frac{13}{\sqrt{4+49}\ \sqrt{9+1}} = \frac{13}{\sqrt{53}\ \sqrt{10}} = \frac{13\sqrt{530}}{530}$$

(8) Find the equation of the ellipse in standard form that has vertices at $(0, \pm 6)$ and foci at $(0, \pm 4)$.

Solution

Given $F_1(0,-4)$, $F_2(0,4)$, $B'(0,-6)$, $B(0,6)$ the corresponding ellipse has center at $(0,0)$, and therefore:

$$(c): \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with $2b = B'B = |6-(-6)| = |6+6| = |12| = 12 \Leftrightarrow b = 6$

$2c = F_1F_2 = |y_{F_2} - y_{F_1}| = |4-(-4)| = |4+4| = |8| = 8 \Leftrightarrow c = 4.$

Since $F_1F_2 \parallel y$-axis $\Rightarrow b > a \Rightarrow b^2 = a^2 + c^2 \Rightarrow$

$\Rightarrow a^2 = b^2 - c^2 = 6^2 - 4^2 = (6-4)(6+4) = 2 \cdot 10 = 20.$

It follows that

$$(c): \frac{x^2}{20} + \frac{y^2}{36} = 1$$

⑨ Find the vertex, focus, directrix of the parabola
(c) : $y = -16x^2$.

<u>Solution</u>

Since (c) : $y = -16x^2 \Leftrightarrow x^2 = (-1/16)\, y \Leftrightarrow$
$\Leftrightarrow x^2 = 4(-1/64)\, y$

It follows that:

- vertex of (c) : $A(0,0)$
- focus of (c) : $F(0, -1/64)$
- directrix of (c) : (ℓ) : $y = +1/64$.

(10) Find the sum of the first N terms of the sequence defined by

$$a_n = \left(\frac{-2}{5}\right)^{n-1}, \quad \text{for } n \in \{1, 2, 3, \dots\}$$

Solution

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} \left(\frac{-2}{5}\right)^{n-1} = \sum_{n=0}^{N-1} \left(\frac{-2}{5}\right)^{n} = \frac{1-(-2/5)^N}{1-(-2/5)} =$$

$$= \frac{1-(-1)^N 2^N/5^N}{1+2/5} = \frac{1-(-1)^N 2^N/5^N}{7/5} =$$

$$= \frac{5}{7}\left[1-(-1)^N \frac{2^N}{5^N}\right] =$$

$$= \frac{5}{7} \, \frac{5^N-(-1)^N 2^N}{5^N} = \frac{5^N-(-1)^N 2^N}{7 \cdot 5^{N-1}}$$

(11) Find the amplitude and period of the motion for the given function model for the displacement of an object moving in a simple harmonic motion given by $y = 3\sin(t/2)$ and sketch the graph over one complete period

Solution

Solve $\dfrac{t}{2} = \dfrac{k\pi}{2} \Longleftrightarrow t = k\pi.$

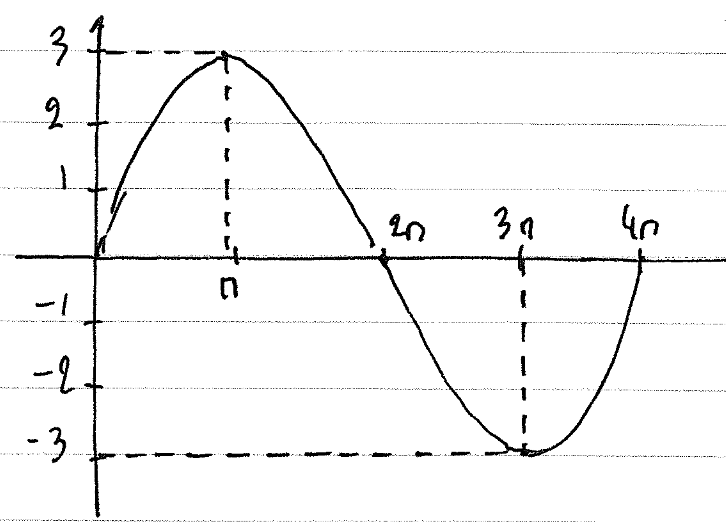
For $k=0$: $\begin{cases} t=0 \\ y=3\sin 0 = 0 \end{cases}$

For $k=1$: $\begin{cases} t=\pi \\ y=1\cdot 3=3 \end{cases}$

For $k=2$: $\begin{cases} t=2\pi \\ y=0\cdot 3=0 \end{cases}$

For $k=3$: $\begin{cases} t=3\pi \\ y=-1\cdot 3=-3 \end{cases}$

For $k=4$: $\begin{cases} t=4\pi \\ y=0\cdot 3=0 \end{cases}$



Amplitude: 3

Period: 4π