CHAPTER 1

THE FOUNDATIONS OF GEOMETRY

▼ Introduction

The development of Geometry begins with the introduction of certain fundamental concepts which we do not attempt to define in terms of other concepts. Instead we introduce a collection of statements, called axioms, that describe the behaviour of the fundamental concepts. These statements therefore are an indirect "definition", an axiomatic definition, of the fundamental concepts. We say that the fundamental concepts and the axioms together are the foundations of geometry.

In this study we will focus primarily on plane geometry. The first axiomatic development of geometry was done by Euclid (330 - 275 BC) in his monumental work "Elements". Hilbert developed a complete and rigorous ~~done~~ axiomatic theory of geometry in 1899 in his classic work ("Grundlagen der Geometrie"). Hilbert's theory is very complicated because he develops geometry completely from scratch. The theory that we will develop here will be much simpler because we will take as given both set theory and real number theory. This choice makes it substantially simpler to deal with issues of continuity.

The axioms of geometry

We introduce the following fundamental concepts:

a) A set of <u>points</u> $\mathcal{E}$, which we call <u>plane</u>

b) A function $\mu : \mathcal{E} \times \mathcal{E} \longrightarrow \mathbb{R}$, which we call <u>distance</u>

c) A set of subsets of $\mathcal{E}$, which we denote as $\mathcal{L}$. ~~and~~
The elements of $\mathcal{L}$ are called <u>lines</u>.

Given these concepts, we assume the following axioms:

Axioms of incidence

<u>Axiom 1</u>: For any two points A, B, there is a line (ε) that passes through these points.

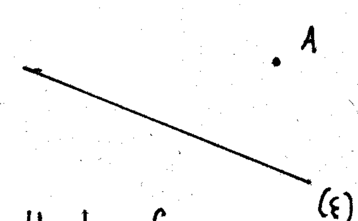
$$\boxed{(\forall A, B \in \mathcal{E})(\exists (\varepsilon) \in \mathcal{L})(A, B \in (\varepsilon))}$$

<u>Axiom 2</u>: If two lines (ε_1) and (ε_2) pass through the same distinct points A, B, then the lines are identical.

$$\boxed{(\forall (\varepsilon_1), (\varepsilon_2) \in \mathcal{L})(A, B \in (\varepsilon_1) \wedge A, B \in (\varepsilon_2) \Longrightarrow (\varepsilon_1) = (\varepsilon_2))}$$

<u>Axiom 3</u>: For every line (ε) there is at least one point A that does not lie on that line.

$$\boxed{(\forall (\varepsilon) \in \mathcal{L})(\exists A \in \mathcal{E})(A \notin (\varepsilon))}$$



A consequence of these three axioms is that for any two lines $(\varepsilon_1), (\varepsilon_2)$ one of the following has to be true:

a) The lines do not intersect: $(\varepsilon_1) \cap (\varepsilon_2) = \varnothing$

b) The lines intersect at only one point: $(\varepsilon_1) \cap (\varepsilon_2) = \{A\}$

c) The lines are identical: $(\varepsilon_1) = (\varepsilon_2)$.

Axiom of parallels

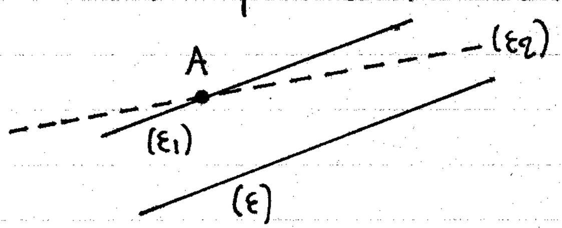
Definition: We say that two lines (ε_1) and (ε_2) are parallel ($(\varepsilon_1) \parallel (\varepsilon_2)$) if and only if they lie on the same plane and they have no points in common.

$$\boxed{(\varepsilon_1) \parallel (\varepsilon_2) \Longleftrightarrow (\varepsilon_1), (\varepsilon_2) \in \mathcal{L} \wedge (\varepsilon_1) \cap (\varepsilon_2) = \varnothing}$$

Axiom 4: If two lines $(\varepsilon_1), (\varepsilon_2)$ pass through the same point A and are both parallel to the same line (ε), then they are identical

$$\boxed{(\varepsilon_1) \cap (\varepsilon_2) = \{A\} \wedge (\varepsilon_1) \parallel (\varepsilon) \wedge (\varepsilon_2) \parallel (\varepsilon) \Rightarrow (\varepsilon_1) = (\varepsilon_2).}$$

This axiom is particularly known as Euclid's axiom and it says that we may not cast more than one parallel line to a given line from the same point. There has been considerable interest in "non-Euclidean" geometries where we drop this particular axiom.



Note that this axiom does not assert the existence of one parallel for any point A. That assertion can be proven as a theorem from the axioms of perpendiculars.

Axioms of distance

The established notation for the distance $\mu(A,B)$ between two points A, B is $\mu(A,B) = AB$. We will reserve however the letter μ to denote distance for a few cases where the established notation is inconvenient.

<u>Axiom 5</u> : The distance between a point and itself is zero.

$$\boxed{(\forall A \in \mathcal{E})(AA = 0)}$$

<u>Axiom 6</u> : Distance between two points is independent of the order of the points (i.e. it is a symmetric function)

$$\boxed{(\forall A, B \in \mathcal{E})(AB = BA)}$$

<u>Axiom 7</u> : (<u>The triangle inequality</u>)

$$\boxed{(\forall A, B, \Gamma \in \mathcal{E})(AB \leqslant A\Gamma + \Gamma B)}$$

A direct consequence of the distance axioms is the following proposition which need not be taken as an axiom itself :

<u>Proposition</u>: The distance between any two points is always positive or zero :

$$\boxed{(\forall A, B \in \mathcal{E})(AB \geqslant 0)}$$

<u>Proof</u>

$$AB = \frac{1}{2}(AB + AB) =$$

$\longrightarrow$ (Axiom 6)

$$= \frac{1}{2}(AB + BA) \geqslant$$

$\longrightarrow$ (Axiom 7)

$$\geqslant \frac{1}{2} AA =$$

$\longrightarrow$ (Axiom 5)

$$= \frac{1}{2} \cdot 0 = 0 \Rightarrow AB \geqslant 0, \ \forall A, B \in \mathcal{E} \qquad \square$$

Axiom of continuity

The axiom of continuity essentially makes the statement that every line $(\varepsilon) \in \mathcal{L}$ is isomorphic, in a certain way, with the set of real numbers. $\mathbb{R}$. The "certain way" is closely related with the notion of distance.

Axiom 8 : Let $(\varepsilon) \in \mathcal{L}$ be any given line. Then, there exists a mapping $\psi : (\varepsilon) \longrightarrow \mathbb{R}$ such that all of the following statements are true:

a) $\psi[(\varepsilon)] = \mathbb{R}$

b) $(\forall A, B \in (\varepsilon))(\psi(A) = \psi(B) \Rightarrow A = B)$ $\quad [\psi \text{ "1-1"}]$

c) $(\forall A, B \in (\varepsilon))(AB = |\psi(A) - \psi(B)|)$

► We call ψ a continuity map of the line (ε). ◄

Properties (a) and (b) mean that ψ is a bijection (every point corresponds to a distinct number and there are points for the entire set of real numbers).

Property (c) means that the mapping ψ is closely related with the distance function.

A direct consequence of the axiom of continuity is that every line has infinite points. More specifically:

Proposition : Every line has cardinality $2^{\aleph_0}$

$$\boxed{\forall (\varepsilon) \in \mathcal{L} : \ |(\varepsilon)| = 2^{\aleph_0}}$$

Proof

Let $(\varepsilon) \in \mathcal{L}$ be given. Let $\psi : (\varepsilon) \longrightarrow \mathbb{R}$ be the continuity map of (ε) (Axiom 8).

$$\left.\begin{array}{l} \psi[(\varepsilon)] = \mathbb{R} \\ \psi \text{ "1-1"} \end{array}\right\} \Rightarrow \psi \text{ is a bijection} \Rightarrow (\varepsilon), \mathbb{R} \text{ are equivalent}$$

$$\Rightarrow |(\varepsilon)| = |\mathbb{R}| = 2^{\aleph_0} \Rightarrow |(\varepsilon)| = 2^{\aleph_0}. \quad \square$$

Axiom of existence

In order for a line to exist at all, we need the existence of two points A, B. However that has never been asserted yet. So, we introduce the axiom of existence:

<u>Axiom 9</u>: There exist at least two distinct points A, B.

$$\boxed{\exists A, B \in \mathcal{E} : A \neq B}$$

Axioms of perpendiculars.

The axioms of perpendiculars are statements about two concepts that are not fundamental, that can be defined by the fundamental concepts: the <u>segment bisector</u> and the <u>perpendicular relation</u>.

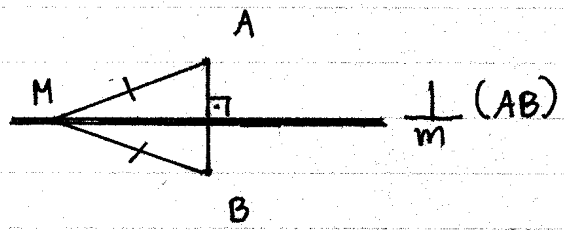
<u>Definition</u>: Let $A, B \in \mathcal{E}$ be two distinct points $A \neq B$. Then the <u>segment bisector</u> $\underset{m}{\perp}(AB)$ is the set of all points that are equidistant from A and B:

$$\boxed{\underset{m}{\perp}(AB) = \{ M \in \mathcal{E} \mid AM = MB \}}$$

<u>Axiom 10</u>: The segment bisector is a line.

$$\boxed{(\forall A, B \in \mathcal{E})\left(\underset{m}{\perp}(AB) \in \mathcal{L}\right)}$$

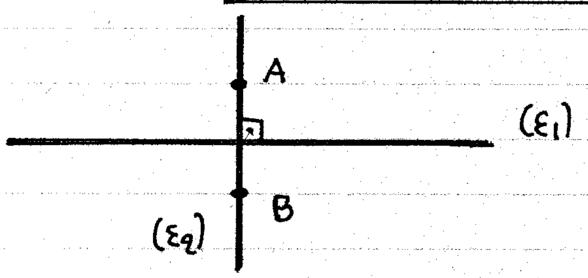
This is an ~~ext~~ extremely powerful statement. It means that all the axioms that we assume for lines also apply on the perpendicular bisector



Axiom 10 makes it possible to define the perpendicular relation:

Definition: Let $(\varepsilon_1), (\varepsilon_2) \in \mathcal{L}$ be two lines. We say that they are perpendicular if and only if there are two points $A, B \in (\varepsilon_2)$ on (ε_2) such that (ε_1) is the segment bisector of A, B.

$$\boxed{(\varepsilon_1) \perp (\varepsilon_2) \Leftrightarrow \exists A, B \in (\varepsilon_2) : (\varepsilon_1) = \underset{m}{\perp}(AB)}$$

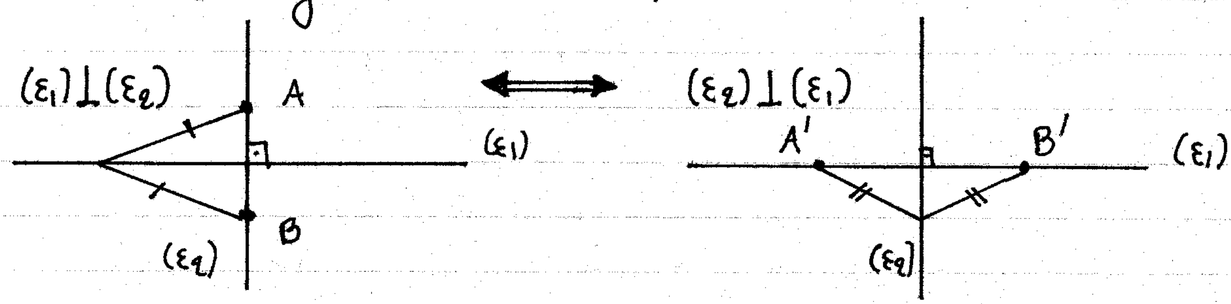


Note that the $\perp$ relation is symmetric. Because it is not possible to prove this from the other axioms, we introduce it as another axiom

Axiom 11: If $(\varepsilon_1) \perp (\varepsilon_2)$, then $(\varepsilon_2) \perp (\varepsilon_1)$

$$\boxed{(\forall (\varepsilon_1), (\varepsilon_2) \in \mathcal{L})\big((\varepsilon_1) \perp (\varepsilon_2) \rightarrow (\varepsilon_2) \perp (\varepsilon_1)\big)}$$

This means that we can find points A', B' on (ε_1) such that (ε_2) is the segment bisector of $A'B'$:

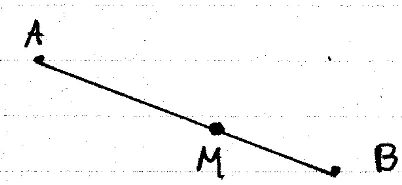


Axiom of half-planes

This axiom states the idea that a line divides the plane into two half-planes.

Definition: Let $A, B \in \mathcal{E}$ be any two points. The <u>segment</u> $\overline{AB}$ is defined as the set of points M such that $AM + MB = AB$:

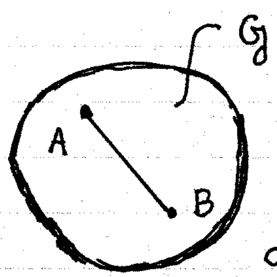
$$\boxed{\overline{AB} = \{ M \in \mathcal{E} \mid AM + MB = AB \}}$$



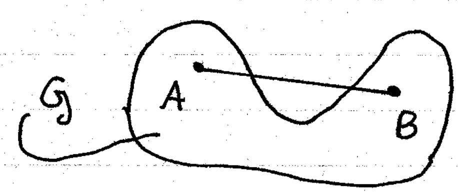
Definition: Let $\mathcal{G} \in \mathcal{P}(\mathcal{E})$ be an arbitrary set of points. We say that $\mathcal{G}$ is <u>convex</u> if and only if • the segment $\overline{AB}$ defined by any points of $\mathcal{G}$ lies entirely in $\mathcal{G}$:

$$\boxed{\mathcal{G} \text{ convex} \Leftrightarrow (\forall A, B \in \mathcal{G})(\overline{AB} \subseteq \mathcal{G})}$$

<u>example</u>



convex



not convex

Axiom 12: For every line $(\varepsilon) \in \mathcal{L}$, there are two sets of points $H_1, H_2 \in \mathcal{P}(\mathcal{E})$ such that the following statements are all true:

a) $(\varepsilon) \cup H_1 \cup H_2 = \mathcal{E}$

b) $H_1 \cap H_2 = \varnothing$, $(\varepsilon) \cap H_1 = (\varepsilon) \cap H_2 = \varnothing$

c) H_1, H_2 are convex

d) $(\forall A \in H_1)(\forall B \in H_2)(\exists \Gamma \in (\varepsilon))(\overline{AB} \cap (\varepsilon) = \{\Gamma\})$

▸ We call the sets H_1, H_2 <u>half-planes</u> and $(\varepsilon)/H_1/H_2$ the <u>half-plane partition</u> of the line (ε). An immediate consequence of axiom 12 is that the half plane partition is unique.

<u>Proposition</u>: Let $(\varepsilon)/H_1/H_2$, $(\varepsilon)/G_1/G_2$ be two half-plane partitions. Then if:

$$A \in H_1 \wedge A \in G_1 \Rightarrow H_1 = G_1 \wedge H_2 = G_2 \quad ◀$$

<u>Proof</u>

First we show that $H_1 = G_1$

<u>$(\Rightarrow)$:</u> Let $B \in H_1 \Rightarrow \overline{AB} \subseteq H_1$ (H_1 convex)

$\Rightarrow \overline{AB} \cap (\varepsilon) = \varnothing$ (bc. $H_1 \cap (\varepsilon) = \varnothing$)

$\Rightarrow B \in G_1$ (otherwise $A \in G_1 \wedge B \in G_2 \Rightarrow$

$\Rightarrow \overline{AB} \cap (\varepsilon) \neq \varnothing$)

therefore $(\forall B \in H_1)(B \in G_1) \Rightarrow H_1 \subseteq G_1$ (1)

<u>$(\Leftarrow)$</u> : Let $B \in G_1 \Rightarrow \overline{AB} \subseteq G_1 \Rightarrow AB \cap (\varepsilon) = \varnothing \Rightarrow B \in H_1$

therefore $(\forall B \in G_1)(B \in H_1) \Rightarrow G_1 \subseteq H_1$ (2)

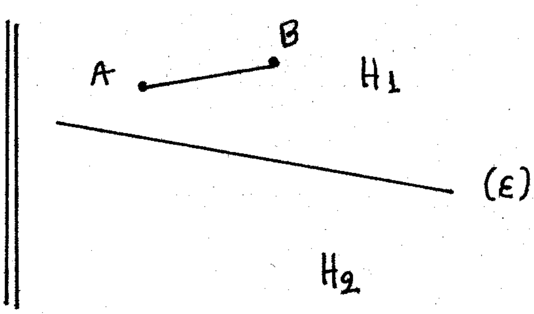
(1),(2) $\Rightarrow$ <u>$H_1 = G_1$</u> ◀

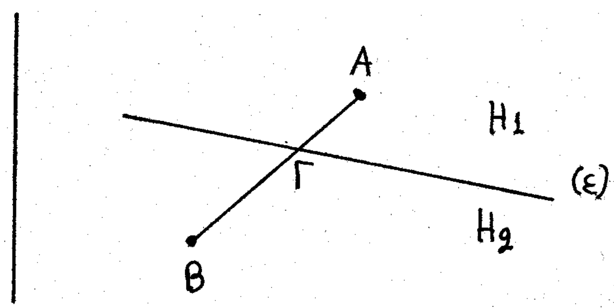
Now we show that $H_2 = G_2$.

Let $B \in H_2 \Leftrightarrow (\exists \Gamma \in (\varepsilon))(\overline{AB} \cap (\varepsilon) = \{\Gamma\})$

$\Leftrightarrow B \in G_2$

therefore $(\forall B)(B \in H_2 \Leftrightarrow B \in G_2) \Rightarrow$ <u>$H_2 = G_2$</u> □





In the following sections we will prove some immediate consequences of our axioms. These statements will appear "intuitively obvious", so the purpose of this development is to show that our choice of axioms is strong enough to encompass all these obvious statements.

Consequences of the continuity axiom

First we develop a lemma that strengthens the continuity axiom:

Lemma: Let $A, B \in (\varepsilon)$ be two points on the line (ε). Then there is a continuity map $\psi : (\varepsilon) \longrightarrow \mathbb{R}$ such that it also has the following properties:
a) $\psi(A) = 0$
b) $\psi(B) > 0$

Proof

- Let ψ_0 be a continuity map of (ε), whose existence is asserted by the axiom of continuity.
- Define: $\lambda = \begin{cases} 1 & , \text{ if } \psi_0(B) > \psi_0(A) \\ -1 & , \text{ if } \psi_0(B) < \psi_0(A). \end{cases}$

(note that $A \neq B \Rightarrow \psi_0(A) \neq \psi_0(B)$).
- Define: $\psi(M) = \lambda[\psi_0(M) - \psi_0(A)]$, $\forall M \in (\varepsilon)$

We can easily verify that:
$\psi(A) = \lambda[\psi_0(A) - \psi_0(A)] = 0$
$\psi(B) = \lambda[\psi_0(B) - \psi_0(A)] = |\psi_0(B) - \psi_0(A)| > 0$
It remains to show that ψ is a continuity map of (ε).
a) $\underline{\psi[(\varepsilon)] = \mathbb{R}}$: Let $x \in \mathbb{R}$ be given.

$$x \in \psi[(\varepsilon)] \Leftrightarrow (\exists M \in (\varepsilon))(\psi(M) = x) \Leftrightarrow$$
$$\Leftrightarrow (\exists M \in (\varepsilon))(\lambda[\psi_0(M) - \psi_0(A)] = x) \Leftrightarrow$$

$$\Leftrightarrow (\exists\, M \in (\varepsilon))\left(\psi_0(M) = \psi_0(A) + \frac{x}{\lambda} \right)$$

$$\Leftrightarrow \left(\psi_0(A) + \frac{x}{\lambda}\right) \in \psi_0[(\varepsilon)] = \mathbb{R} \leftarrow \text{true} \quad \forall x \in \mathbb{R}.$$

Therefore: $\forall x \in \mathbb{R} : x \in \psi[(\varepsilon)] \Rightarrow \mathbb{R} \subseteq \psi[(\varepsilon)]$. (1)

By construction also $\psi[(\varepsilon)] \subseteq \mathbb{R} \underset{(1)}{\Rightarrow} \psi[(\varepsilon)] = \mathbb{R}$.

b) Let $\Gamma, \Delta \in (\varepsilon)$. be given.

$$\psi(\Gamma) = \psi(\Delta) \Rightarrow \lambda[\psi_0(\Gamma) - \psi_0(A)] = \lambda[\psi_0(\Delta) - \psi_0(A)] \Rightarrow$$
$$\Rightarrow \psi_0(\Gamma) = \psi_0(\Delta) \Rightarrow$$
$$\Rightarrow \Gamma = \Delta$$

therefore $(\forall\, \Gamma, \Delta \in (\varepsilon))(\psi(\Gamma) = \psi(\Delta) \Rightarrow \Gamma = \Delta)$.

c) Let $\Gamma, \Delta \in (\varepsilon)$ be given. Recall that $|\lambda| = 1$. Then

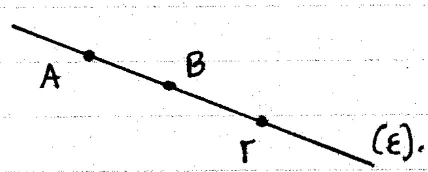
$$|\psi(\Gamma) - \psi(\Delta)| = |\lambda(\psi_0(\Gamma) - \psi_0(A)) - \lambda(\psi_0(\Delta) - \psi_0(A))| =$$
$$= |\lambda| \cdot |\psi_0(\Gamma) - \psi_0(A) - \psi_0(\Delta) + \psi_0(A)| =$$
$$= 1 \cdot |\psi_0(\Gamma) - \psi_0(\Delta)| = \Gamma\Delta, \quad \forall\, \Gamma, \Delta \in (\varepsilon).$$

From (a),(b),(c) it follows that ψ_0 is indeed a continuity map. □

We now use this lemma to prove a few results about "colinearity".

Definition: We say that three points $A, B, \Gamma \in \mathcal{E}$ are colinear if and ~~only~~ only if there is a line (ε) that passes through all of these three points:

$$\boxed{A, B, \Gamma \in \mathcal{E} \text{ are colinear} \Leftrightarrow (\exists\, (\varepsilon) \in \mathcal{L})(A, B, \Gamma \in (\varepsilon)).}$$



← Three collinear points.

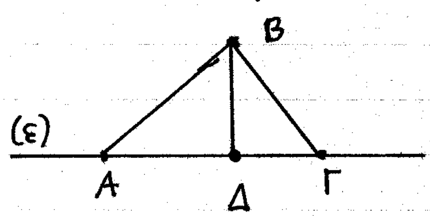
<u>Theorem</u>: Let $A, B, \Gamma \in \mathcal{E}$ be 3 points of the plane. Then:

$$\boxed{AB + B\Gamma = A\Gamma \Rightarrow A, B, \Gamma \text{ collinear}}$$

<u>Proof</u>

Suppose that A, B, Γ are not collinear

- Let (ε) be the unique line defined by A, Γ. by axiom 1.
- Let $\psi : (\varepsilon) \longrightarrow \mathbb{R}$ be a continuity map such that $\psi(A) = 0$ and $\psi(\Gamma) > 0$
- Let $\Delta = \psi^{-1}(AB) \Rightarrow \underline{\psi(\Delta) = AB}.$ (1)



Note that A, B, Γ non-collinear $\Rightarrow$
$\Rightarrow B \notin (\varepsilon) \Rightarrow B \neq \Delta$, because $\Delta \in (\varepsilon) \Rightarrow$
$\Rightarrow \underline{B\Delta > 0}$ (2).

Also note that

$(1) \Rightarrow \psi(\Delta) = AB \geqslant 0 \Rightarrow \psi(\Delta) \geqslant 0$

$\psi(\Delta) = AB = A\Gamma - B\Gamma \leqslant A\Gamma = |\psi(\Gamma) - \psi(A)| = |\psi(\Gamma)| = \psi(\Gamma) \Rightarrow$
$\Rightarrow \psi(\Delta) \leqslant \psi(\Gamma)$

therefore $\underline{\psi(A) \leqslant \psi(\Delta) \leqslant \psi(\Gamma).}$ (3)

To derive a contradiction note that:

$A\Delta = |\psi(\Delta) - \psi(A)| = |AB - 0| = AB \Rightarrow \underline{A \in \perp_m (B\Delta).}$

$\Gamma\Delta = |\psi(\Delta) - \psi(\Gamma)| = \psi(\Gamma) - \psi(\Delta) =$
$= (\psi(\Gamma) - \psi(A)) - AB = |\psi(\Gamma) - \psi(A)| - AB =$
$= A\Gamma - AB = (AB + B\Gamma) - AB = B\Gamma = \Gamma B \Rightarrow \underline{\Gamma \in \perp_m (B\Delta).}$

However, by axiom 2 we have

$A, \Gamma \in \perp_m (B\Delta) \wedge A, \Gamma \in (\varepsilon) \Rightarrow \perp_m (B\Delta) = (\varepsilon) \Rightarrow \Delta \in \perp_m (B\Delta)$, bc $\Delta \in (\varepsilon)$

$\Rightarrow B\Delta = \Delta\Delta = 0 \longleftarrow$ contradiction because by (2) $B\Delta > 0$. $\square$

The contrapositive of this theorem is the <u>triangle inequality</u>:

<u>Theorem</u> : Let $A, B, \Gamma \in \mathcal{E}$ be 3 points of the plane. Then:

$$\boxed{A, B, \Gamma \text{ non-collinear} \Rightarrow AB + B\Gamma > A\Gamma}$$

<u>Proof</u>

The contrapositive of the previous theorem states:

A, B, Γ non-collinear $\Rightarrow AB + B\Gamma \neq A\Gamma$ (1)

By axiom 7 : $AB + B\Gamma \geq A\Gamma$ (2)

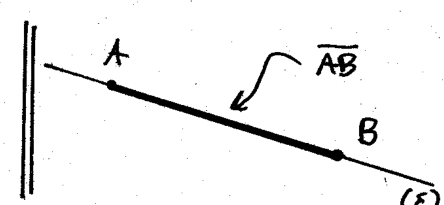
$(1) \wedge (2) \Rightarrow AB + B\Gamma > A\Gamma$ □

Another important consequence is that a line is a convex set; every segment $\overline{AB}$ defined by two points $A, B \in (\varepsilon)$ lies entirely on (ε) :

<u>Theorem</u> : Let $(\varepsilon) \in \mathcal{L}$ be a line. Then:

$$\boxed{(\forall A, B \in (\varepsilon))(\overline{AB} \subseteq (\varepsilon))}$$

<u>Proof</u>



$A, B \in (\varepsilon)$

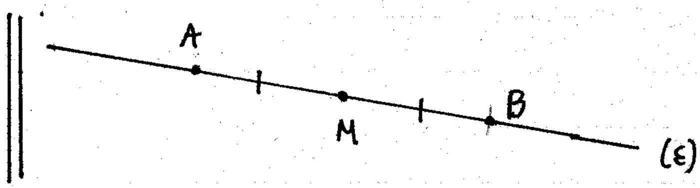
Let ~~$M \in \mathcal{E}$~~ be given. Since

$$\overline{AB} = \{M \in \mathcal{E} \mid AM + MB = AB\} \subseteq$$
$$\subseteq \{M \in \mathcal{E} \mid A, M, B \text{ collinear}\} = \quad (AB = AM + MB \Rightarrow A, M, B \text{ collinear})$$
$$= \{M \in \mathcal{E} \mid (\exists (\delta) \in \mathcal{L})(A, M, B \in (\delta))\} =$$
$$= \{M \in \mathcal{E} \mid A, M, B \in (\varepsilon)\} = \quad (\text{by axiom } 2)$$
$$= \{M \in \mathcal{E} \mid M \in (\varepsilon)\} = (\varepsilon) \Rightarrow$$
$$\Rightarrow \overline{AB} \subseteq (\varepsilon) \quad \square$$

Finally another fundamental result is that given a point $M \in (\varepsilon)$ there are <u>only two</u> points $A \neq B \in (\varepsilon)$ such that $AM = MB = x$ where $x > 0$ is given. This is also an immediate consequence of the axiom of continuity. Equivalently, if $M\Gamma = x$ for some point $\Gamma \in (\varepsilon)$, then Γ is either A or B.

Theorem: Let $(\varepsilon) \in \mathcal{L}$ be a line, $M \in (\varepsilon)$, and $x \in (0, +\infty)$. Then

$$\boxed{(\exists A, B \in (\varepsilon))\big(A \neq B \wedge M \in \overline{AB} \wedge (\forall \Gamma \in (\varepsilon))(M\Gamma = x \Leftrightarrow \Gamma = A \vee \Gamma = B)\big)}$$



Proof

Let $\psi : (\varepsilon) \longrightarrow \mathbb{R}$ be a continuity map. Then

$$\begin{aligned} M\Gamma = x &\Leftrightarrow |\psi(\Gamma) - \psi(M)| = x \Leftrightarrow \\ &\Leftrightarrow \psi(\Gamma) - \psi(M) = x \;\vee\; \psi(\Gamma) - \psi(M) = -x \Leftrightarrow \\ &\Leftrightarrow \psi(\Gamma) = \psi(M) + x \;\vee\; \psi(\Gamma) = \psi(M) - x \Leftrightarrow \\ &\Leftrightarrow \Gamma = \psi^{-1}(\psi(M) + x) \;\vee\; \Gamma = \psi^{-1}(\psi(M) - x) \qquad (1) \end{aligned}$$

Choose $A = \psi^{-1}(\psi(M) + x)$ and
$B = \psi^{-1}(\psi(M) - x)$

Then from (1): $(\forall \Gamma \in (\varepsilon))(M\Gamma = x \Leftrightarrow \Gamma = A \vee \Gamma = B)$. Also:

$x > 0 \Rightarrow \psi(M) + x \neq \psi(M) - x \Rightarrow A \neq B$.

Finally:

$$\begin{aligned} AM + MB &= |\psi(M) - \psi(A)| + |\psi(B) - \psi(M)| = \\ &= |\psi(M) - (\psi(M) + x)| + |(\psi(M) - x) - \psi(M)| = \\ &= |x| + |-x| = 2x = (\psi(M) + x) - (\psi(M) - x) = \\ &= \psi(A) - \psi(B) = |\psi(A) - \psi(B)| = AB \Rightarrow M \in \overline{AB} \qquad \square \end{aligned}$$

▼ Hilbert's betweeness relation

David Hilbert introduced "betweeness", a ternary relation over points, as a fundamental concept. In our formulation betweeness can be studied as a defined concept.

Definition: Let A, B, Γ be any three points. We say that B is between A and Γ, and denote it as $A-B-\Gamma$, if and only if $B \in \overline{A\Gamma}$ and $B \neq A$, $B \neq \Gamma$.

$$\boxed{A-B-\Gamma \Leftrightarrow B \in \overline{A\Gamma} \wedge B \neq A \wedge B \neq \Gamma}$$

The following are trivial consequences of the definition:

a) $A-B-\Gamma \Rightarrow \Gamma-B-A$

b) $A-B-\Gamma \Rightarrow A, B, \Gamma$ collinear

c) $A-B-\Gamma \Rightarrow A \neq \Gamma$

The more involved results follow from the following proposition that relates betweeness with the continuity map. First we prove an auxilliary lemma about absolute values:

Lemma: $\boxed{(\forall a, b \in \mathbb{R})(\,|a|+|b| = |a+b| \Leftrightarrow ab \geq 0\,)}$

Proof

Let $a, b \in \mathbb{R}$ be given.

$$|a|+|b| = |a+b| \Leftrightarrow (|a|+|b|)^2 = |a+b|^2 \Leftrightarrow$$
$$\Leftrightarrow |a|^2 + 2|a||b| + |b|^2 = (a+b)^2 \Leftrightarrow$$
$$\Leftrightarrow a^2 + 2|a||b| + b^2 = a^2 + 2ab + b^2 \Leftrightarrow$$
$$\Leftrightarrow |a||b| = ab \Leftrightarrow |ab| = ab \Leftrightarrow ab \geq 0 \qquad \square$$

<u>Proposition</u>: Let $A, B, \Gamma \in \mathcal{E}$, let (ε) be the line defined by A, Γ, and let $B \in (\varepsilon)$. Let $\psi : (\varepsilon) \to \mathbb{R}$ be a continuity map. Then the following equivalence holds:

$$\boxed{A-B-\Gamma \iff \psi(A) < \psi(B) < \psi(\Gamma) \vee \psi(\Gamma) < \psi(B) < \psi(A).}$$

<u>Proof</u>

First note that:

$$\begin{aligned}
B \in \overline{A\Gamma} &\iff AB + B\Gamma = A\Gamma \iff \\
&\iff |\psi(B)-\psi(A)| + |\psi(\Gamma)-\psi(B)| = |\psi(\Gamma)-\psi(A)| \iff \\
&\iff |\psi(B)-\psi(A)| + |\psi(\Gamma)-\psi(B)| = |(\psi(\Gamma)-\psi(B))+(\psi(B)-\psi(A))| \\
&\iff (\psi(B)-\psi(A))(\psi(\Gamma)-\psi(B)) \geqslant 0 \quad (\text{by lemma}) \\
&\iff [(\psi(B)-\psi(A) \geqslant 0 \wedge \psi(\Gamma)-\psi(B) \geqslant 0) \\
&\qquad \vee (\psi(B)-\psi(A) \leqslant 0 \wedge \psi(\Gamma)-\psi(B) \leqslant 0)] \iff \\
&\iff \psi(A) \leqslant \psi(B) \leqslant \psi(\Gamma) \vee \psi(\Gamma) \leqslant \psi(B) \leqslant \psi(A) \qquad (1)
\end{aligned}$$

It follows that

$$\begin{aligned}
A-B-\Gamma &\iff B \in \overline{A\Gamma} \wedge B \neq A \wedge B \neq \Gamma \\
\stackrel{(1)}{} &\iff (\psi(A) \leqslant \psi(B) \leqslant \psi(\Gamma) \vee \psi(\Gamma) \leqslant \psi(B) \leqslant \psi(A)) \wedge \psi(B) \neq \psi(A) \wedge \psi(B) \neq \psi(\Gamma) \\
&\iff \psi(A) < \psi(B) < \psi(\Gamma) \vee \psi(\Gamma) < \psi(B) < \psi(A) \qquad \square
\end{aligned}$$

<u>Theorem</u>: Let $A, B, \Gamma \in (\varepsilon)$ be any three points on a line. Then:

$$\boxed{A \neq B \neq \Gamma \neq A \Rightarrow A-B-\Gamma \,\underline{\vee}\, B-\Gamma-A \,\underline{\vee}\, \Gamma-A-B}$$

<u>Proof</u>

Let $\psi : (\varepsilon) \to \mathbb{R}$ be a continuity map of (ε).

$$\begin{aligned}
A \neq B \neq \Gamma \neq A &\Rightarrow \psi(A) \neq \psi(B) \neq \psi(\Gamma) \neq \psi(A) \Rightarrow \\
&\Rightarrow \begin{cases} \psi(A) < \psi(B) \,\underline{\vee}\, \psi(B) < \psi(A) \\ \psi(B) < \psi(\Gamma) \,\underline{\vee}\, \psi(\Gamma) < \psi(B) \\ \psi(\Gamma) < \psi(A) \,\underline{\vee}\, \psi(A) < \psi(\Gamma) \end{cases} \Rightarrow
\end{aligned}$$

$$\Rightarrow (\psi(A)<\psi(B)<\psi(\Gamma) \vee \psi(\Gamma)<\psi(B)<\psi(A))$$
$$\underline{\vee}\ (\psi(B)<\psi(\Gamma)<\psi(A) \vee \psi(A)<\psi(\Gamma)<\psi(B))$$
$$\underline{\vee}\ (\psi(\Gamma)<\psi(A)<\psi(B) \vee \psi(B)<\psi(A)<\psi(\Gamma)) \Rightarrow$$

$$\Rightarrow A-B-\Gamma \ \underline{\vee}\ B-\Gamma-A \ \underline{\vee}\ \Gamma-A-B \qquad \square$$

This theorem states that for any 3 distinct points on a line, one and only one is between the other two.

Now we show an existence result that is often used in geometric arguments.

<u>Theorem</u> : (<u>Existence of between points</u>).

Let $A, B \in (\varepsilon)$ be two points on a line with $A \neq B$.
Let $x \in (0,+\infty)$ be a real positive number. Then:

a) There is a <u>unique</u> point $\Gamma \in (\varepsilon)$ such that $A-\Gamma-B$ and $A\Gamma/\Gamma B = x$

$$\boxed{|\{\Gamma \in (\varepsilon) \mid A-\Gamma-B \wedge A\Gamma/\Gamma B = x\}| = 1}$$

b) There is a <u>unique</u> point $\Gamma \in (\varepsilon)$ such that $A-B-\Gamma$ and $AB/B\Gamma = x$

$$\boxed{|\{\Gamma \in (\varepsilon) \mid A-B-\Gamma \wedge AB/B\Gamma = x\}| = 1}$$

c) There is a unique point $\Gamma \in (\varepsilon)$ such that $\Gamma-A-B$ and $\Gamma A/AB = x$

$$\boxed{|\{\Gamma \in (\varepsilon) \mid \Gamma-A-B \wedge \Gamma A/AB = x\}| = 1}$$

<u>Proof</u>

Let $\psi : (\varepsilon) \to \mathbb{R}$ be a continuity map such that $\psi(A) = 0$ and $\psi(B) > 0$.
Then $AB = |\psi(B) - \psi(A)| = |\psi(B)| = \psi(B) \Rightarrow \underline{AB = \psi(B)}$ (1)

a) Assume $A-\Gamma-B$ and solve $A\Gamma/\Gamma B = x$ for Γ.

$$A-\Gamma-B \Rightarrow \psi(A)<\psi(\Gamma)<\psi(B) \vee \psi(A)>\psi(\Gamma)>\psi(B) \Rightarrow$$
$$\Rightarrow \psi(A)<\psi(\Gamma)<\psi(B) \qquad (1)$$

therefore:

$$A\Gamma = |\psi(\Gamma)-\psi(A)| = |\psi(\Gamma)| = \psi(\Gamma) \qquad (2)$$
$$\Gamma B = |\psi(\Gamma)-\psi(B)| = \psi(B)-\psi(\Gamma). \qquad (3)$$

so:

$$\frac{A\Gamma}{\Gamma B} = x \Leftrightarrow \frac{\psi(\Gamma)}{\psi(B)-\psi(\Gamma)} = x \Leftrightarrow \psi(\Gamma) = x\psi(B) - x\psi(\Gamma) \Leftrightarrow$$
$$\Leftrightarrow \psi(\Gamma) = \frac{x\psi(B)}{1+x} \Leftrightarrow \Gamma = \psi^{-1}\left[\frac{x\psi(B)}{1+x}\right] \qquad (4)$$

It follows that

$$A-\Gamma-B \wedge \frac{A\Gamma}{\Gamma B} = x \Rightarrow \Gamma = \psi^{-1}\left[\frac{x\psi(B)}{1+x}\right] \qquad \blacktriangleleft$$

($\Leftarrow$): Conversely, let $\Gamma = \psi^{-1}\left[\frac{x\psi(B)}{1+x}\right]$. Then

$$x>0 \Rightarrow 0 < \frac{x}{1+x} < 1 \Rightarrow 0 < \frac{x\psi(B)}{1+x} < \psi(B) \Rightarrow$$
$$\Rightarrow \psi(A)<\psi(\Gamma)<\psi(B) \Rightarrow A-\Gamma-B.$$

Given that, from the converse of (4): $A\Gamma/\Gamma B = x$ $\blacktriangleleft$

Therefore:

$$\begin{cases} A-\Gamma-B \\ A\Gamma/\Gamma B = x \end{cases} \Leftrightarrow \Gamma = \psi^{-1}\left[\frac{x\psi(B)}{1+x}\right]$$

so

$$|\{\Gamma \in (\varepsilon) \mid A-\Gamma-B \wedge A\Gamma/\Gamma B = x\}| = \left|\left\{\psi^{-1}\left[\frac{x\psi(B)}{1+x}\right]\right\}\right| = 1$$

b) ($\Rightarrow$): Assume $A-B-\Gamma$ and solve $AB/B\Gamma = x$.

$$A-B-\Gamma \Rightarrow \psi(A)<\psi(B)<\psi(\Gamma) \vee \psi(A)>\psi(B)>\psi(\Gamma) \Rightarrow$$
$$\Rightarrow \psi(A)<\psi(B)<\psi(\Gamma).$$
$$\Rightarrow B\Gamma = |\psi(\Gamma)-\psi(B)| = \psi(\Gamma)-\psi(B)$$

So, we have:

$$\frac{AB}{B\Gamma} = x \Leftrightarrow \frac{\psi(B)}{\psi(\Gamma)-\psi(B)} = x \Leftrightarrow \psi(B) = x\psi(\Gamma) - x\psi(B) \Leftrightarrow$$

$$\Leftrightarrow \psi(\Gamma) = \frac{(1+x)\psi(B)}{x} \Leftrightarrow \Gamma = \psi^{-1}\left[\frac{(1+x)\psi(B)}{x}\right] \qquad (5)$$

therefore $A-B-\Gamma \wedge \frac{AB}{B\Gamma} = x \Rightarrow \Gamma = \psi^{-1}\left[\frac{(1+x)\psi(B)}{x}\right]$

$\underline{(\Leftarrow)}$: ~~Co~~nversely, let $\Gamma = \psi^{-1}\left[\frac{(1+x)\psi(B)}{x}\right]$. Then.

$$x>0 \Rightarrow \frac{1+x}{x} > 1 \Rightarrow \psi(\Gamma) = \frac{(1+x)\psi(B)}{x} > \psi(B) > \psi(A) \Rightarrow$$

$$\Rightarrow \psi(A) < \psi(B) < \psi(\Gamma) \Rightarrow A-B-\Gamma.$$

Given this, it follows from (5) that $AB/B\Gamma = x$ ◀

Therefore $\begin{cases} A-B-\Gamma \\ AB/B\Gamma = x \end{cases}$ has a unique solution. for Γ.

c) Can be proved by reusing (b).

$$m = |\{\Gamma \in (\varepsilon) \mid \Gamma-A-B \wedge \Gamma A/AB = x\}| =$$
$$= |\{\Gamma \in (\varepsilon) \mid B-A-\Gamma \wedge BA/A\Gamma = 1/x\}| = \quad (b)$$
$$= 1$$

□

A corollary of this theorem is the existence and uniqueness of the <u>midpoint</u> of a segment.

<u>Definition</u>: Let $A, B \in (\varepsilon)$ be two points of the line (ε). A point $M \in \overline{AB}$ is called the <u>midpoint</u> of the segment $\overline{AB}$ if and ~~oly~~ only if $AM = MB$.

$$\boxed{M \text{ midpoint of } \overline{AB} \Leftrightarrow M \in \overline{AB} \wedge AM = MB}$$

<u>Corollary</u>: Every segment $\overline{AB}$ has a unique midpoint M.

$$\boxed{(\forall A, B \in \mathcal{E})(\exists!\, M \in \overline{AB})(M \text{ midpoint of } \overline{AB})}$$

► Because the midpoint is unique, we will denote it as $M = \mathcal{m}(\overline{AB})$ ◄

A weaker restatement of the theorem we proved says that for any two points $A, B \in (\varepsilon)$ on a line, there is a point between them, a point "on the right" of $\overline{AB}$, and a point on the left of $\overline{AB}$

<u>Corollary</u>:

a) $(\forall A, B \in (\varepsilon))(\exists\, \Gamma \in (\varepsilon))(A - \Gamma - B)$

b) $(\forall A, B \in (\varepsilon))(\exists\, \Gamma \in (\varepsilon))(A - B - \Gamma)$

c) $(\forall A, B \in (\varepsilon))(\exists\, \Gamma \in (\varepsilon))(\Gamma - A - B)$.

▼ Consecutive points on a line

The "betweeness" relation can be generalized for any number of points with the notion of consecutivity.

Definition: Let $[n] = \{1, 2, \ldots, n\}$ the n first natural numbers. We call a finite sequence of points any map $\mathcal{A} : [n] \to \mathcal{E}$

This definition can be generalized to transfinite sizes as follows:

Definition: A sequence of points is any map $\mathcal{A} : \mathfrak{a} \to \mathcal{E}$ where $\mathfrak{a} \in On$ is any ordinal number.

▸ notation: It is common to denote sequences as

$$\mathcal{A} = \langle A_i \mid i \in \mathfrak{a} \rangle$$

This means that the domain of $\mathcal{A}$ is $dom(\mathcal{A}) = \mathfrak{a}$ and that $(\forall i \in \mathfrak{a})(\mathcal{A}(i) = A_i)$, that is, A_i is the value of the sequence for a given $i \in \mathfrak{a}$.

Definition: Let $\mathcal{A} = \langle A_i \mid i \in \mathfrak{a} \rangle$ be a sequence of points. We say that $\mathcal{A}$ is consecutive if and only if for any $i < j < k$, $A_i - A_j - A_k$:

$$\boxed{\mathcal{A} \text{ consecutive} \iff (\forall i, j, k \in \mathfrak{a})(i < j < k \iff A_i - A_j - A_k)}$$

Although it is possible to discuss transfinite sequences of consecutive points, only finite sequences are of real practical use, so we will limit our development only to ~~finite~~ those.

Criterion for consecutivity

Theorem: Let $\mathcal{A} = \langle A_i \mid i \in [n] \rangle$ be a finite sequence of points. Then:

$$\boxed{\mathcal{A} \text{ consecutive} \Leftrightarrow \sum_{i=1}^{n-1} A_i A_{i+1} = A_1 A_n}$$

Proof:

($\Rightarrow$): Suppose that $\mathcal{A}$ consecutive.
Let (ε) be the line defined by the points A_i and
let $\psi : (\varepsilon) \longrightarrow \mathbb{R}$ be the continuity map. of (ε), such that $\psi(A_1) < \psi(A_n)$.
$\mathcal{A}$ consecutive $\Rightarrow (\forall i \in [n-2]) \bullet (A_i - A_{i+1} - A_{i+2}) \Rightarrow$
$\Rightarrow (\forall i \in [n-2]) \, (\psi(A_i) < \psi$