Math 1450.03

Exam 4 Solution

Find $\cos\vartheta$ of the angle ϑ between the vectors $\vec{a} = (-1, 2)$ and $\vec{b} = (-3, \sqrt{2})$.

Solution

$$\cos\vartheta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{(-1,2)\cdot(-3,\sqrt{2})}{|(-1,2)||(-3,\sqrt{2})|} = \frac{(-1)(-3)+2\sqrt{2}}{\sqrt{(-1)^2+2^2}\,\sqrt{(-3)^2+(\sqrt{2})^2}}$$

$$= \frac{3+2\sqrt{2}}{\sqrt{1+4}\,\sqrt{9+2}} = \frac{3+2\sqrt{2}}{\sqrt{5}\,\sqrt{11}} = \frac{(3+2\sqrt{2})\sqrt{55}}{55}$$

Evaluate $c = |\sqrt{3}\vec{a} + 2\vec{b}|$ for $\vec{a} = (-1, -\sqrt{2})$ and $b = (+1, -2\sqrt{2})$

<u>Solution</u>

$$c = |\sqrt{3}\vec{a} + 2\vec{b}| = |\sqrt{3}(-1, -\sqrt{2}) + 2(1, -2\sqrt{2})| =$$
$$= |(-\sqrt{3}, -\sqrt{6}) + (2, -4\sqrt{2})| = |(2-\sqrt{3}, -\sqrt{6} - 4\sqrt{2})|$$
$$= \sqrt{(2-\sqrt{3})^2 + (-\sqrt{6} - 4\sqrt{2})^2} =$$
$$= \sqrt{2^2 - 2\cdot 2\sqrt{3} + (\sqrt{3})^2 + (\sqrt{6})^2 + 2\sqrt{6}(4\sqrt{2}) + (4\sqrt{2})^2}$$
$$= \sqrt{4 - 4\sqrt{3} + 3 + 6 + 8\sqrt{12} + 32}$$
$$= \sqrt{45 - 4\sqrt{3} + 16\sqrt{3}}$$
$$= \sqrt{45 + 12\sqrt{3}}$$

Use basic sums to show that

$1^2+3^2+\dots+(2n-1)^2 = (1/3)n(2n-1)(2n+1)$

<u>Solution</u>

$$1^2+3^2+\dots+(2n-1)^2 = \sum_{k=1}^{n}(2k-1)^2 = \sum_{k=1}^{n}(4k^2-4k+1) =$$

$$= 4\sum_{k=1}^{n}k^2 - 4\sum_{k=1}^{n}k + n = 4S_2(n) - 4S_1(n) + n =$$

$$= 4(1/6)n(n+1)(2n+1) - 4(1/2)n(n+1) + n$$
$$= (2/3)n(n+1)(2n+1) - 2n(n+1) + n$$
$$= (1/3)n\left[2(n+1)(2n+1) - 6(n+1) + 3\right] =$$
$$= (1/3)n\left[2(2n^2+n+2n+1) - 6n - 6 + 3\right] =$$
$$= (1/3)n(4n^2+2n+4n+2-6n-6+3)$$
$$= (1/3)n(4n^2+(2+4-6)n+(2-6+3))$$
$$= (1/3)n(4n^2-1) = (1/3)n(2n-1)(2n+1)$$

Evaluate the following geometric series: $\sum_{k=0}^{n} (\sqrt{2})^{2k-1}$

<u>Solution</u>

$$\sum_{k=0}^{n} (\sqrt{2})^{2k-1} = \frac{1}{\sqrt{2}} \sum_{k=0}^{n} (\sqrt{2})^{2k} = \frac{1}{\sqrt{2}} \sum_{k=0}^{n} 2^{k} =$$

$$= \frac{1}{\sqrt{2}} \frac{1-2^{n+1}}{1-2} = \frac{2^{n+1}-1}{\sqrt{2}} =$$

$$= \frac{\sqrt{2}\,(2^{n+1}-1)}{2}$$

Evaluate the following infinite geometric series

$$\sum_{k=2}^{+\infty} (-1)^k \left(\frac{1}{\sqrt{3}}\right)^k$$

<u>Solution</u>

$$\sum_{k=2}^{+\infty} (-1)^k \left(\frac{1}{\sqrt{3}}\right)^k = \sum_{k=0}^{+\infty} \left(\frac{-1}{\sqrt{3}}\right)^k - \left(\frac{-1}{\sqrt{3}}\right)^0 - \left(\frac{-1}{\sqrt{3}}\right)^1 =$$

$$= \frac{1}{1-(-1/\sqrt{3})} - 1 - \left(\frac{-1}{\sqrt{3}}\right) =$$

$$= \frac{\sqrt{3}}{\sqrt{3}+1} - 1 + \frac{1}{\sqrt{3}} =$$

$$= \frac{\sqrt{3}\sqrt{3} - \sqrt{3}(\sqrt{3}+1) + (\sqrt{3}+1)}{\sqrt{3}(\sqrt{3}+1)} = \frac{3-3-\sqrt{3}+\sqrt{3}+1}{\sqrt{3}(\sqrt{3}+1)} =$$

$$= \frac{1}{\sqrt{3}(\sqrt{3}+1)} = \frac{\sqrt{3}-1}{\sqrt{3}(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{\sqrt{3}-1}{\sqrt{3}(3-1)} =$$

$$= \frac{\sqrt{3}(\sqrt{3}-1)}{6}$$