Exam 3 Solution

Math 1450.03

Find all solutions to the following equation $\cos(\pi/6+5x)+\sin(-3x)=0$.

Solution

$$\cos(\pi/6+5x)+\sin(-3x)=0 \Leftrightarrow \cos(\pi/6+5x)=-\sin(-3x) \Leftrightarrow$$
$$\Leftrightarrow \cos(\pi/6+5x)=\sin(3x) \Leftrightarrow \cos(\pi/6+5x)=\cos(\pi/2-3x) \Leftrightarrow$$
$$\Leftrightarrow \pi/6+5x=2k\pi+(\pi/2-3x) \vee \pi/6+5x=2k\pi-(\pi/2-3x) \Leftrightarrow$$
$$\Leftrightarrow 5x+3x=2k\pi+\pi/2-\pi/6 \vee \pi/6+5x=2k\pi-\pi/2+3x \Leftrightarrow$$
$$\Leftrightarrow 8x=2k\pi+\pi/3 \vee 5x-3x=2k\pi-\pi/6-\pi/2$$
$$\Leftrightarrow 8x=2k\pi+\pi/3 \vee 2x=2k\pi-2\pi/3$$
$$\Leftrightarrow x=\frac{k\pi}{4}+\frac{\pi}{24} \vee x=k\pi-\frac{\pi}{3}$$

Find all solutions to the following equation

$2\sin^2 x + \sqrt{3} = (2+\sqrt{3})\sin x$

Solution

Define $y = \sin x$. Then

$2\sin^2 x + \sqrt{3} = (2+\sqrt{3})\sin x \Leftrightarrow 2y^2 + \sqrt{3} = (2+\sqrt{3})y \Leftrightarrow$

$\Leftrightarrow 2y^2 - (2+\sqrt{3})y + \sqrt{3} = 0 \Leftrightarrow 2y^2 - 2y - \sqrt{3}y + \sqrt{3} = 0 \Leftrightarrow$

$\Leftrightarrow 2y(y-1) - \sqrt{3}(y-1) = 0 \Leftrightarrow (2y-\sqrt{3})(y-1) = 0 \Leftrightarrow$

$\Leftrightarrow 2y - \sqrt{3} = 0 \vee y - 1 = 0 \Leftrightarrow y = \sqrt{3}/2 \vee y = 1$

$\Leftrightarrow \sin x = \sqrt{3}/2 = \sin(\pi/3) \vee \sin x = 1$

$\Leftrightarrow x = 2k\pi + \pi/3 \vee x = (2k+1)\pi - \pi/3 \vee x = 2k\pi + \pi/2$

Find all solutions to the following equation:

$\sin(2x) = \sin^3 x$

Solution

$\sin(2x) = \sin^3 x \Leftrightarrow \sin(2x) - \sin^3 x = 0 \Leftrightarrow 2\sin x \cos x - \sin^3 x = 0$

$\Leftrightarrow \sin x\,(2\cos x - \sin^2 x) = 0 \Leftrightarrow \sin x\,[2\cos x - (1-\cos^2 x)] = 0$

$\Leftrightarrow \sin x\,(2\cos x - 1 + \cos^2 x) = 0 \Leftrightarrow$

$\Leftrightarrow \sin x = 0 \vee \cos^2 x + 2\cos x - 1 = 0. \qquad (1)$

We note that $\sin x = 0 \Leftrightarrow x = k\pi$

Define $y = \cos x$. Then $\cos^2 x + 2\cos x - 1 = 0 \Leftrightarrow y^2 + 2y - 1 = 0$

$\Delta = 2^2 - 4\cdot 1\cdot(-1) = 4 + 4 = 8 \Rightarrow$

$\Rightarrow y_{1,2} = \dfrac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2} \begin{cases} -1+\sqrt{2} \\ -1-\sqrt{2} \end{cases}$

and note that

$y = -1+\sqrt{2} \Leftrightarrow \cos x = -1+\sqrt{2} \Leftrightarrow x = 2k\pi + \mathrm{Arccos}(-1+\sqrt{2}) \vee$

$\vee\, x = 2k\pi - \mathrm{Arccos}(-1+\sqrt{2})$

$y = -1-\sqrt{2} \Leftrightarrow \cos x = -1-\sqrt{2} \leftarrow$ no solutions.

It follows that

Eq. $(1) \Leftrightarrow x = k\pi \vee x = 2k\pi + \mathrm{Arccos}(\sqrt{2}-1) \vee$

$\vee\, x = 2k\pi - \mathrm{Arccos}(\sqrt{2}-1)$

Find all solutions to the following equation:

$3\sin x - \sqrt{3}\cos x = 3$

Solution

$3\sin x - \sqrt{3}\cos x = 3 \Leftrightarrow \sin x - (\sqrt{3}/3)\cos x = 1 \Leftrightarrow$

$\Leftrightarrow \sin x - \tan(\pi/6)\cos x = 1 \Leftrightarrow \sin x - \dfrac{\sin(\pi/6)}{\cos(\pi/6)}\cos x = 1 \Leftrightarrow$

$\Leftrightarrow \sin x\cos(\pi/6) - \sin(\pi/6)\cos x = \cos(\pi/6) \Leftrightarrow \sin(x-\pi/6) = \cos(\pi/6)$

$\Leftrightarrow \sin(x-\pi/6) = \sin(\pi/2-\pi/6) \Leftrightarrow \sin(x-\pi/6) = \sin(\pi/3) \Leftrightarrow$

$\Leftrightarrow x-\pi/6 = 2k\pi+\pi/3 \vee x-\pi/6 = (2k+1)\pi - \pi/3 \Leftrightarrow$

$\Leftrightarrow x = 2k\pi+\pi/3+\pi/6 \vee x = (2k+1)\pi - \pi/3+\pi/6 \Leftrightarrow$

$\Leftrightarrow x = 2k\pi+\pi/2 \vee x = (2k+1)\pi - \pi/6$

Given a triangle with $b=1$, $c=1+\sqrt{3}$, $\hat{A}=60°$, find the exact value of a.

Solution

$a^2 = b^2+c^2-2bc\cos\hat{A} = 1^2+(1+\sqrt{3})^2-2\cdot1\cdot(1+\sqrt{3})\cdot(1/2)$

$= 1+(1+2\sqrt{3}+3)-(1+\sqrt{3}) =$

$= 1+1+2\sqrt{3}+3-1-\sqrt{3} = 4+\sqrt{3} \Rightarrow$

$\Rightarrow a = \sqrt{4+\sqrt{3}}$

Given a triangle with $\hat{B} = 30°$, $\hat{C} = 45°$, $b = \sqrt{2}$, find the exact value of a.

Solution

We note that

$$\hat{A} = 180° - \hat{B} - \hat{C} = 180° - 30° - 45° \Rightarrow$$

$$\Rightarrow \sin\hat{A} = \sin(180° - 30° - 45°) = -\sin(-30° - 45°) = \sin(30° + 45°)$$

$$= \sin 30° \cos 45° + \sin 45° \cos 30° =$$

$$= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} = \frac{\sqrt{2}(1+\sqrt{3})}{4}$$

and therefore:

$$\frac{a}{\sin\hat{A}} = \frac{b}{\sin\hat{B}} \Leftrightarrow a\sin\hat{B} = b\sin\hat{A} \Leftrightarrow$$

$$\Leftrightarrow a = \frac{b\sin\hat{A}}{\sin\hat{B}} = \frac{\sqrt{2}\left(\frac{\sqrt{2}(1+\sqrt{3})}{4}\right)}{\frac{1}{2}} = \frac{2(\sqrt{2})^2(1+\sqrt{3})}{4} =$$

$$= 1 + \sqrt{3}.$$