Math 1450.01

Exam 4 Solution

Find the ~~angle~~ $\cos\vartheta$ of the angle ϑ between the vectors $\vec{a} = (2,5)$ and $\vec{b} = (-1, \sqrt{3})$

Solution

$$\cos\vartheta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{(2,5)\cdot(-1,\sqrt{3})}{|(2,5)|\cdot|(-1,\sqrt{3})|} =$$

$$= \frac{2\cdot(-1)+5\sqrt{3}}{\sqrt{2^2+5^2}\ \sqrt{(-1)^2+(\sqrt{3})^2}} = \frac{-2+5\sqrt{3}}{\sqrt{4+25}\ \sqrt{1+3}} =$$

$$= \frac{-2+5\sqrt{3}}{\sqrt{29}\ \sqrt{4}} = \frac{(-2+5\sqrt{3})\sqrt{29}}{29\cdot 2}$$

Evaluate $c = |2\vec{a} + \sqrt{3}\,\vec{b}|$ for $\vec{a} = (1,-2)$ and $\vec{b} = (-2,-\sqrt{3})$

Solution

$$c = |2\vec{a} + \sqrt{3}\,\vec{b}| = |2(1,-2) + \sqrt{3}(-2,-\sqrt{3})| =$$
$$= |(2,-4) + (-2\sqrt{3}, -3)| = |(2-2\sqrt{3}, -4-3)| =$$
$$= |(2-2\sqrt{3}, -7)| = \sqrt{(2-2\sqrt{3})^2 + (-7)^2} =$$
$$= \sqrt{2^2 - 2\cdot 2\cdot 2\sqrt{3} + (2\sqrt{3})^2 + 49}$$
$$= \sqrt{4 - 8\sqrt{3} + 12 + 49} = \sqrt{65 - 8\sqrt{3}}$$

Use Basic sums to show that

$1\cdot 2+2\cdot 5+\cdots+n(3n-1)=n^2(n+1)$

<u>Solution</u>

$$1\cdot 2+2\cdot 5+\cdots+n(3n-1)=\sum_{k=1}^{n}k(3k-1)=\sum_{k=1}^{n}(3k^2-k)=$$

$$=3\sum_{k=1}^{n}k^2-\sum_{k=1}^{n}k=3S_2(n)-S_1(n)=$$

$$=\frac{3n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}=\frac{n(n+1)}{2}\left[(2n+1)-1\right]$$

$$=\frac{n(n+1)(2n+1-1)}{2}=\frac{2n^2(n+1)}{2}=n^2(n+1)$$

Evaluate the following geometric series:

$$\sum_{k=0}^{n} (-1)^k (1/2)^{2k+1}$$

<u>Solution</u>

$$\sum_{k=0}^{n} (-1)^k (1/2)^{2k+1} = \sum_{k=0}^{n} (-1)^k (1/2)^{2k} (1/2) = \sum_{k=0}^{n} (1/2)(-1)^k (1/4)^k$$

$$= \frac{1}{2} \sum_{k=0}^{n} (-1/4)^k = \frac{1}{2} \frac{1-(-1/4)^{n+1}}{1-(-1/4)} =$$

$$= \frac{1}{2} \frac{1-(-1)^{n+1}(1/4)^{n+1}}{5/4} = \frac{1}{2} \frac{4}{5} \left[1+(-1)^n \frac{1}{4^{n+1}} \right]$$

$$= \frac{2}{5} \left[\frac{4^{n+1}+(-1)^n}{4^{n+1}} \right]$$

Evaluate the following infinite geometric series

$$\sum_{k=1}^{+\infty} (-1)^k \left(\frac{1}{\sqrt{2}}\right)^{k+1}$$

Solution

$$\sum_{k=1}^{+\infty} (-1)^k \left(\frac{1}{\sqrt{2}}\right)^{k+1} = \frac{1}{\sqrt{2}} \sum_{k=1}^{+\infty} \left(\frac{-1}{\sqrt{2}}\right)^k = \frac{1}{\sqrt{2}} \left[\sum_{k=0}^{+\infty} \left(\frac{-1}{\sqrt{2}}\right)^k - \left(\frac{-1}{\sqrt{2}}\right)^0 \right]$$

$$= \frac{1}{\sqrt{2}} \left[\frac{1}{1-(-1/\sqrt{2})} - 1 \right] = \frac{1}{\sqrt{2}} \left[\frac{1}{1+1/\sqrt{2}} - 1 \right] =$$

$$= \frac{1}{\sqrt{2}} \left[\frac{\sqrt{2}}{\sqrt{2}+1} - 1 \right] = \frac{1}{\sqrt{2}} \left[\frac{\sqrt{2}-(\sqrt{2}+1)}{\sqrt{2}+1} \right] =$$

$$= \frac{1}{\sqrt{2}} \frac{\sqrt{2}-\sqrt{2}-1}{\sqrt{2}+1} = \frac{-1}{\sqrt{2}(\sqrt{2}+1)} = \frac{-(\sqrt{2}-1)}{\sqrt{2}(\sqrt{2}+1)(\sqrt{2}-1)} =$$

$$= \frac{1-\sqrt{2}}{\sqrt{2}((\sqrt{2})^2-1^2)} = \frac{1-\sqrt{2}}{(2-1)\sqrt{2}} = \frac{1-\sqrt{2}}{\sqrt{2}} =$$

$$= \frac{\sqrt{2}(1-\sqrt{2})}{2} = \frac{\sqrt{2}-2}{2}$$

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