Ossifrage and Algebra

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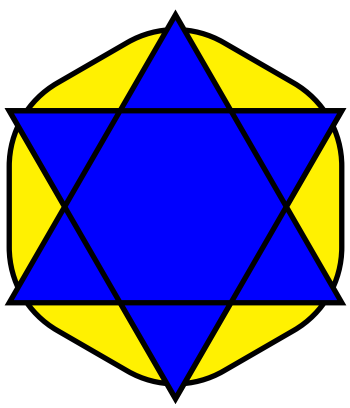
If there should be another flood
Hither for refuge fly
Were the whole world to be submerged
This book would still be dry.[1]



[1]Anonymous annotation on the fly leaf of an Algebra book.

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-Maldición anónima contra los ladrones de libros en el monasterio de San Pedro, Barcelona.

Preface

Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country. *-David HILBERT*

These notes started during the Spring of 2002. I would like to thank José Mason and John Majewicz for numerous suggestions. These notes have borrowed immensely from their work and suggestions.

I have not had the time to revise, hence errors will abound, especially in the homework answers. I will be grateful to receive an email pointing out corrections.

I would like to thank Margaret Hitczenko. I have used some of her ideas from the CEMEC project here. I would also like to thank Iraj Kalantari, Lasse Skov, and Don Stalk for alerting me of numerous typos/errors/horrors.

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Part I

First Impressions

1 Why Study Algebra?

The good Christian should beware of Mathematicians and all those who make empty prophesies. The danger already exists that the Mathematicians have made a covenant with the Devil to darken the spirit and to confine man in the bonds of Hell. *-St. Augustine*

This chapter is optional. It nevertheless tries to answer two important queries which are often heard:

- Why am I learning this material?
- Where will I ever use this material again?

Mathematicians can give—what they consider—reasonable explanations to the questions above. But any reasonable explanation imposes duties on both the person asking the question and the mathematician answering it. To understand the given answer, you will need *patience*. If you dislike Mathematics at the outset, no amount of patience or reasonable explanations will do.

Some of the concepts that we will mention in the history section might be unfamiliar to you. Do not worry. The purpose of the section is to give you a point of reference as to how old many of these ideas are.

1.1 Illiteracy and Innumeracy

No more impressive warning can be given to those who would confine knowledge and research to what is apparently useful, than the reflection that conic sections were studied for eighteen hundred years merely as an abstract science, without regard to any utility other than to satisfy the craving for knowledge on the part of mathematicians, and that then at the end of this long period of abstract study, they were found to be the necessary with which to attain the knowledge of the most important laws of nature. *-Alfred North WHITEHEAD*

Illiteracy is the lack of ability to read and write. *Innumeracy* is the lack of familiarity with mathematical concepts and methods. It is a truism that people would be embarrassed to admit their illiteracy, but no so to admit their innumeracy.

Mathematicians like to assert the plurality of applications of their discipline to real world problems, by presenting modelisations from science, engineering, business, etc. For example, chemical compounds must obey certain geometric arrangements that in turn specify how they behave. By means of mathematical studies called *Group Theory* and *Pólya Theory of Counting*, these geometric arrangements can be fully catalogued. Modern genetics relies much in two branches of mathematics—*Combinatorics* and *Probability Theory*—in order to explain the multiform combinations among genes. Some financial firms utilise a mathematical theory called *Brownian Motion* to explain the long term behaviour of markets. By associating the way virus strands twist using *Knot Theory*, medical research is now better understanding the behaviour of viruses.[1]

These applications are perhaps too intricate for a novice to master, and many years of Mathematics beyond Algebra are required in order to comprehend them. Hence, why must a person who is not planning to become an engineer, a scientist, a business analyst, etc., learn Algebra? A brief answer is the following:

[1] More such applications can be found in the popularisation by Keith Devlin [Devl].

- Algebra provides a first example of an *abstract system*.
- Algebra strengthens deductive reasoning.
- Algebra is a gateway course, used by other disciplines as a hurdle for admission.
- Algebra is part of our cultural legacy, much like Art and Music, and its mastery is expected by all who want to be considered educated.

What now, if you appreciate all the above reasons, but still claim that *you cannot learn Algebra*? That Algebra *is not for you*? That *no grown adult needs* Algebra, since after all, Algebra does not cure obesity, does not stop poverty, does not stop war, does not alleviate famine, etc.? Observe that all these reasons can also be given for learning how to read, or learning a foreign language. Unlike foreign languages, Algebra is a more universal language, much like Music, but to fully appreciate its power you must be willing to learn it. Most of your teachers will assert that any non mentally-challenged person can learn Algebra. Again, most of us will assert that the major difficulties in learning Algebra stem from previous difficulties with Arithmetic. Hence, if you are reasonably versed in Arithmetic, you should not confront much trouble with Algebra. All these so-called reasons against Algebra are balderdash, since lacking clairvoyance, how can one claim not to need a discipline in the future?

Homework

Problem 1.1.1 *Comment on the following assertion:* There is no need to learn Mathematics, since nowadays all calculations can be carried out by computers.

Problem 1.1.2 *Comment on the following assertion:* I was never good at Maths. I will never pass this Algebra class.

Problem 1.1.3 *Comment on the following assertions:* Only nutritionists should know about the basics of nutrition since that is their trade. Only medical doctors should know about the basics of health, since that is their trade. Only mathematicians should know about Algebra since that is their trade.

1.2 Brief History

> ***I have no fault to find with those who teach geometry. That science is the only one which has not produced sects; it is founded on analysis and on synthesis and on the calculus; it does not occupy itself with probable truth; moreover it has the same method in every country.***
> *-Fredrick the Great*

Algebra is a very old discipline. Already by 2000 B.C. the ancient Babylonians were solving quadratic equations by completing squares [Eves, pp. 31-32]. [2] The early Egyptians and Babylonians developed arithmetic and geometry for purely practical reasons—essentially for the necessities of commerce and land surveying.

Around 100 BC, in the Chinese mathematics book *Jiuzhang suanshu* (The Nine Chapters on the Mathematical Art), linear equations are solved using the method of *regula falsa* and systems of linear equations are solved by the equivalent of modern matrix methods. Around the same time in India, *Bakhshali Manuscript* introduces the use of letters and other signs in the resolution of problems. Cubic and quartic equations are treated, as well as linear equations with up to five unknowns.

The Greek civilisation assimilated the Egyptian and Babylonian practical knowledge and develops mathematics as an abstract and deductive field. Geometry and Number Theory are extensively studied by Euclid, whose *Elements* (written approximately 300 BC) is perhaps the most successful textbook ever written, being one of the very first works to be printed after the printing press was invented and used as the basic text on geometry throughout the Western world for about 2,000 years. A first step in the construction of algebra as a formal body comes with the work of Diophantus of Alexandria,

[2] Equations of the type $ax^2 + bx + c = 0$, where x is the unknown quantity and a, b, c are known constants.

whose *Arithmetic* (written approximately 300 AD) studied what we now call diophantine equations. One of Diophantus' innovations was to introduce symbolic notation for arithmetical quantities. For example, he denoted the square of a quantity by $\Delta\Upsilon$, the cube of a quantity by $K\Upsilon$, the fourth power of a quantity by $\Delta\Upsilon\Delta$, and the fifth power by $\Delta K\Upsilon$. Before him, these quantities were treated rhetorically (verbally). Diophantus also knew how to manipulate positive and negative exponents, he represented addition of quantities by juxtaposition and subtraction with the symbol **:**. Diophantus methods were not geometrical, like the methods of most Greek mathematicians, but they used the properties of the numbers involved. For this and more, Diophantus is considered the father of algebra.

In 628 AD the Indian mathematician Brahmagupta, in his treatise *Brahma Sputa Siddhanta*, gives rules for solving linear and quadratic equations. He also discovers that quadratic equations have two roots, including both negative as well as irrational roots.

Inspired in the work of Brahmagupta, there appears in what is now Uzbekistan the mathematician Muhammad bin Musa al-Khuwarazmi (d. 847 AD), from whose book *Kitab al-mukhasar fi hisab al-jabr wa'l muqabala* (The Book of summary concerning calculating by transposition and reduction), the word *algebra* (اَلْجَبْر) comes from. The word is roughly translated as "rearranging" or "transposing."[3] In a way, al-Khuwarazmi's work was a regression from Diophantus, since his treatment was rhetorical rather than symbolical. al-Khuwarazmi's however, treated quantities formally, much like Brhamagupta, rather than geometrically, like the Greeks. Relying on the work of al-Khuwarazmi, Abul Kamil introduces radicals, the use of irrational quantities, and systems of equations. Around 1072 AD the Persian mathematician Omar Khayyam in his *Treatise on Demonstration of Problems of Algebra*, gives a complete classification of cubic equations with general geometric solutions found by means of intersecting conic sections.

With the advent of the Renaissance in Europe, the ancient Greek and Hindu works are known through Arabic translations. Algebra is then developed at a rapid pace by the Italian mathematicians Cardano, Tartaglia, Ferrari, and Bombelli. In 1557 Robert Recorde introduces the sign = for equality. In the same century Widmann introduces the + for addition and the sign − for subtraction. William Oughtred in 1631 uses the letter x to denote the unknown of an equation.

Homework

Problem 1.2.1 *Who is called the* Father of Algebra *and why?*

Problem 1.2.2 *Is Algebra an Arab invention?*

Problem 1.2.3 *What is the contraposition alluded to in the title of this book,* Ossifrage and Algebra*?*

1.3 What is Elementary Algebra All About?

> ***I was just going to say, when I was interrupted, that one of the many ways of classifying minds is under the heads of arithmetical and algebraical intellects. All economical and practical wisdom is an extension of the following arithmetical formula:*** $2+2=4$***. Every philosophical proposition has the more general character of the expression*** $a+b=c$***. We are mere operatives, empirics, and egotists until we learn to think in letters instead of figures.*** *-Oliver Wendell HOLMES*

Elementary algebra generalises arithmetic by treating quantities in the abstract. Thus where in an arithmetic problem you may assert that $2+3=3+2$ and that $4+1=1+4$, in elementary algebra you assert that for any two numbers a, b we have

$$a+b=b+a.$$

[3]The word, of course, had a non-mathematical connotation before the popularity of al-Khuwarazmi's book. In Moorish Spain an "algebrista" was a bonesetter—a reuniter of broken bones—and so, many barbers of the time were called *algebristas*.

It allows the formulation of problems and their resolution by treating an unknown quantity formally. For example, we will learn later on how to calculate

$$123456789^2 - (123456787)(123456791)$$

without using a calculator. But the power of algebra goes beyond these curiosities. Back in 1994, Thomas Nicely, a number theorist, found an error in the Intel Pentium chip.[4] This means that computers with this chip were carrying out incorrect calculations. One example given at the time was the following:

$$4195835.0 \div 3145727.0 = 1.333820449136241000 \quad \text{(Correct value)}$$

$$4195835.0 \div 3145727.0 = 1.333739068902037589 \quad \text{(Flawed Pentium)}$$

The lesson here: computers cannot be trusted!

Problem 1.3.1 *You start with* 100. *You give* 20% *to your friend. But it turns out that you need the* $\$100$ *after all in order to pay a debt. By what percent should you increase your current amount in order to restore the* $\$100$? *The answer is* not 20%*!*

Problem 1.3.2 *A bottle of wine and its cork cost* $\$1$. *The bottle of wine costs* 80¢ *more than than the cork. What is the price of the cork, in cents?*

1.4 Puzzles

Mathematics possesses not only truth, but supreme beauty—a beauty cold and austere, like that of a sculpture, and capable of stern perfection, such as only great art can show. *-Bertrand RUSSELL*

The purpose of the puzzles below is to evince some techniques of mathematical problem-solving: working backwards, search for patterns, case by case analysis, etc.

1 Example A frog is in a 10 ft well. At the beginning of each day, it leaps 5 ft up, but at the end of the day it slides 4 ft down. After how many days, if at all, will the frog escape the well?

Solution: ► *The frog will escape after seven days. At the end of the sixth day, the frog has leaped* 6 *feet. Then at the beginning of the seventh day, the frog leaps* 5 *more feet and is out of the well.* ◄

2 Example Dale should have divided a number by 4, but instead he subtracted 4. He got the answer 48. What should his answer have been?

Solution: ► *We work backwards. He obtained* 48 *from* $48 + 4 = 52$. *This means that he should have performed* $52 \div 4 = 13$. ◄

3 Example When a number is multiplied by 3 and then increased by 16, the result obtained is 37. What is the original number?

Solution: ► *We work backwards as follows. We obtained* 37 *by adding* 16 *to* $37 - 16 = 21$. *We obtained this* 21 *by multiplying by* 3 *the number* $21 \div 3 = 7$. *Thus the original number was a* 7. ◄

4 Example You and I play the following game. I tell you to write down three 2-digit integers between 10 and 89. Then I write down three 2-digit integers of my choice. The answer comes to 297, no matter

[4] One may find more information here: http://www.trnicely.net/pentbug/pentbug.html.

which three integers you choose (my choice always depends on yours). For example, suppose you choose $12, 23, 48$. Then I choose $87, 76, 51$. You add

$$12 + 23 + 48 + 87 + 76 + 51 = 297.$$

Again, suppose you chose $33, 56, 89$. I then choose 66, 43, 10. Observe that

$$33 + 56 + 89 + 66 + 43 + 10 = 297.$$

Explain how I choose my numbers so that the answer always comes up to be 297 (!!!).

Solution: ► *Notice that I always choose my number so that when I add it to your number I get* 99*, therefore, I end up adding* 99 *three times and* $3 \times 99 = 297$.[5] ◄

5 Example What is the sum

$$1 + 2 + 3 + \cdots + 99 + 100$$

of all the positive integers from 1 to 100?

Solution: ► *Pair up the numbers into the fifty pairs*[6]

$$(100 + 1) = (99 + 2) = (98 + 3) = \cdots = (50 + 51).$$

Thus we have 50 *pairs that add up to* 101 *and so the desired sum is* $101 \times 50 = 5050$. *Another solution will be given in example 249.* ◄

Homework

Problem 1.4.1 *What could St. Augustine mean by mathematicians making* prophecies*? Could he have meant some other profession than mathematician?*

Problem 1.4.2 *Can we find five even integers whose sum is* 25*?*

Problem 1.4.3 *Iblis entered an elevator in a tall building. She went up* 4 *floors, down* 6 *floors, up* 8 *floors and down* 10 *floors. She then found herself on the* 23*rd floor. In what floor did she enter the elevator?*

Problem 1.4.4 *A natural number is called a* palindrome *if it is read forwards as backwards, e.g.,* 1221, 100010001, *etc., are palindromes. The palindrome* 10001 *is strictly between two other palindromes. Which two?*

Problem 1.4.5 *Each square represents a digit.*[7] *Find the value of each missing digit.*

$$\begin{array}{cccccc} & \blacksquare & 7 & 5 & \blacksquare & 6 \\ - & & \blacksquare & 5 & 6 & \blacksquare \\ \hline & 2 & 4 & \blacksquare & 7 & 5 \end{array}$$

Problem 1.4.6 *Is it possible to replace the letter* a *in the square below so that every row has the same sum of every column?*

1	2	5
3	3	2
a	3	1

Problem 1.4.7 *Fill each square with exactly one number*

[5] Using algebraic language, observe that if you choose x, y, z, then I choose $(99 - x), (99 - y), (99 - z)$. This works because

$$x + (99 - x) + y + (99 - y) + z + (99 - z) = 3(99) = 297.$$

[6] This trick is known as *Gauß' trick*, after the German mathematician Karl Friedrich Gauß (1777-1855). Presumably, when Gauß was in first grade, his teacher gave this sum to the pupils in order to keep them busy. To the amazement of the teacher, Gauß came up with the answer almost instantaneously.

[7] A *digit*, from the Latin *digitum* (finger), is one of the ten numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

from

$$\{1,2,3,4,5,6,7,8,9\}$$

so that the square becomes a magic square, that is, a square where every row has the same sum as every column, and as every diagonal.

Is there more than one solution?

Problem 1.4.8 *Vintik and Shpuntik agreed to go to the fifth car of a train. However, Vintik went to the fifth car from the beginning, but Shpuntik went to the fifth car from the end. How many cars has the train if the two friends got to one and the same car?*

Problem 1.4.9 *Bilbo and Frodo have just consumed a plateful of cherries. Each repeats the rhyme 'Tinker, tailor, soldier, sailor, rich man, poor man, beggar man, thief' over and over again as he runs through his own heap of cherry stones. Bilbo finishes on 'sailor', whereas Frodo finishes on 'poor man'. What would they have finished on if they had run through both heaps together?*

Problem 1.4.10 *A boy and a girl collected* 24 *nuts. The boy collected twice as many nuts as the girl. How many did each collect?*

Part II

Arithmetic Review

2 Arithmetic Operations

"Reeling and Writhing, of course, to begin with," the Mock Turtle replied, "and then the different branches of Arithmetic: Ambition, Distraction, Uglification, and Derision." *-Lewis CARROLL*

In this chapter we review the operations of addition, subtraction, multiplication, and division of numbers. We also introduce exponentiation and root extraction. We expect that most of the material here will be familiar to the reader. We nevertheless will present arithmetic operations in such a way so that algebraic generalisations can be easily derived from them.

2.1 Symbolical Expression

Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different. *-GOETHE*

We begin our study of Algebra by interpreting the meaning of its symbols. We will use *letters* to denote arbitrary numbers. This will free us from a long enumeration of cases. For example, suppose we notice that

$$1+0=1, \quad 2+0=2, \quad \frac{1}{2}+0=\frac{1}{2}, \ldots,$$

etc. Since numbers are infinite, we could not possibly list all cases. Here abstraction provides some economy of thought: we could say that if x is a number, then

$$x+0=x,$$

with no necessity of knowing what the arbitrary number x is.

We will normally associate the words *increase, increment, augment,* etc., with addition. Thus if x is an unknown number, the expression "a number increased by seven" is translated into symbols as $x+7$. We will later see that we could have written the equivalent expression $7+x$.

We will normally associate the words *decrease, decrement, diminish, difference* etc., with subtraction. Thus if x is an unknown number, the expression "a certain number decreased by seven" is translated into symbols as $x-7$. We will later see that this differs from $7-x$, which is "seven decreased by a certain number."

We will normally associate the word *product* with multiplication. Thus if x is an unknown number, the expression "the product of a certain number and seven" is translated into symbols as $7x$. Notice here that we use *juxtaposition* to denote the multiplication of a letter and a number, that is, we do not use the $\times$ (times) symbol, or the $\cdot$ (central dot) symbol. This will generally be the case, and hence the following are all equivalent,

$$7x, \quad 7\cdot x, \quad 7(x), \quad (7)(x), \quad 7\times x.$$

Notice again that a reason for *not* using $\times$ when we use letters is so that we do not confuse this symbol with the letter x. We could have also have written $x7$, but this usage is just plain weird.

☞ *We do need symbols in order to represent the product of two numbers. Thus we write the product* $5\cdot 6=5\times 6=(5)(6)=30$ *so that we do not confuse this with the number* 56.

A few other words are used for multiplication by a specific factor. If the unknown quantity is a, then *twice* the unknown quantity is represented by $2a$. *Thrice* the unknown quantity is represented by $3a$. To *treble* a quantity is to triple it, hence "treble a" is $3a$. The *square* of a quantity is that quantity multiplied by itself, so for example, the square of a is aa, which is represented in short by a^2. Here a is the *base* and 2 is the *exponent*. The *cube* of a quantity is that quantity multiplied by its square, so for example, the cube of a is aaa, which is represented in short by a^3. Here a is the *base* and 3 is the *exponent*.

The word *quotient* will generally be used to denote division. For example, the quotient of a number and 7 is denoted by $x \div 7$, or equivalently by $\frac{x}{7}$ or $x/7$. The *reciprocal* of a number x is $\frac{1}{x}$.

Here are some more examples.

6 Example If a number x is trebled and if to this new number we add five, we obtain $3x+5$.

7 Example If x is the larger between x and y, the difference between x and y is $x-y$. However, if y is the larger between x and y, the difference between x and y is $y-x$.

8 Example If a and b are two numbers, then their product is ab, which we will later see that it is the same as ba.

9 Example The sum of the squares of x and y is x^2+y^2. However, the square of the sum of x and y is $(x+y)^2$.

10 Example The expression $2x-\frac{1}{x^2}$ can be translated as "twice a number is diminished by the reciprocal of its square."

11 Example If n is an integer, its predecessor is $n-1$ and its successor is $n+1$.

12 Example You begin the day with E eggs. During the course of the day, you fry O omelettes, each requiring A eggs. How many eggs are left?

Solution: ► *$E-OA$, since OA eggs are used in frying O omelettes.* ◄

13 Example An even natural number has the form $2a$, where a is a natural number. An odd natural number has the form $2a+1$, where a is a natural number.

14 Example A natural number divisible by 3 has the form $3a$, where a is a natural number. A natural number leaving remainder 1 upon division by 3 has the form $3a+1$, where a is a natural number. A natural number leaving remainder 2 upon division by 3 has the form $3a+2$, where a is a natural number.

15 Example Find a formula for the n-th term of the arithmetic progression

$$2, 7, 12, 17, \ldots.$$

Solution: ► *We start with 2, and then keep adding 5, thus*

$$2 = 2+5\cdot 0, \qquad 7 = 2+5\cdot 1, \qquad 12 = 2+5\cdot 2, \qquad 17 = 2+5\cdot 3, \ldots.$$

The general term is therefore of the form $2+5(n-1)$, where $n = 1, 2, 3, \ldots$ is a natural number. ◄

16 Example Find a formula for the n-th term of the geometric progression

$$6, 12, 24, 48, \ldots.$$

Solution: ► *We start with* 6*, and then keep multiplying by* 2*, thus*

$$6 = 6 \cdot 2^0, \qquad 12 = 6 \cdot 2^1, \qquad 24 = 6 \cdot 2^2, \qquad 48 = 6 \cdot 2^3, \ldots.$$

The general term is therefore of the form $3 \cdot 2^{n-1}$*, where* $n = 1, 2, 3, \ldots$ *is a natural number.* ◄

17 Example Identify the law of formation and conjecture a general formula:

$$1 = 1,$$

$$1 + 2 = \frac{(2)(3)}{2},$$

$$1 + 2 + 3 = \frac{(3)(4)}{2},$$

$$1 + 2 + 3 + 4 = \frac{(4)(5)}{2},$$

$$1 + 2 + 3 + 4 + 5 = \frac{(5)(6)}{2}.$$

Solution: ► *Notice that the right hand side consists of the last number on the left times its successor, and this is then divided by* 2*. Thus we are asserting that*

$$1 + 2 + 3 + \cdots + (n-1) + n = \frac{(n)(n+1)}{2}.$$

◄

Homework

Problem 2.1.1 *If a person is currently* N *years old, what was his age* 20 *years ago?*

Problem 2.1.2 *If a person is currently* N *years old, what will his age be in* 20 *years?*

Problem 2.1.3 *You start with* x *dollars. Then you treble this amount and finally you increase what you now have by* 10 *dollars. How many dollars do you now have?*

Problem 2.1.4 *You start with* x *dollars. Then you add \$10 to this amount and finally you treble what you now have. How many dollars do you now have?*

Problem 2.1.5 *A knitted scarf uses three balls of wool. I start the day with* b *balls of wool and knit* s *scarves. How many balls of wool do I have at the end of the day?*

Problem 2.1.6 *Think of a number. Double it. Add* 10*. Half your result. Subtract your original number. After these five steps, your answer is* 5 *regardless of your original number! If* x *is the original number, explain by means of algebraic formulæ each step.*

Problem 2.1.7 *What is the general form for a natural number divisible by* 4*? Leaving remainder* 1 *upon division by* 4*? Leaving remainder* 2 *upon division by* 4*? Leaving remainder* 3 *upon division by* 4*?*

Problem 2.1.8 *Find a general formula for the* n*-th term of the arithmetic progression*

$$1, 7, 13, 19, 25, \ldots.$$

Problem 2.1.9 *Identify the law of formation and conjecture a general formula:*

$$1^2 = \frac{(1)(2)(3)}{6},$$

$$1^2 + 2^2 = \frac{(2)(3)(5)}{6},$$

$$1^2 + 2^2 + 3^2 = \frac{(3)(4)(7)}{6},$$

$$1^2 + 2^2 + 3^2 + 4^2 = \frac{(4)(5)(9)}{6},$$

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \frac{(5)(6)(11)}{6}.$$

Problem 2.1.10 *Identify the law of formation and conjecture a general formula.*

$$1 = 1^2,$$
$$1 + 3 = 2^2,$$
$$1 + 3 + 5 = 3^2,$$
$$1 + 3 + 5 + 7 = 4^2,$$
$$1 + 3 + 5 + 7 + 9 = 5^2.$$

Problem 2.1.11 *You start the day with q quarters and d dimes. How much money do you have? Answer in cents. If by the end of the day you have lost a quarters and b dimes, how much money do you now have? Answer in cents.*

Problem 2.1.12 *Let x be an unknown quantity. How would you translate into symbols the expression "the cube of a quantity is reduced by its square and then what is left is divided by 8"?*

Problem 2.1.13 *A man bought a hat for h dollars. He then bought a jacket and a pair of trousers. If the jacket is thrice as expensive as the hat and the trousers are 8 dollars cheaper than jacket, how much money did he spend in total for the three items?*

Problem 2.1.14 *A camel merchant sold a camel for a dollars and gained b dollars as profit. What is the real cost of the camel?*[1]

Problem 2.1.15 *A merchant started the year with m dollars; the first month he gained x dollars, the next month he lost y dollars, the third month he gained b dollars, and the fourth month lost z dollars. How much had he at the end of that month?*

2.2 The Natural Numbers

> ***Numbers are the beginning and end of thinking. With thoughts were numbers born. Beyond numbers thought does not reach.*** *-Magnus Gustaf MITTAG-LEFFLER*

This section gives an overview of the natural numbers. We start with two symbols, 0 and 1, and an operation $+$, adjoining the elements

$$1+1, \quad 1+1+1, \quad 1+1+1+1, \quad 1+1+1+1+1, \quad \ldots.$$

Observe that this set is *infinite* and *ordered*, that is, you can compare any two elements and tell whether one is larger than the other. We define the symbols

$$2 = 1+1, \quad 3 = 1+1+1, \quad 4 = 1+1+1+1, \quad 5 = 1+1+1+1+1, \quad 6 = 1+1+1+1+1+1,$$

$$7 = 1+1+1+1+1+1+1, \quad 8 = 1+1+1+1+1+1+1+1, \quad 9 = 1+1+1+1+1+1+1+1+1.$$

Beyond 9 we reuse these symbols by also attaching a meaning to their place. Thus

$$10 = 1+1+1+1+1+1+1+1+1+1, \quad 11 = 1+1+1+1+1+1+1+1+1+1+1, \quad 12 = 1+1+1+1+1+1+1+1+1+1+1+1, \quad \text{etc.}$$

18 Definition A *positional notation* or *place-value notation system* is a numeral system in which each position is related to the next by a constant multiplier of that numeral system. Each position is represented by a limited set of symbols. The resultant value of each position is the value of its symbol or symbols multiplied by a power of the base.

As you know, we use base-10 positional notation. For example, in

$$1234 = 1 \cdot 1000 + 2 \cdot 100 + 3 \cdot 10 + 4 \cdot 1,$$

1 does not mean "1," but 1000; 2 does not mean "2," but 200, etc.

Before positional notation became standard, simple additive systems (sign-value notation) were used such as the value of the Hebrew letters, the value of the Greek letters, Roman Numerals, etc. Arithmetic with these systems was incredibly cumbersome.[2]

[1] The answer is not *priceless!*

[2] Try multiplying 123 by 321, say, using Roman numerals!

19 Definition The collection of all numbers defined by the recursion method above is called the set of *natural numbers*, and we represent them by the symbol $\mathbb{N}$, that is,

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

Natural numbers are used for two main reasons:

1. counting, as for example, "there are 10 sheep in the herd",
2. or ordering, as for example, "Los Angeles is the second largest city in the USA."

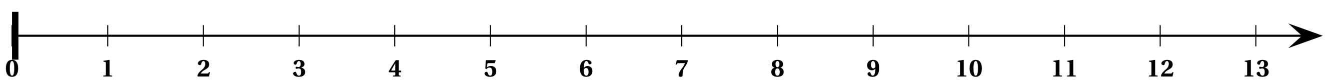


Figure 2.1: The Natural Numbers $\mathbb{N}$.

We can interpret the natural numbers as a linearly ordered set of points, as in figure 2.1. This interpretation of the natural numbers induces an order relation as defined below.

20 Definition Let a and b be two natural numbers. We say that a *is (strictly) less than* b, if a is to the left of b on the natural number line. We denote this by $a < b$.

☞ *In what follows, the symbol* $\in$ *is used to indicate that a certain element belongs to a certain set. The negation of* $\in$ *is* $\not\in$. *For example,* $1 \in \mathbb{N}$ *because* 1 *is a natural number, but* $\frac{1}{2} \not\in \mathbb{N}$.

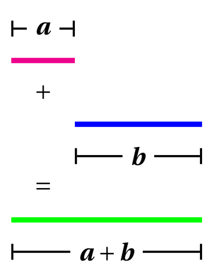


Figure 2.2: Addition in $\mathbb{N}$.

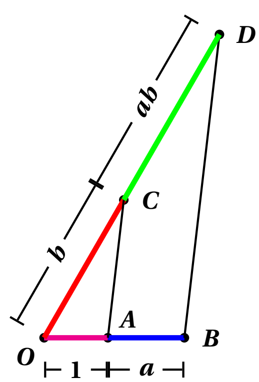


Figure 2.3: Multiplication in $\mathbb{N}$.

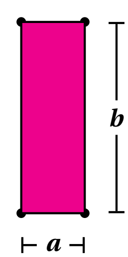


Figure 2.4: Multiplication in $\mathbb{N}$.

We can think of addition of natural numbers as concatenation of segment lengths. For example, if we add a segment whose length is b units to a segment whose length is a units, we obtain a segment whose length is $a+b$ units. See for example figure 2.2. Multiplication is somewhat harder to interpret. Form $\triangle OAC$ with $OA = 1$ and $OC = b$. Extend the segment $[OA]$ to B, with $AB = a$. Through B draw a line parallel to $[AC]$, meeting $[OC]$-extended at D. By the similarity of $\triangle OAC$ and OBD, $CD = ab$. Another possible interpretation of multiplication occurs in figure 2.4, where a rectangle of area ab (square units) is formed having sides of a units by b units.

Observe that if we add or multiply any two natural numbers, the result is a natural numbers. We encode this observation in the following axiom.

21 Axiom (Closure) $\mathbb{N}$ is *closed under addition*, that is, if $a \in \mathbb{N}$ and $b \in \mathbb{N}$ then also $a+b \in \mathbb{N}$. $\mathbb{N}$ is *closed under multiplication*, that is, if $x \in \mathbb{N}$ and $y \in \mathbb{N}$ then also $xy \in \mathbb{N}$.

If 0 is added to any natural number, the result is unchanged. Similarly, if a natural number is multiplied by 1, the result is also unchanged. This is encoded in the following axiom.

22 Axiom (Additive and Multiplicative Identity) $0 \in \mathbb{N}$ is the *additive identity* of $\mathbb{N}$, that it, it has the property that for all $x \in \mathbb{N}$ it follows that

$$x = 0 + x = x + 0.$$

$1 \in \mathbb{N}$ is the *multiplicative identity* of $\mathbb{N}$, that it, it has the property that for all $a \in \mathbb{N}$ it follows that

$$a = 1a = a1.$$

Again, it is easy to see that when two natural numbers are added or multiply, the result does not depend on the order. This is encoded in the following axiom.

23 Axiom (Commutativity) Let $a \in \mathbb{N}$ and $b \in \mathbb{N}$. Then $a+b = b+a$ and $ab = ba$.

Two other important axioms for the natural numbers are now given.

24 Axiom (Associativity) Let a, b, c be natural numbers. Then the order of parentheses when performing addition is irrelevant, that is,

$$a + (b + c) = (a + b) + c = a + b + c.$$

Similarly, the order of parentheses when performing multiplication is irrelevant,

$$a(bc) = (ab)c = abc.$$

25 Axiom (Distributive Law) Let a, b, c be natural numbers. Then

$$a(b + c) = ab + ac,$$

and

$$(a + b)c = ac + bc.$$

We now make some further remarks about addition and multiplication. The product mn is simply put, stenography for addition. That is, we have the equivalent expressions

$$mn = \underbrace{n + n + \cdots + n}_{m \text{ times}} = \underbrace{m + m + \cdots + m}_{n \text{ times}}.$$

Thus when we write $(3)(4)$ we mean

$$(3)(4) = 3 + 3 + 3 + 3 = 4 + 4 + 4 = 12.$$

Hence if we encounter an expression like

$$(3)(5) + (6)(4)$$

we must clearly perform the multiplication first and then the addition, obtaining

$$(3)(5) + (6)(4) = 5 + 5 + 5 + 6 + 6 + 6 + 6 = 39,$$

or more succinctly

$$(3)(5) + (6)(4) = 15 + 24 = 39.$$

In turn, a stenographic form for multiplication by the same number is exponentiation, which we will now define.

26 Definition (Exponentiation) If n is a natural number greater than or equal to 1 then the *n-th power of* a is defined by

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}$$

Here, a is the *base*, and n is the *exponent.*. If a is any number different from 0 then we define

$$a^0 = 1.$$

We do not attach any meaning to 0^0.[3]

27 Example Powers of 2 permeate computer culture. A *bit* is a binary digit taking a value of either 0 (electricity does not pass through a circuit) or 1 (electricity passes through a circuit). We have,

$$\begin{array}{ll} 2^1 = 2 & 2^6 = 64 \\ 2^2 = 4, & 2^7 = 128 \\ 2^3 = 8, & 2^8 = 256 \\ 2^4 = 16, & 2^9 = 512 \\ 2^5 = 32, & 2^{10} = 1024 \end{array}$$

Since $2^{10} \approx 1000$, we call 2^{10} a *kilobit*.[4]

28 Example Notice that $2^3 = 8$ and $3^2 = 9$ are consecutive powers. A 150 year old problem, called *Catalan's Conjecture* asserted that these were the only strictly positive consecutive powers. This conjecture was proved by the number theorist Preda Mihailescu on 18 April 2002. This is one more example that not "everything" has been discovered in Mathematics, that research still goes on today.

Notice that a^b **is not** ab. *Thus* $2^3 = (2)(2)(2) = 8$, *and* **not** $(2)(3) = 6$.

In any expression containing addition and exponentiation, we perform the exponentiation first, since it is really a shortcut for writing multiplication.

29 Example We have

$$(2)(4) + 3^3 = 8 + 27 = 35,$$

$$3^2 + 2^3 = 9 + 8 = 17,$$

$$(3^2)(4)(5) = (9)(4)(5) = 180.$$

The order of operations can be coerced by means of grouping symbols, like parentheses ( ), brackets [ ], or braces { }.

30 Example We have

$$(3+2)(5+3) = (5)(8) = 40,$$

$$(3+2)^2 = (5)^2 = 25,$$

$$(5 + (3 + 2(4))^2)^3 = (5 + (3 + 8)^2)^3 = (5 + (11)^2)^3 = (5 + 121)^3 = 126^3 = 2000376.$$

[3] Much to the chagrin of logicians and other spawns of Satan.
[4] From the Greek *kilo*, meaning *thousand*.

☞ *Observe that* $(3+2)^2 = 25$ *but that* $3^2 + 2^2 = 9 + 4 = 13$. *Thus exponentiation does not distribute over addition.*

31 Example Each element of the set

$$\{10, 11, 12, \ldots, 19, 20\}$$

is multiplied by each element of the set

$$\{21, 22, 23, \ldots, 29, 30\}.$$

If all these products are added, what is the resulting sum?

Solution: ► *This is asking for the product* $(10+11+\cdots+20)(21+22+\cdots+30)$ *after all the terms are multiplied. But* $10+11+\cdots+20 = 165$ *and* $21+22+\cdots+30 = 255$. *Therefore we want* $(165)(255) = 42075$.
◄

32 Definition To *evaluate* an expression with letters means to substitute the values of its letters by the equivalent values given.

33 Example Evaluate $a^3 + b^3 + c^3 + 3abc$ when $a = 1$, $b = 2$, $c = 3$.

Solution: ► *Substituting,*

$$1^3 + 2^3 + 3^3 + 3(1)(2)(3) = 1 + 8 + 27 + 18 = 54.$$

◄

We introduce now the operation of extracting roots. Notice that we will introduce this "new" operation by resorting to the "reverse" of an "old" operation. This is often the case in Mathematics.

34 Definition (Roots) Let m be a natural number greater than or equal to 2, and let a and b be any natural numbers. We write that $\sqrt[m]{a} = b$ if $a = b^m$. In this case we say that b is the *m-th root* of a. The number m is called the *index* of the root.

☞ *In the special case when* $m = 2$, *we do not write the index. Thus we will write* $\sqrt{a}$ *rather than* $\sqrt[2]{a}$. *The number* $\sqrt{a}$ *is called the* square root of a. *The number* $\sqrt[3]{a}$ *is called the* cubic root of a.

35 Example We have

$$\begin{array}{lll} \sqrt{1} = 1 & \text{because} & 1^2 = 1, \\ \sqrt{4} = 2 & \text{because} & 2^2 = 4, \\ \sqrt{9} = 3 & \text{because} & 3^2 = 9, \\ \sqrt{16} = 4 & \text{because} & 4^2 = 16, \\ \sqrt{25} = 5 & \text{because} & 5^2 = 25, \\ \sqrt{36} = 6 & \text{because} & 6^2 = 36. \end{array}$$

36 Example We have

$$\sqrt[10]{1} = 1 \quad \text{because} \quad 1^{10} = 1,$$

$$\sqrt[5]{32} = 2 \quad \text{because} \quad 2^5 = 32,$$

$$\sqrt[3]{27} = 3 \quad \text{because} \quad 3^3 = 27,$$

$$\sqrt[3]{64} = 4 \quad \text{because} \quad 4^3 = 64,$$

$$\sqrt[3]{125} = 5 \quad \text{because} \quad 5^3 = 125.$$

$$\sqrt[10]{1024} = 2 \quad \text{because} \quad 2^{10} = 1024.$$

Having now an idea of what it means to add and multiply natural numbers, we define subtraction and division of natural numbers by means of those operations. This is often the case in Mathematics: we define a new procedure in terms of old procedures.

37 Definition (Definition of Subtraction) Let m, n, x be natural numbers. Then the statement $m - n = x$ means that $m = x + n$.

38 Example To compute $15 - 3$ we think of which number when added 3 gives 15. Clearly then $15 - 3 = 12$ since $15 = 12 + 3$.

39 Definition (Definition of Division) Let m, n, x be natural numbers, with $n \neq 0$. Then the statement $m \div n = x$ means that $m = xn$.

40 Example Thus to compute $15 \div 3$ we think of which number when multiplied 3 gives 15. Clearly then $15 \div 3 = 5$ since $15 = 5 \cdot 3$.

Neither subtraction nor division are closed in $\mathbb{N}$. For example, $3 - 5$ is not a natural number, and neither is $3 \div 5$. Again, the operations of subtraction and division misbehave in the natural numbers, they are not commutative. For example, $5 - 3$ is not the same as $3 - 5$ and $20 \div 4$ is not the same as $4 \div 20$.

Homework

Problem 2.2.1 *Find the numerical value of* $1^1 2^2 3^3$.

Problem 2.2.2 *Find the numerical value of*

$$(\sqrt{36} - \sqrt{25})^2.$$

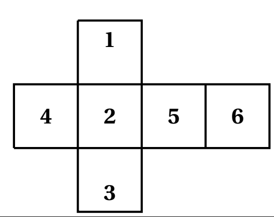
Problem 2.2.3 *Find the numerical value of* $3 \cdot 4 + 4^2$.

Problem 2.2.4 *Evaluate* $(a + b)(a - b)$ *when* $a = 5$ *and* $b = 2$.

Problem 2.2.5 *Evaluate* $(a^2 + b^2)(a^2 - b^2)$ *when* $a = 2$ *and* $b = 1$.

Problem 2.2.6 *If today is Thursday, what day will it be* 100 *days from now?*

Problem 2.2.7 *If the figure shewn is folded to form a cube, then three faces meet at every vertex. If for each vertex we take the product of the numbers on the three faces that meet there, what is the largest product we get?*



Problem 2.2.8 *A book publisher must bind* 4500 *books. One contractor can bind all these books in* 30 *days and another contractor in* 45 *days. How many days would be needed if both contractors are working simultaneously?*

Problem 2.2.9 *For commencement exercises, the students of a school are arranged in nine rows of twenty eight students per row. How many rows could be made with thirty six students per row?*

Problem 2.2.10 *Oscar rides his bike, being able to cover* 6 *miles in* 54 *minutes. At that speed, how long does it take him to cover a mile?*

Problem 2.2.11 *For which values of the natural number* n *is* $36 \div n$ *a natural number?*

Problem 2.2.12 *Doing only one multiplication, prove that*

$$\begin{aligned}&(666)(222)+(1)(333)+(333)(222)\\&+(666)(333)+(1)(445)+(333)(333)\\&+(666)(445)+(333)(445)+(1)(222) = 1000000.\end{aligned}$$

Problem 2.2.13 *A car with five tyres (four road tyres and a spare tyre) travelled* $30,000$ *miles. If all five tyres were used equally, how many miles' wear did each tyre receive?*

Problem 2.2.14 *A quiz has* 25 *questions with four points awarded for each correct answer and one point deducted for each incorrect answer, with zero for each question omitted. Anna scores* 77 *points. How many questions did she omit?*

Problem 2.2.15 *A certain calculator gives as the result of the product*

$$987654 \times 745321$$

the number $7.36119E11$, *which means* $736,119,000,000$. *Explain how to find the last six missing digits.*

Problem 2.2.16 *How many digits does* $4^{16}5^{25}$ *have?*

Problem 2.2.17 *As a publicity stunt, a camel merchant has decided to pose the following problem: "If one gathers all of my camels into groups of* 4, 5 *or* 6, *there will be no remainder. But if one gathers them into groups of* 7 *camels, there will be* 1 *camel left in one group." The number of camels is the smallest positive integer satisfying these properties. How many camels are there?*

Problem 2.2.18 *Create a new arithmetic operation* $\oplus$ *by letting* $a \oplus b = 1 + ab$.

1. *Compute* $1 \oplus (2 \oplus 3)$.
2. *Compute* $(1 \oplus 2) \oplus 3$.
3. *Is your operation associative. Explain.*
4. *Is the operation commutative? Explain.*

2.3 Fractions

> ***I continued to do arithmetic with my father, passing proudly through fractions to decimals. I eventually arrived at the point where so many cows ate so much grass, and tanks filled with water in so many hours I found it quite enthralling.*** *-Agatha CHRISTIE*

In this section we review some of the arithmetic pertaining fractions.

41 Definition A *(positive numerical) fraction* is a number of the form $m \div n = \dfrac{m}{n}$ where m and n are natural numbers and $n \neq 0$. Here m is the *numerator* of the fraction and n is the *denominator* of the fraction.

Given a natural number $n \neq 0$, we divide the interval between consecutive natural numbers k and $k+1$ into n equal pieces. Figures 2.5 2.6, and 2.7, shew examples with $n = 2$, $n = 3$, and $n = 4$, respectively. Notice that the larger n is, the finer the partition.

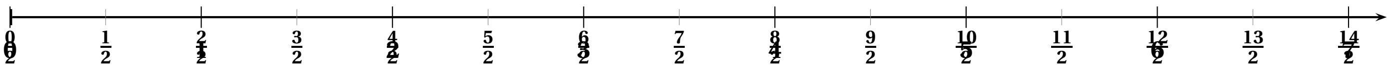


Figure 2.5: Fractions with denominator 2.

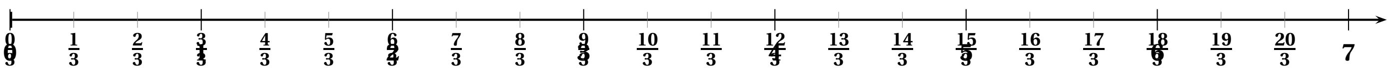


Figure 2.6: Fractions with denominator 3.

You will notice that in many of the fractions written above, the numerator is larger than the denominator. In primary school vocabulary these fractions are commonly called improper, *and perhaps, you were taught to convert them to a* mixed number. *For example, when we write* $\frac{9}{2}$, *your primary school teacher might have preferred to write* $4\frac{1}{2}$. *We will, however, prefer to use improper fractions as this is the common usage in Algebra and in modern computer algebra programmes.*

By doing more of the diagrams above you may have noticed that there are multiple names for, say, the natural number 2. For example

$$2 = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \frac{10}{5} = \frac{12}{6},$$

etc. This observation is a particular case of the following result.

42 Theorem (Cancellation Law) Let m, n, k be natural numbers with $n \neq 0$ and $k \neq 0$. Then

$$\frac{mk}{nk} = \frac{m}{n}.$$

Proof: *We will prove this for* $m \leq n$. *For* $m > n$ *the argument is similar. Divide the interval* $[0;1]$ *into* nk *pieces. Consider the* k*-th,* $2k$*-th,* $3k$*-th, ...,* nk*-th markers. Since* $\frac{nk}{nk} = 1$, *the* nk*-th marker has to be* 1. *Thus the* n *markers* k*-th,* $2k$*-th,* $3k$*-th, ...,* nk*-th, form a division of* $[0;1]$ *into* n *equal spaces. It follows that the* k*-th marker is* $\frac{1}{n}$, *that is,* $\frac{k}{nk} = \frac{1}{n}$, *the* $2k$*-th marker is* $\frac{2}{n}$, *that is,* $\frac{2k}{nk} = \frac{2}{n}$, *etc., and so the* mk*-th marker is* $\frac{m}{n}$, *that is,* $\frac{mk}{nk} = \frac{m}{n}$, *as we wanted to prove.* ❑

Thus given a fraction, if the numerator and the denominator have any common factors greater than 1, that is, any non-trivial factors, we may reduce the fraction and get an equal fraction.

43 Definition Two fractions such that $\frac{a}{b} = \frac{x}{y}$ are said to be *equivalent*. If $b < y$, then $\frac{a}{b}$ is said to be a *reduced form* of $\frac{x}{y}$.

It is possible to prove that any fraction has a unique reduced form with minimal denominator, the so called equivalent fraction in lowest terms. *This depends on the fact that the natural numbers can be factored uniquely into primes, and hence, though we will accept this result, we will not prove it here.*

44 Example To reduce $\frac{104}{120}$ to lowest terms, observe that

$$\frac{104}{120} = \frac{104 \div 4}{120 \div 4} = \frac{26 \div 2}{30 \div 2} = \frac{13}{15}.$$

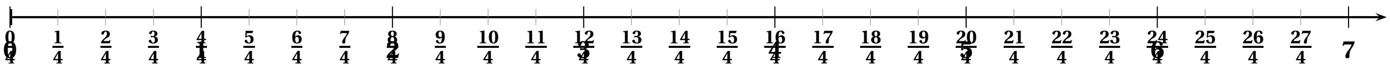


Figure 2.7: Fractions with denominator 4.

Since 13 and 15 do not have a non-trivial factor in common, $\dfrac{13}{15}$ is the desired reduction.

☞ *The reduction steps above are not unique. For example, if we had seen right away that* 8 *was a common factor, we would have obtained* $\dfrac{104}{120} = \dfrac{104 \div 8}{120 \div 8} = \dfrac{13}{15}$*, obtaining the same result. Hence, no matter how many steps you take, as long as there is valid cancellation to do and you perform them all, you will always obtain the right result.*

45 Example Find a fraction with denominator 120 equivalent to $\dfrac{11}{24}$.

Solution: ► *Observe that* $120 \div 24 = 5$. *Thus*

$$\frac{11}{24} = \frac{11 \cdot 5}{24 \cdot 5} = \frac{55}{120}.$$

◄

Homework

Problem 2.3.1 *Four comrades are racing side by side down a dusty staircase. Frodo goes down two steps at a time, Gimli three, Legolas four, and Aragorn five. If the only steps with all four's footprints are at the top and the bottom, how many steps have just one footprint?*

Problem 2.3.2 *What fraction of an hour is* 36 *minutes?*

Problem 2.3.3 *Express* $\dfrac{102}{210}$ *in least terms.*

Problem 2.3.4 *Find an equivalent fraction to* $\dfrac{102}{210}$ *with denominator* 3990.

Problem 2.3.5 *Arrange in increasing order:* $\dfrac{2}{3}, \dfrac{3}{5}, \dfrac{8}{13}$.

Problem 2.3.6 *Four singers take part in a musical round of* 4 *equal lines, each finishing after singing the round through* 3 *times. The second singer begins when the first singer begins the second line, the third singer begins when the first singer begins the third line, the fourth singer begins when the first singer begins the fourth line. What is the fraction of the total singing time when all the singers are singing simultaneously?*

2.4 Operations with Fractions

A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator the smaller the fraction.
-Count Lev Nikolgevich TOLSTOY

We now define addition of fractions. We would like this definition to agree with our definition of addition of natural numbers. Recall that we defined addition of natural numbers x and y as the concatenation of two segments of length x and y. We thus define the addition of two fractions $\dfrac{a}{b}$ and $\dfrac{c}{d}$ as the concatenation of two segments of length $\dfrac{a}{b}$ and $\dfrac{c}{d}$. The properties of associativity, commutativity, etc., stem from this definition. The problem we now have is how to concretely apply this definition to find the desired sum? From this concatenation definition it follows that natural numbers a and $b \neq 0$,

$$\frac{a}{b} = \underbrace{\frac{1}{b} + \cdots + \frac{1}{b}}_{a \text{ times}}. \tag{2.1}$$

It follows from the above property that

$$\frac{x}{b} + \frac{y}{b} = \frac{x+y}{b}. \tag{2.2}$$

We now determine a general formula for adding fractions of different denominators.

46 Theorem (Sum of Fractions) Let a, b, c, d be natural numbers with $b \neq 0$ and $d \neq 0$. Then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}.$$

Proof: *From the Cancellation Law (Theorem 42),*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd},$$

proving the theorem. ❑

The formula obtained in the preceding theorem agrees with that of (2.2) when the denominators are equal. For, using the theorem,

$$\frac{x}{b} + \frac{y}{b} = \frac{xb}{b \cdot b} + \frac{yb}{b \cdot b} = \frac{xb+by}{b \cdot b} = \frac{b(x+y)}{b \cdot b} = \frac{x+y}{b},$$

where we have used the distributive law.

Observe that the trick for adding the fractions in the preceding theorem was to convert them to fractions of the same denominator.

47 Definition To express two fractions in a *common denominator* is to write them in the same denominator. The smallest possible common denominator is called the *least common denominator*.

48 Example Add: $\frac{3}{5} + \frac{4}{7}$.

Solution: ► *A common denominator is* $5 \cdot 7 = 35$. *We thus find*

$$\frac{3}{5} + \frac{4}{7} = \frac{3 \cdot 7}{5 \cdot 7} + \frac{4 \cdot 5}{7 \cdot 5} = \frac{21}{35} + \frac{20}{35} = \frac{41}{35}.$$

◄

☞ *In the preceding example,* 35 *is not the only denominator that we may have used. Observe that* $\frac{3}{5} = \frac{42}{70}$ *and* $\frac{4}{7} = \frac{40}{70}$. *Adding,*

$$\frac{3}{5} + \frac{4}{7} = \frac{3 \cdot 14}{5 \cdot 14} + \frac{4 \cdot 10}{7 \cdot 10} = \frac{42}{70} + \frac{40}{70} = \frac{82}{70} = \frac{82 \div 2}{70 \div 2} = \frac{41}{35}.$$

This shews that it is not necessary to find the least common denominator *in order to add fractions, simply a common denominator.*

In fact, let us list the multiples of 5 and of 7 and let us circle the common multiples on these lists:

The multiples of 5 are $5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, \ldots,$

The multiples of 7 are $7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, \ldots.$

The sequence

$$35, 70, 105, 140, \ldots,$$

is a sequence of common denominators for 5 and 7.

49 Example To perform the addition $\frac{2}{7}+\frac{1}{5}+\frac{3}{2}$, observe that $7\cdot 5\cdot 2 = 70$ is a common denominator. Thus

$$\begin{aligned}\frac{2}{7}+\frac{1}{5}+\frac{3}{2} &= \frac{2\cdot 10}{7\cdot 10}+\frac{1\cdot 14}{5\cdot 14}+\frac{3\cdot 35}{2\cdot 35}\\ &= \frac{20}{70}+\frac{14}{70}+\frac{105}{70}\\ &= \frac{20+14+105}{70}\\ &= \frac{139}{70}.\end{aligned}$$

A few words about subtraction of fractions. Suppose $\frac{a}{b} \geq \frac{c}{d}$. Observe that this means that $ad \geq bc$, which in turns means that $ad - bc \geq 0$. Then from a segment of length $\frac{a}{b}$ we subtract one of $\frac{b}{d}$. In much the same manner of addition of fractions then

$$\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}.$$

Now, since $ad - bc \geq 0$, $ad - bc$ is a natural number and hence $\frac{ad-bc}{bd}$ is a fraction.

We now approach multiplication of fractions. We saw that a possible interpretation for the product xy of two natural numbers x and y is the area of a rectangle with sides of length x and y. We would like to extend this interpretation in the case when x and y are fractions. That is, given a rectangle of sides of length $x = \frac{a}{b}$ and $y = \frac{c}{d}$, we would like to deduce that $\frac{ac}{bd}$ is the area of this rectangle.

50 Theorem (Multiplication of Fractions) Let a, b, c, d be natural numbers with $b \neq 0$ and $d \neq 0$. Then

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}.$$

Proof: *First consider the case when $a = c = 1$. Start with a unit square and cut it horizontally into b equal segments. Then cut it vertically into d equal segments. We have now bd equal pieces, each one having an area of $\frac{1}{bd}$. Since each piece is in dimension $\frac{1}{b}$ by $\frac{1}{d}$, we have shewn that*

$$\frac{1}{b}\cdot\frac{1}{d}=\frac{1}{bd}.$$

Construct now a rectangle of length $\frac{a}{b}$ and width $\frac{c}{d}$. Such a rectangle is obtained by concatenating along its length a segments of length $\frac{1}{b}$ and along its width c segments of length $\frac{1}{d}$. This partitions the large rectangle into ac sub-rectangles, each of area $\frac{1}{bd}$. Hence the area of the $\frac{a}{b}$ by $\frac{c}{d}$ rectangle is $ab\left(\frac{1}{bd}\right)$, from where

$$\frac{a}{b}\cdot\frac{c}{d}=ac\left(\frac{1}{bd}\right)=\frac{ac}{bd},$$

proving the theorem. ❑

51 Example We have

$$\frac{2}{3}\cdot\frac{3}{7}=\frac{6}{21}=\frac{2}{7}.$$

Alternatively, we could have cancelled the common factors, as follows,

$$\frac{2}{\cancel{3}}\cdot\frac{\cancel{3}}{7}=\frac{2}{7}.$$

52 Example Find the exact value of the product

$$\left(1-\frac{2}{5}\right)\left(1-\frac{2}{7}\right)\left(1-\frac{2}{9}\right)\cdots\left(1-\frac{2}{99}\right)\left(1-\frac{2}{101}\right).$$

Solution: ► *We have,*

$$\begin{aligned}\left(1-\frac{2}{5}\right)\left(1-\frac{2}{7}\right)\left(1-\frac{2}{9}\right)\cdots\left(1-\frac{2}{99}\right)\left(1-\frac{2}{101}\right) &= \frac{3}{5}\cdot\frac{5}{7}\cdot\frac{7}{9}\cdot\frac{9}{11}\cdots\frac{97}{99}\cdot\frac{99}{101}\\ &= \frac{3}{101}.\end{aligned}$$

◄

We now tackle division of fractions. Recall that we defined division of natural numbers as follows. If $n \neq 0$ and m, x are natural numbers then $m \div n = x$ means that $m = xn$. We would like a definition of fraction division compatible with this definition of natural number division. Hence we give the following definition.

53 Definition Let a, b, c, d be natural numbers with $b \neq 0$, $c \neq 0$, $d \neq 0$. We define the fraction division

$$\frac{a}{b}\div\frac{c}{d}=\frac{x}{y} \iff \frac{a}{b}=\frac{x}{y}\cdot\frac{c}{d}.$$

We would like to know what $\frac{x}{y}$ above is in terms of a, b, c, d. For this purpose we have the following theorem.

54 Theorem (Division of Fractions) Let a, b, c, d be natural numbers with $b \neq 0$, $c \neq 0$, $d \neq 0$. Then

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc},$$

that is, $\frac{x}{y}$ in definition 53 is $\frac{x}{y}=\frac{ad}{bc}$.

Proof: *Let us prove that $\frac{x}{y}=\frac{ad}{bc}$ satisfies the definition of division of fractions. Observe that*

$$\frac{x}{y}\cdot\frac{c}{d}=\frac{ad}{bc}\cdot\frac{c}{d}=\frac{adc}{bcd}=\frac{a}{b},$$

and hence $\frac{x}{y}$ so chosen is the right result for the division of fractions. Could there be another fraction, say $\frac{x'}{y'}$ that satisfies definition 53? Suppose

$$\frac{a}{b}=\frac{x'}{y'}\cdot\frac{c}{d}.$$

Then

$$\frac{x}{y} = \frac{a}{b}\cdot\frac{d}{c} = \frac{x'}{y'}\cdot\frac{c}{d}\cdot\frac{d}{c} = \frac{x'}{y'}.$$

Hence $\frac{x}{y} = \frac{a}{b}\cdot\frac{d}{c}$ *is the only fraction that satisfies the definition of fraction division.* ❑

55 Definition Let $c \neq 0$, $d \neq 0$ be natural numbers. The *reciprocal* of the fraction $\frac{c}{d}$ is the fraction $\frac{d}{c}$.

Theorem 54 says that in order to divide two fractions we must simply multiply the first one by the reciprocal of the other.

56 Example We have,

$$\frac{24}{35} \div \frac{20}{7} = \frac{24}{35}\cdot\frac{7}{20} = \frac{4\cdot 6}{7\cdot 5}\cdot\frac{7\cdot 1}{4\cdot 5} = \frac{6\cdot 1}{5\cdot 5} = \frac{6}{25}.$$

Homework

Problem 2.4.1 *Complete the "fraction puzzle" below.*

$\frac{1}{3}$	$+$		$=$	2
$+$		$-$		
$\frac{3}{4}$	$\times$		$=$	
$=$		$=$		
	$\div$	$\frac{2}{3}$	$=$	

Problem 2.4.2 *Find the exact numerical value of*

$$\left(\frac{2}{51}\right)\div\left(\frac{3}{17}\right)\cdot\left(\frac{7}{10}\right).$$

Problem 2.4.3 *Find the exact numerical value of*

$$\frac{\frac{4}{7}-\frac{2}{5}}{\frac{4}{7}+\frac{2}{5}}.$$

Problem 2.4.4 *Find the value of*

$$\frac{10+10^2}{\frac{1}{10}+\frac{1}{100}}.$$

Problem 2.4.5 *Find the exact numerical value of*

$$\frac{1^3+2^3+3^3-3(1)(2)(3)}{(1+2+3)(1^2+2^2+3^2-1\cdot 2-2\cdot 3-3\cdot 1)}.$$

Problem 2.4.6 *If*

$$\frac{1}{1+\frac{1}{5}} = \frac{a}{b},$$

where the fraction $\frac{a}{b}$ *is in least terms, find* a^2+b^2.

Problem 2.4.7 *Find the exact numerical value of*

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{99}\right)\left(1-\frac{1}{100}\right).$$

Problem 2.4.8 *John takes* 2 *hours to paint a room, whereas Bill takes* 3 *hours to paint the same room. How long would it take if both of them start and are working simultaneously?*

Problem 2.4.9 *Naomi has* 16 *yards of gift-wrap in order to wrap the gifts for the Festival of Lights at the community centre. Each gift requires* $1\frac{7}{8}$ *yards of paper. How many gifts can she wrap?*

Problem 2.4.10 *Evaluate*

$$\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2}}}}.$$

Problem 2.4.11 *Find the value of*

$$1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4}}}.$$

Problem 2.4.12 *At a certain college* 99% *of the* 100 *students are female, but only* 98% *of the students living on campus are female. If some females live on campus, how many students live off campus?*

Problem 2.4.13 *What would be the price of a* $5\frac{1}{2}$*-mile trip with the following taxi-cab company?*

SMILING CAMEL TAXI SERVICES	
First $\frac{1}{4}$ *mi*	*\$ .85*
Additional $\frac{1}{4}$ *mi*	*\$ .40*

2.5 The Integers

God created the integers. Everything else is the work of Man. *-Leopold KROENECKER*

The introduction of fractions in the preceding section helped solve the problem that the natural numbers are not closed under division. We now solve the problem that the natural numbers are not closed under subtraction.

57 Definition A natural number not equal to $\mathbf{0}$ is said to be *positive*. The set

$$\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \ldots\}$$

is called the set of *positive integers*.

58 Definition Given a natural number n, we define its opposite $-n$ as the unique number $-n$ such that

$$n + (-n) = (-n) + n = \mathbf{0}.$$

The collection

$$\{-\mathbf{1}, -\mathbf{2}, -\mathbf{3}, -\mathbf{4}, -\mathbf{5}, \ldots\}$$

of all the opposites of the natural numbers is called the set of *negative integers*. The collection of natural numbers together with the negative integers is the set of *integers*, which we denote by the symbol[5] $\mathbb{Z}$.

A graphical representation of the integers is given in figure 2.8.

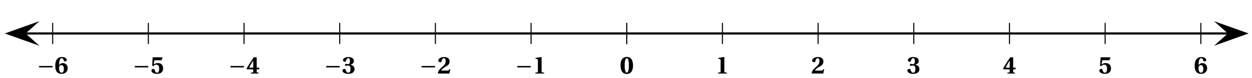


Figure 2.8: The Integers $\mathbb{Z}$.

There seems to be no evidence of usage of negative numbers by the Babylonians, Pharaonic Egyptians, or the ancient Greeks. It seems that the earliest usage of them came from China and India. In the 7th Century, negative numbers were used for bookkeeping in India. The Hindu astronomer Brahmagupta, writing around A.D. 630, shews a clear understanding of the usage of negative numbers.

Thus it took humans a few millennia to develop the idea of negative numbers. Since, perhaps, our lives are more complex now, it is not so difficult for us to accept their existence and understand the concept of negative numbers.

Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$. If $a > \mathbf{0}$, then $-a < \mathbf{0}$. If $b < \mathbf{0}$, then $-b > \mathbf{0}$. Thus either the number, or its "mirror reflexion" about $\mathbf{0}$ is positive, and in particular, for any $a \in \mathbb{Z}$, $-(-a) = a$. This leads to the following definition.

[5]From the German word for number: *Zählen*.

59 Definition Let $a \in \mathbb{Z}$. The *absolute value* of a is defined and denoted by

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0, \end{cases}$$

60 Example $|5| = 5$ since $5 > 0$. $|-5| = -(-5) = 5$, since $-5 < 0$.

Letters have no idea of the sign of the numbers they represent. Thus it is a **mistake** *to think, say, that $+x$ is always positive and $-x$ is always negative.*

We would like to define addition, subtraction, multiplication and division in the integers in such a way that these operations are consistent with those operations over the natural numbers and so that they again closure, commutativity, associativity, and distributivity under addition and multiplication.

We start with addition. Recall that we defined addition of two natural numbers and of two fractions as the concatenation of two segments. We would like this definition to extend to the integers, but we are confronted with the need to define what a “negative segment” is. This we will do as follows. If $a < 0$, then $-a > 0$. We associate with a a segment of length $|-a|$, but to the left of 0 on the line, as in figure 2.9.

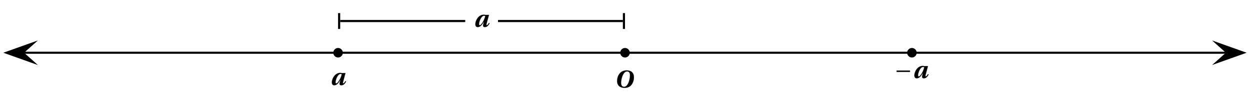


Figure 2.9: A negative segment. Here $a < 0$.

Hence we define the addition of integers a, b, as the concatenation of segments. Depending on the sign of a and b, we have four cases. (We exclude the cases when at least one of a or b is zero, these cases being trivial.)

61 Example (Case $a > 0, b > 0$) To add b to a, we first locate a on the line. From there, we move b units right (since $b > 0$), landing at $a + b$. Notice that this case reduces to addition of natural numbers, and hence, we should obtain the same result as for addition of natural numbers. This example is illustrated in figure 2.10. For a numerical example (with $a = 3$, $b = 2$), see figure 2.11.

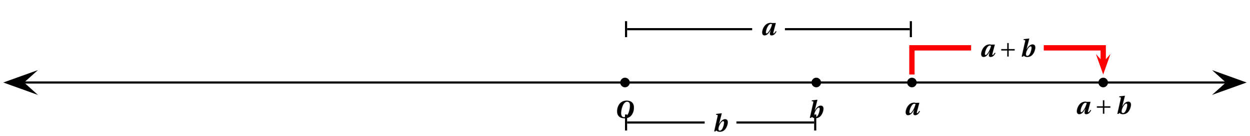


Figure 2.10: $a + b$ with $a > 0$, $b > 0$.

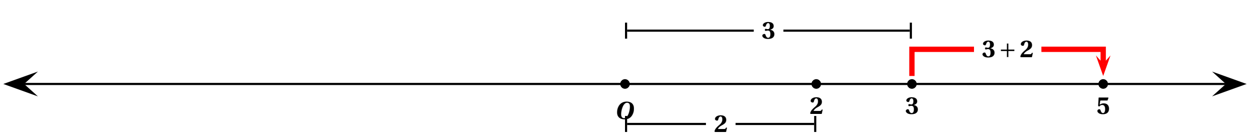


Figure 2.11: $3 + 2$.

62 Example (Case $a<0, b<0$) To add b to a, we first locate a on the line. From there, we move b units left (since $b<0$), landing at $a+b$. This example is illustrated in figure 2.12. For a numerical example (with $a=-3$, $b=-2$), see figure 2.13.

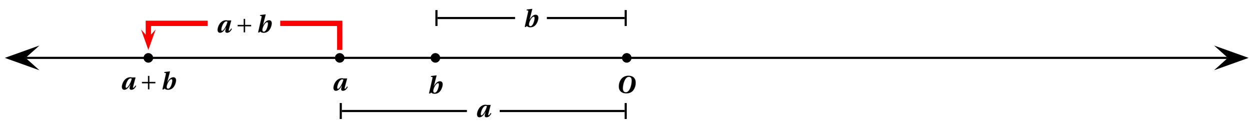


Figure 2.12: $a+b$ with $a<0$, $b<0$.

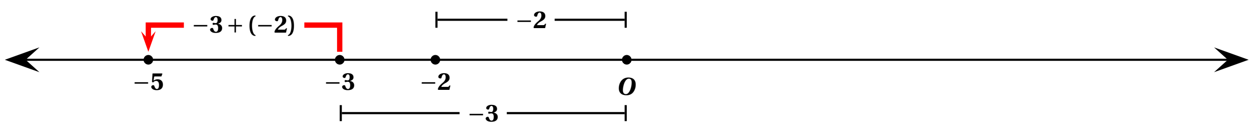


Figure 2.13: $-3+(-2)$.

Examples 61 and 62 conform to the following intuitive idea. If we associate positive numbers to "gains" and negative numbers to "losses" then a "gain" plus a "gain" is a "larger gain" and a "loss" plus a "loss" is a "larger loss."

63 Example We have,

$$(+1)+(+3)+(+5)=+9,$$

since we are adding three gains, and we thus obtain a larger gain.

64 Example We have,

$$(-11)+(-13)+(-15)=-39,$$

since we are adding three losses, and we thus obtain a larger loss.

We now tackle the cases when the summands have opposite signs. In this case, borrowing from the preceding remark, we have a "gain" plus a "loss." In such a case it is impossible to know before hand whether the result is a gain or a loss. The only conclusion we could gather, again, intuitively, is that the result will be in a sense "smaller", that is, we will have a "smaller gain" or a smaller loss." Some more thinking will make us see that if the "gain" is larger than the loss, then the result will be a "smaller gain," and if the "loss" is larger than the "gain" then the result will be a "smaller loss."

65 Example (Case $a<0, b>0$) To add b to a, we first locate a on the line. Since $a<0$, it is located to the left of O. From there, we move b units right (since $b>0$), landing at $a+b$. This example is illustrated in figure 2.14. For a numerical example (with $a=-3$, $b=2$), see figure 2.15. Again, we emphasise, in the sum $(-3)+(+2)$, the "loss" is larger than the "gain." Hence when adding, we expect a "smaller loss", fixing the sign of the result to be minus.

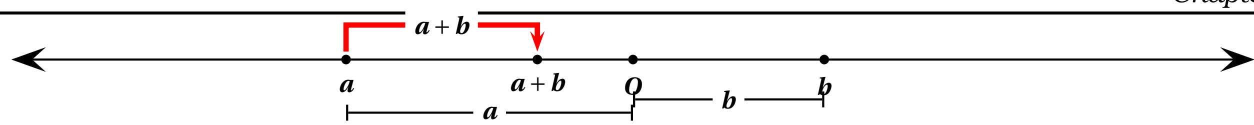


Figure 2.14: $a+b$ with $a<0$, $b>0$.

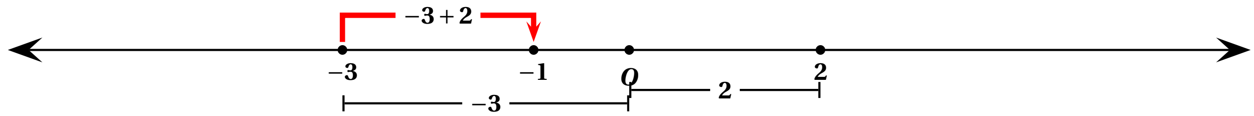


Figure 2.15: $-3+2$.

66 Example (Case $a>0, b<0$) To add b to a, we first locate a on the line. From there, we move b units left (since $b<0$), landing at $a+b$. This example is illustrated in figure 2.16. For a numerical example (with $a=3$, $b=-2$), see figure 2.17.

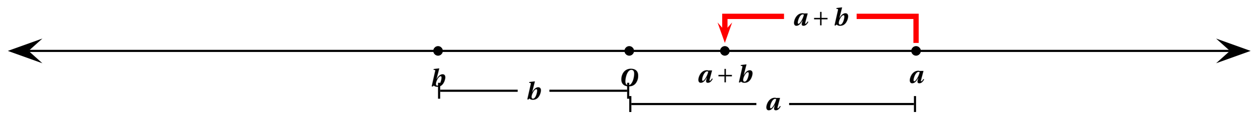


Figure 2.16: $a+b$ with $a>0$, $b<0$.

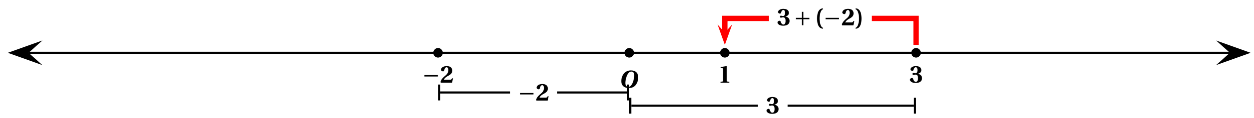


Figure 2.17: $3+(-2)$.

67 Example We have,

$$(+19)+(-21)=-2,$$

since the loss of 21 is larger than the gain of 19 and so we obtain a loss.

68 Example We have,

$$(-100)+(+210)=+110,$$

since the loss of 100 is smaller than the gain of 210 and so we obtain a gain.

We now turn to subtraction. We define subtraction in terms of addition.

69 Definition Subtraction is defined as

$$a-b=a+(-b).$$

70 Example We have, $(+8)-(+5)=(+8)+(-5)=3$.

71 Example We have, $(-8)-(-5)=(-8)+(+5)=-3$.

72 Example We have, $(+8)-(-5)=(+8)+(+5)=13$.

73 Example We have, $(-8)-(+5)=(-8)+(-5)=-13$.

We now explore multiplication. Again, we would like the multiplication rules to be consistent with those we have studied for natural numbers. This entails that, of course, that if we multiply two positive integers, the result will be a positive integer. What happens in other cases? Suppose $a > 0$ and $b < 0$. We would like to prove that $ab < 0$. Observe that

$$a(b-b) = 0 \implies ab - ab = 0 \implies ab + a(-b) = 0 \implies ab = -a(-b).$$

Since $-b > 0$, $a(-b)$ is the product of two positive integers, and hence positive. Thus $-a(-b)$ is negative, and so $ab = -a(-b) < 0$. We have proved that the product of a positive integer and a negative integer is negative. Using the same trick we can prove that

$$(-x)(-y) = xy.$$

If $x < 0$, $y < 0$, then both $-x > 0$, $-y > 0$, hence the product of two negative integers is the same as the product of two positive integers, and hence positive. We have thus proved the following rules:

$$(+)(+) = (-)(-) = +, \qquad (+)(-) = (-)(+) = -.$$

Intuitively, you may think of a negative sign as a reversal of direction on the real line. Thus the product or quotient of two integers different sign is negative. Two negatives give two reversals, which is to say, no reversal at all, thus the product or quotient of two integers with the same sign is positive. The sign rules for division are obtained from and are identical from those of division.

74 Example We have,

$$(-2)(5) = -10, \qquad (-2)(-5) = +10, \qquad (+2)(-5) = -10, \qquad (+2)(+5) = +10.$$

75 Example We have,

$$(-20) \div (5) = -4, \qquad (-20) \div (-5) = +4, \qquad (+20) \div (-5) = -4, \qquad (+20) \div (+5) = +4.$$

The rules of operator precedence discussed in the section of natural numbers apply.

76 Example We have,

$$\begin{aligned} \frac{(-8)(-12)}{3} + \frac{30}{((-2)(3))} &= \frac{96}{3} + \frac{30}{(-6)} \\ &= 32 + (-5) \\ &= 27. \end{aligned}$$

77 Example We have,

$$\begin{aligned} (5-12)^2 - (-3)^3 &= (-7)^2 - (-27) \\ &= 49 + 27 \\ &= 76. \end{aligned}$$

As a consequence of the rule of signs for multiplication, a product containing an odd number of minus signs will be negative and a product containing an even number of minus signs will be positive.

78 Example

$$(-2)^2 = 4, \qquad (-2)^3 = -8, \qquad (-2)^{10} = 1024.$$

☞ *Notice the difference between, say,* $(-a)^2$ *and* $-a^2$. $(-a)^2$ *is the square of* $-a$, *and hence it is always non-negative. On the other hand,* $-a^2$ *is the opposite of* a^2, *and therefore it is always non-positive.*

79 Example We have,

$$5+(-4)^2 = 5+16 = 21,$$

$$5-4^2 = 5-16 = -11,$$

$$5-(-4)^2 = 5-16 = -11.$$

Homework

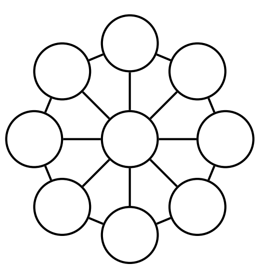
Problem 2.5.1 *Perform the following operations mentally.*

1. $(-9)-(-17)$
2. $(-17)-(9)$
3. $(-9)-(17)$
4. $(-1)-(2)-(-3)$
5. $(-100)-(101)+(-102)$
6. $|-2|-|-2|$
7. $|-2|-(-|2|)$
8. $|-100|+(-100)-(-(-100))$

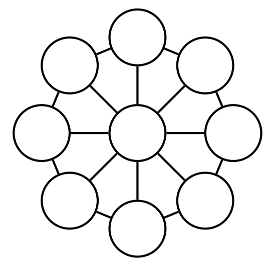
Problem 2.5.2 *Place the nine integers*

$$\{-4,-3,-2,-1,0,1,2,3,4\}$$

exactly once in the diagram below so that every diagonal sum be the same.



Problem 2.5.3 *Place the nine integers* $\{-2,-1,0,1,2,3,4,5,6\}$ *exactly once in the diagram below so that every diagonal sum be the same.*



Problem 2.5.4 *Complete the "crossword" puzzle with* 1*'s or* -1*'s.*

-1	$\times$		$=$	
$\times$	■	$\times$	■	$\times$
	$\times$	-1	$=$	
$=$	■	$=$	■	$=$
	$\times$		$=$	-1

Problem 2.5.5 *Evaluate the expression*

$$\frac{a^3+b^3+c^3-3abc}{a^2+b^2+c^2-ab-bc-ca}$$

when $a=2$, $b=-3$, *and* $c=5$.

2.6 Rational, Irrational, and Real Numbers

Bridges would not be safer if only people who knew the proper definition of a real number were allowed to design them. *-N. David MERMIN*

80 Definition The set of *negative fractions* is the set

$$\left\{-\frac{a}{b} : a \in \mathbb{N}, a > 0, b \in \mathbb{N}, b > 0\right\}.$$

The set of positive fractions together with the set of negative fractions and the number 0 form the set of *rational numbers*, which we denote by $\mathbb{Q}$.

The rules for operations with rational numbers derive from those of operations with fractions and with integers. Also, the rational numbers are closed under the four operations of addition, subtraction, multiplication, and division. A few examples follow.

81 Example We have

$$\begin{aligned}
\frac{2}{5}\cdot\frac{15}{12}-\frac{7}{10}\div\frac{14}{15} &= \frac{2}{5}\cdot\frac{15}{12}-\frac{7}{10}\cdot\frac{15}{14}\\
&= \frac{\cancel{2}}{\cancel{5}}\cdot\frac{\cancel{5}\cdot\cancel{3}}{\cancel{2}\cdot 2\cdot\cancel{3}}-\frac{\cancel{7}}{2\cdot\cancel{5}}\cdot\frac{3\cdot\cancel{5}}{2\cdot\cancel{7}}\\
&= \frac{1}{2}-\frac{3}{4}\\
&= \frac{2}{4}-\frac{3}{4}\\
&= -\frac{1}{4}.
\end{aligned}$$

It can be proved that any rational number has a decimal expansion which is either periodic (repeats) or terminates, and that viceversa, any number with either a periodic or a terminating expansion is a rational number. For example, $\frac{1}{4} = 0.25$ has a terminating decimal expansion, and $\frac{1}{11} = 0.0909090909\ldots = 0.\overline{09}$ has a repeating one. By long division you may also obtain

$$\frac{1}{7} = 0.\overline{142857}, \qquad \frac{1}{17} = 0.\overline{0588235294117647},$$

and as you can see, the periods may be longer than what your calculator can handle.

What about numbers whose decimal expansion is infinite and does not repeat? This leads us to the following definition.

82 Definition A number whose decimal expansion is infinite and does not repeat is called an *irrational number.*

From the discussion above, an irrational number is one that cannot be expressed as a fraction of two integers.

83 Example Consider the number

$$0.1010010001000010000010000001\ldots,$$

where the number of 0's between consecutive 1's grows in sequence: $1, 2, 3, 4, 5, \ldots$. Since the number of 0's is progressively growing, this infinite decimal does not have a repeating period and hence must be an irrational number.

Using my computer, when I enter $\sqrt{2}$ I obtain as an answer

$$1.4142135623730950488016887242097.$$

Is this answer exact? Does this decimal repeat? It can be proved that the number $\sqrt{2}$ is irrational, hence the above answer is only an approximation and the decimal does not repeat. The first proof of the irrationality of $\sqrt{2}$ is attributed to Hippasus of Metapontum, one of the disciples of Pythagoras (c 580 BC–c 500 BC).[6] The Greek world view at that time was that all numbers where rational, and hence this discovery was anathema to the Pythagoreans who decided to drown Hipassus for his discovery.

☞ *It can be proved that if n is a natural number that is not a perfect square, then $\sqrt{n}$ is irrational. Hence $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}$, etc., are all irrational.*

In 1760, Johann Heinrich Lambert (1728 - 1777) proved that π is irrational.

☞ *In particular, then, it would be incorrect to write $\pi = 3.14$, or $\pi = \frac{22}{7}$, or $\pi = \frac{355}{113}$, etc., since π is not rational. All of these are simply approximations, and hence we must write $\pi \approx 3.14$, $\pi \approx \frac{22}{7}$, or $\pi \approx \frac{355}{113}$, etc.*

84 Definition The set of *real numbers*, denoted by $\mathbb{R}$, is the collection of rational numbers together with the irrational numbers.

Homework

Problem 2.6.1 *Evaluate the expression*

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} - \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^2$$

when $x = -1$, $y = 2$, and $z = -3$.

Problem 2.6.2 *Evaluate the following expressions when $x = -\frac{2}{3}$ and $y = \frac{3}{5}$.*

1. $2x + 3y$
2. $xy - x - y$
3. $x^2 + y^2$

Problem 2.6.3 *In this problem, you are allowed to use any of the operations $+, -, \div, \cdot, !$, and exponentiation. You must use exactly four 4's. Among your fours you may also use $.4$. The $n!$ (factorial) symbol means that you multiply all the integers up to n. For example, $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.* **With these rules, write every integer, from 1 to 20 inclusive.** *For example,*

$$11 = \frac{4}{.4} + \frac{4}{4}, \qquad 15 = \frac{44}{4} + 4, \qquad 20 = \frac{4}{.4} + \frac{4}{.4}, \qquad 13 = 4! - \frac{44}{4}.$$

Problem 2.6.4 *Without using any of the signs $+, -, \div, \cdot$, but exponentiation being allowed, what is the largest number that you can form using three 4's? Again, you must explain your reasoning.*

Problem 2.6.5 *Find the value of $\frac{5}{6} - \left(-\frac{5}{6}\right)^2$.*

Problem 2.6.6 *Suppose that you know that $\frac{1}{3} = 0.333333\ldots = 0.\overline{3}$. What should $0.1111\ldots = 0.\overline{1}$ be?*

Problem 2.6.7 *Find the value of $121(0.\overline{09})$.*

Problem 2.6.8 *Use a calculator to round $\sqrt{2} + \sqrt{3} + \sqrt{5}$ to two decimal places.*

Problem 2.6.9 *Use a calculator to round $\sqrt{2} \cdot \sqrt{3} \cdot \sqrt{5}$ to two decimal places.*

Problem 2.6.10 *Let a, b be positive real numbers. Is it always true that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$?*

[6] The Pythagoreans were akin to religious cults of today. They forbade their members to eat beans, dedicated their lives to Mathematics and Music, and believed that the essence of everything in the world was *number*.

Part III

Algebraic Operations

3 Addition and Subtraction

3.1 Terms and Algebraic Expressions

The study of non-Euclidean Geometry brings nothing to students but fatigue, vanity, arrogance, and imbecility. "Non-Euclidean" space is the false invention of demons, who gladly furnish the dark understanding of the "non-Euclideans" with false knowledge. The "Non-Euclideans," like the ancient sophists, seem unaware that their understandings have become obscured by the promptings of the evil spirits. *-Matthew RYAN*

If in one room we had a group of 7 Americans, 8 Britons, and 3 Canadians, and if in another room we had 4 Americans, 4 Britons and 1 Canadian, we could write—in shorthand–addressing the total of people by nationality, that we have

$$(7A+8B+3C)+(4A+4B+1C)=11A+12B+4C.$$

The procedure of *collecting like terms* that will be shortly explained, draws essentially from the concept utilised in this example.

Again, consider the following way of adding 731 and 695. Since

$$731=7\cdot 10^2+3\cdot 10+1, \qquad 6\cdot 10^2+9\cdot 10+5,$$

we could add in the following fashion, without worrying about carrying:

$$(7\cdot 10^2+3\cdot 10+1)+(6\cdot 10^2+9\cdot 10+5)=13\cdot 10^2+12\cdot 10+6=1300+120+6=1426.$$

85 Definition An *algebraic expression* is a collection of symbols (letters and/or numbers). An algebraic expression may have one or more *terms*, which are separated from each other by the signs + (plus) or − (minus).

86 Example The expression $18a+3b-5$ consists of three terms.

87 Example The expression $18ab^2c$ consists of one term. Notice in this case that since no sign precedes $18ab^2c$, the + sign is tacitly understood. In other words, $18ab^2c=+18ab^2c$.

88 Example The expression $a+3a^2-4a^3-8ab+7$ consists of five terms.

89 Definition When one of the factors of a term is a number we call this number the *numerical coefficient* (or *coefficient* for short) of the term. In the expression a^b, a is the *base* and b is the exponent.

90 Example In the expression $13a^2b^3$, 13 is the numerical coefficient of the term, 2 is the exponent of a and 3 is the exponent of b.

91 Example In the expression a^2b^d, 1 is the numerical coefficient of the term, 2 is the exponent of a and d is the exponent of b.

92 Example In the expression $-a^c$, -1 is the numerical coefficient of the term, and c is the exponent of a. Notice then that the expression $-1a^c$ is equivalent to the expression $-a^c$.

93 Definition If two terms

1. agree in their letters, and
2. for each letter appearing in them the exponent is the same,

then we say that the terms are *like terms*.

94 Example The terms $-a, 7a, 5a$, are all like terms.

95 Example The terms $4a^2b^3, 7b^3a^2, 5a^2b^3$, are all like terms. Notice that by commutativity of multiplication, $7b^3a^2 = 7a^2b^3$.

96 Example The terms $4a$ and $4a^2$, are unlike terms, since the exponent of a in $4a$ is 1 and the exponent of a in $4a^2$ is 2.

97 Example The terms $4ab^2$ and $4ab^3$, are unlike terms, since the exponent of b in $4ab^2$ is 2 and the exponent of b in $4ab^3$ is 3.

To add or collect like terms, we simply add the coefficients of the like terms.

98 Example We have,

$$(a+b)+(a+b)+(a+b) = a+a+a+b+b+b = 3a+3b$$

☞ *By the commutative law, we may also write this as* $3b+3a$.

99 Example We have,

$$\begin{aligned}(-8a)+9a+(-15a)+4a &= a+(-15a)+4a \\ &= (-14a)+4a \\ &= -10a\end{aligned}$$

100 Example We have,

$$\begin{aligned}(5\clubsuit-7\spadesuit)+(-5\clubsuit-7\spadesuit) &= (5\clubsuit)+(-5\clubsuit)+(-7\spadesuit)+(-7\spadesuit) \\ &= -14\spadesuit.\end{aligned}$$

101 Example We have,

$$\frac{5a}{7}+\frac{(-6a)}{7}+\frac{(-18a)}{7} = \frac{5a+(-6a)+(-18a)}{7} = \frac{-19a}{7}.$$

102 Example We have,

$$\frac{5a}{7}+\frac{6a}{7}+\frac{(-18a)}{7} = \frac{5a+6a+(-18a)}{7} = \frac{-7a}{7} = -a.$$

103 Example We have,

$$\begin{aligned}(-13a+5b)+(18a+(-18b)) &= -13a+18a+5b+(-18b)\\ &= 5a+(-13b)\\ &= 5a-13b\end{aligned}$$

☞ *By the commutative law, we may also write this as* $-13b+6a$.

104 Example We have,

$$\begin{aligned}(5-4x+3x^2)+(-1-8x-3x^2) &= 5+(-1)-4x+(-8x)+3x^2+(-3x^2)\\ &= 4+(-12x)+0\\ &= 4+(-12x)\\ &= 4-12x.\end{aligned}$$

☞ *By the commutative law, we may also write this as* $-12x+4$.

105 Example We have,

$$(x^2+xy+y^2+xy^2)+(-x^2+2yx-3y^2-x^2y) = 3xy-2y^2+xy^2-x^2y.$$

This last expression is one of the many possible ways to write the final result, thanks to the commutative law.

Homework

Problem 3.1.1 *A boy buys* a *marbles, wins* b*, and loses* c*. How many marbles has he then? Write one or two sentences explaining your solution.*

Problem 3.1.2 *Collect like terms* mentally.

1. $(-9a)+(+17a)-8a$
2. $(-17a)+(-9a)+20a$
3. $-a-2a-3a-4a$
4. $10a-8a+6a-4a+2a$
5. $-5a-3b-9a-10b+10a+17b$

Problem 3.1.3 *Collect like terms:*

$(a+b^2+c+d^2)+(a^2-b-c^2-d)+(a^2-b^2+c-d)+(a-b-c^2-d^2)$.

Problem 3.1.4 *Collect like terms.*

1. $(a+2b+3c)+(3a+2b+c)$
2. $(a+2b+3c)+(3a-2b-c)$
3. $(a+2b+3c)+(3a+2b+c)+(a+b+c)$
4. $(a+b+c)+(a-b+c)+(-a-b-c)$
5. $(x^2-2x+1)+(x^2+2x+1)$
6. $(x^3-x^2+x-1)+(2x^2-x+2)$
7. $(2x^3-2x^2-2)+(2x^3-2x-1)+(2x^3-x^2-x+1)$
8. $\dfrac{(-5a)}{8}+\dfrac{19a}{8}+\dfrac{(-23a)}{8}$

Problem 3.1.5 *Find the value of the expression*

$$(2x^2+x-2)+(-x^2-x+2)$$

when $x=2$.

Problem 3.1.6 *Is it always true that* $a+a=2a$?

Problem 3.1.7 *Is it always true that* $a+a=2a^2$?

Problem 3.1.8 *Is it always true that* $a+a^2=2a^3$*?*

Problem 3.1.9 *If*

$$(8x-5)^5 = Ax^5+Bx^4+Cx^3+Dx^2+Ex+F,$$

find $A+B+C+D+E+F$.

Problem 3.1.10 *Give three different equivalent forms for the expression* $3yx-x^2y$.

3.2 More Suppression of Parentheses

He is unworthy of the name of* Man *who is ignorant of the fact that the diagonal of a square is incommensurable with its side. *-Plato*

A minus sign preceding an expression in parentheses negates every term of the expression. Thus

$$-(a+b) = -a-b.$$

106 Example We have,

$$\begin{aligned}(a-2b)-(-a+5b) &= a-2b+a-5b\\ &= a+a-2b-5b\\ &= 2a-7b\end{aligned}$$

107 Example We have,

$$\begin{aligned}(3-2x-x^2)-(5-6x-7x^2) &= 3-2x-x^2-5+6x+7x^2\\ &= 3-5-2x+6x-x^2+7x^2\\ &= -2+4x+6x^2\end{aligned}$$

108 Example We have,

$$\begin{aligned}(a-b)-(2a-3b)-(-a+3b) &= a-b-2a+3b+a-3b\\ &= a-2a+a-b+3b-3b\\ &= -b\end{aligned}$$

109 Example We have,

$$\begin{aligned}(x^2-x-1)-(-x-3x^2)+(2x^2-8) &= x^2-x-1+x+3x^2+2x^2-8\\ &= x^2+3x^2+2x^2-x+x-1-8\\ &= 6x^2-9.\end{aligned}$$

Recall that the distributive law states that for real numbers a, b, c,

$$a(b+c) = ab+ac.$$

We now apply this to algebraic expressions.

110 Example Multiply term by term: $2(a+2b-3c)$.

Solution: ► *We have,*

$$2(a+2b-3c) = 2a+2(2b)+2(-3c) = 2a+4b-6c.$$

◄

111 Example Multiply term by term: $-2(a+2b-3c)$.

Solution: ► *We have,*

$$-2(a+2b-3c) = -2a+(-2)(2b)+(-2)(-3c) = -2a-4b+6c.$$

◄

112 Example Divide term by term: $\dfrac{2a-4b+7c}{2}$.

Solution: ► *We have,*

$$\frac{2a-4b+7c}{2} = \frac{2a}{2}+\frac{-4b}{2}+\frac{7c}{2} = a-2b+\frac{7c}{2}.$$

◄

☞ *We prefer to use improper fractions and write* $\dfrac{7c}{2}$ *rather than* $3\frac{1}{2}c$. *Also, the expressions* $\dfrac{7}{2}c$ *and* $\dfrac{7c}{2}$ *are identical, that is,* $\dfrac{7}{2}c = \dfrac{7c}{2}$.

We may combine the distributive law and combining like terms.

113 Example Combine like terms: $2(a-2b)+3(-a+2b)$.

Solution: ► *We have,*

$$\begin{aligned} 2(a-2b)+3(-a+2b) &= 2a-4b+(-3a)+6b \\ &= -a+2b. \end{aligned}$$

◄

114 Example Combine like terms: $2(a-2b)-3(-a+2b)$.

Solution: ► *We have,*

$$\begin{aligned} 2(a-2b)-3(-a+2b) &= 2a-4b+3a-6b \\ &= 5a-10b. \end{aligned}$$

◄

115 Example Combine like terms: $$-3(1-2x+4x^2)+\frac{9x^2-3x+6}{-3}.$$

Solution: ► *We have,*

$$\begin{aligned}-3(1-2x+4x^2)+\frac{9x^2-3x+6}{-3} &= -3+6x-12x^2-3x^2+x-2\\ &= -15x^2+7x-5.\end{aligned}$$

◄

116 Example Combine like terms: $$\frac{3}{2}\left(\frac{x}{3}-2\right)-2\left(\frac{x}{12}+2\right).$$

Solution: ► *We have,*

$$\begin{aligned}\frac{3}{2}\left(\frac{x}{3}-2\right)-2\left(\frac{x}{12}+2\right) &= \frac{3x}{6}-\frac{6}{2}-\frac{2x}{12}-4\\ &= \frac{6x}{12}-\frac{2x}{12}-3-4\\ &= \frac{4x}{12}-7\\ &= \frac{x}{3}-7.\end{aligned}$$

◄

Homework

Problem 3.2.1 *From a rod $a+b$ inches long, $b-c$ inches are cut off; how much remains? Write one or two sentences explaining your solution.*

Problem 3.2.2 *A group of x people is going on an excursion and decides to charter a plane. The total price of renting the plane is D dollars, and everyone will pay an equal share. What is this share? Now, the day of departure it turns out that p people are no-shews. How much more will the people who shew up have to pay in order to cover the original cost?*

Problem 3.2.3 *Suppress parentheses and collect like terms:*

$$(a+b^2+c+d^2)-(a^2-b-c^2-d)+(a^2-b^2+c-d)-(a-b-c^2-d^2).$$

Problem 3.2.4 *Expand and simplify:*

$$2(a+b-2c)-3(a-b-c).$$

Problem 3.2.5 *Suppress parentheses and collect like terms.*

1. $(a+2b+3c)-(3a+2b+c)$
2. $(a+2b+3c)+(3a+2b+c)-(a+b+c)$
3. $(a+b+c)-(a-b+c)-(-a-b-c)$
4. $(x^3-x^2+x-1)-(2x^2-x+2)$
5. $(2x^3-2x^2-2)-(2x^3-2x-1)-(2x^3-x^2-x+1)$
6. $(8.1a-9.2b)-(-0.2a-3.1b)-(-a+3b)$
7. $-\left(-\frac{14b}{a}+\frac{15b}{a}-\frac{1}{a}\right)-\left(-\frac{4b}{a}+\frac{5}{a}-\frac{8b}{a}\right)$
8. $-\left(\frac{4}{a}-\frac{5}{a}-2\right)-\left(\frac{4}{a}+\frac{5}{a}-8\right)$
9. $(5\clubsuit-7\spadesuit)-(-5\clubsuit-7\spadesuit)$

Problem 3.2.6 *Collect like terms and write as a single fraction:*

$$\frac{a}{3}-\frac{a^2}{5}+\frac{a}{2}+\frac{5a^2}{6}.$$

Problem 3.2.7 *Combine like terms.*

1. $2(-x+2y)+3(x-2y)-(x+y)+(y-2x)$
2. $\frac{14x-7}{7}-\frac{1}{3}(-3x+12)$
3. $\frac{4x-12x^2+6}{-2}-2(1-2x+x^2)$

Problem 3.2.8 *Match each expression on the left with the equivalent expression on the right.*

____ 1.	$x-y$	A.	$y-x$
____ 2.	$x+y$	B.	$-y+x$
____ 3.	$-(x-y)$	C.	$x-(-y)$

Problem 3.2.9 *You start the day with* A *dollars. Your uncle Bob gives you enough money to double your amount. Your aunt Rita gives you* 10 *dollars. You have to pay* B *dollars in fines, and spent* 12 *dollars fueling your camel with gas. How much money do you have now?*

Problem 3.2.10 *Simplify each of the following expressions as much as possible*

1. $(2-4)x-5(-6)x$
2. $5-2(2x)$
3. $(5-2)(1-2^2)^3x$

Problem 3.2.11 *Remove all redundant parentheses. If removing parentheses changes the expression, state so.*

1. $(6t)+x$
2. $6(t+x)$
3. $(ab)^2$
4. $(ab-c)^2$

Problem 3.2.12 *Fill in the box to make the expression true.*

1. $\frac{a-2b}{c}=\frac{a}{c}+\boxed{}$
2. $\frac{t+3}{t-3}=(t+3)\boxed{}$

Problem 3.2.13 *Fill in the box to make the expression true.*

1. $1-t=\boxed{}t+1$
2. $3x+y-t=\boxed{}+3x$

Problem 3.2.14 *What is the opposite of* $4-y+2$*?*

Problem 3.2.15 *What is the additive inverse of* $\frac{-a}{-(b-c)}$*?*

4 Multiplication

4.1 Laws of Exponents

Anyone who cannot cope with mathematics is not fully human. At best he is a tolerable subhuman who has learned to wear shoes, bathe, and not make messes in the house. *-Robert A. HEINLEIN*

Recall that if a is a real number and if n is a natural number then

$$a^n = \underbrace{a \cdot a \cdots a}_{n\ a'\text{s}},$$

with the interpretation that $a^0 = 1$ if $a \neq 0$.

Suppose we wanted to compute a^2a^5. We proceed as follows, using the fact that $a^2 = aa$ and that $a^5 = aaaaa$:

$$(a^2)(a^5) = (aa)(aaaaa) = aaaaaaa = a^7,$$

since the penultimate expression consists of seven a's. In general, we have the following.

117 Theorem (First Law of Exponents) Let a be a real number and m, n natural numbers. Then

$$a^m a^n = a^{m+n}.$$

Proof: *We have*

$$\begin{aligned} a^m a^n &= \underbrace{a \cdot a \cdots a}_{m\ a'\text{s}} \cdot \underbrace{a \cdot a \cdots a}_{n\ a'\text{s}} \\ &= \underbrace{a \cdot a \cdots a}_{m+n\ a'\text{s}} \\ &= a^{m+n}. \end{aligned}$$

❑

118 Example $(ab^2c^3)(a^3b^3c) = a^{1+3}b^{2+3}c^{3+1} = a^4b^5c^4$.

119 Example $(a^x)(b^{2y})(a^{2x}b^{2y}) = a^{x+2x}b^{2y+2y} = a^{3x}b^{4y}$.

120 Example $(a^{x-2y+z})(a^{2x-y-z})(a^{x+y+z}) = a^{x-2y+z+2x-y-z+x+y+z} = a^{4x-2y+z}$.

121 Example

$$5^5 + 5^5 + 5^5 + 5^5 + 5^5 = 5(5^5) = 5^{1+5} = 5^6 = 15625.$$

We may use associativity and commutativity in the case when the numerical coefficients are different from 1.

122 Example $(5x^2)(6x^3) = 30x^5$.

123 Example $(-2ab^2c^3)(3abc^4) = -6a^2b^3c^7$.

We now tackle division. As a particular case, observe that

$$\frac{a^5}{a^2} = \frac{aaaaa}{aa} = \frac{aaa\cancel{aa}}{\cancel{aa}} = a^3,$$

and, of course, $a^3 = a^{5-2}$. This generalises as follows.

124 Theorem (Second Law of Exponents) Let $a \neq 0$ be a real number and m, n natural numbers, such that $m \geq n$. Then

$$\frac{a^m}{a^n} = a^{m-n}.$$

Proof: *We have*

$$\begin{aligned}\frac{a^m}{a^n} &= \frac{\underbrace{a \cdot a \cdots a}_{m\ a\text{'s}}}{\underbrace{a \cdot a \cdots a}_{n\ a\text{'s}}} \\ &= \underbrace{a \cdot a \cdots a}_{m-n\ a\text{'s}} \\ &= a^{m-n}.\end{aligned}$$

❑

125 Example

$$\frac{2^9}{2^4} = 2^{9-4} = 2^5 = 32.$$

126 Example

$$\frac{-24a^{12}b^9c^5d^2}{2a^6b^3c^4d^2} = -12a^{12-6}b^{9-3}c^{5-4}d^{2-2} = -12a^6b^6c^1d^0 = -12a^6b^6c.$$

127 Example $\dfrac{(5^a)(5^b)}{5^{a+b-2}} = 5^{a+b-(a+b-2)} = 5^2 = 25.$

128 Example $\dfrac{15x^5y^7z^4}{5x^2y^2z^2} = 3x^3y^5z^2.$

129 Example $\dfrac{16b^2yx^2}{-2xy} = -8b^2x$

130 Example $\dfrac{7a^2bc}{-7a^2bc} = -1.$

131 Example $\dfrac{(a+b)^4(2+a+b)^6}{(a+b)^3(2+a+b)^4} = (a+b)(2+a+b)^2$

132 Example $\dfrac{a^5b^4(a+b)^3}{a^3b^4(a+b)} = a^2(a+b)^2.$

Suppose we wanted to compute $(a^2)^5$. Since a quantity raised to the fifth power is simply the quantity multiplied by itself five times, we have:

$$(a^2)^5 = (a^2)(a^2)(a^2)(a^2)(a^2) = a^{2+2+2+2+2} = a^{10},$$

upon using the first law of exponents. In general we have the following law dealing with "exponents of exponents."

133 Theorem (Third Law of Exponents) Let a be a real number and m, n natural numbers. Then

$$(a^m)^n = a^{mn}.$$

Proof: *We have*

$$\begin{aligned}(a^m)^n &= \underbrace{a^m \cdot a^m \cdots a^m}_{n\ a^m\text{'s}} \\ &= a^{\overbrace{m+m+\cdots+m}^{n\ a'\text{s}}} \\ &= a^{mn}.\end{aligned}$$

❑

134 Theorem (Fourth Law of Exponents) Let a and b be a real numbers and let m be a natural number. Then

$$(ab)^m = a^m b^m.$$

Proof: *We have*

$$\begin{aligned}(ab)^m &= \underbrace{ab \cdot ab \cdots ab}_{m\ ab\text{'s}} \\ &= \underbrace{a \cdot a \cdots a}_{m\ a\text{'s}}\underbrace{b \cdot b \cdots b}_{m\ b\text{'s}} \\ &= a^m b^m.\end{aligned}$$

❑

135 Example We have,

$$\begin{aligned}(2ab^2c^3)^2(-3a^3bc^2)^3 &= (2^2(a^1)^2(b^2)^2(c^3)^2)((-3)^3(b^1)^3(c^2)^3) \\ &= (4a^2b^4c^6)(-27a^9b^3c^6) \\ &= -108a^{11}b^7c^{12}.\end{aligned}$$

136 Example $(a^{x^3})^{2x} = a^{(x^3)(2x)} = a^{2x^4}$

137 Example $25^3 4^3 = (25 \cdot 4)^3 = 100^3 = 1000000.$

Homework

Problem 4.1.1 *Simplify:* $\dfrac{24xyz^3}{-3z^2}$

Problem 4.1.2 *Simplify:* $\dfrac{-168a^2b^2cx^2}{-7abx^2}$

Problem 4.1.3 *Find the exact numerical value of:* $\frac{20^6}{25^5}$.

Problem 4.1.4 *Find the exact numerical value of:*

$$\frac{25^2+25^2+25^2+25^2}{25^4}.$$

Problem 4.1.5 *If*

$$\left(\frac{4^5+4^5+4^5+4^5}{3^5+3^5+3^5}\right)\left(\frac{6^5+6^5+6^5+6^5+6^5+6^5}{2^5+2^5}\right)=2^n,$$

find n.

Problem 4.1.6 *If*

$$3^{2001}+3^{2002}+3^{2003}=a3^{2001},$$

find a.

Problem 4.1.7 *Demonstrate that*

$$(a^x b^y)\left(\frac{b^{2x}}{a^{-y}}\right)=a^{x+y}b^{y+2x}.$$

Problem 4.1.8 *Divide:* $\frac{1}{x}\div\frac{1}{x^2}$.

Problem 4.1.9 *Multiply:* $\left(\frac{1}{x}\right)\left(\frac{1}{x^2}\right)$.

Problem 4.1.10 *Divide:* $\frac{1}{s-1}\div\frac{1}{(s-1)^2}$.

Problem 4.1.11 *Multiply:* $\left(\frac{1}{s-1}\right)\left(\frac{1}{(s-1)^2}\right)$.

Problem 4.1.12 *Divide:* $\frac{a^2}{x}\div\frac{x}{a^2}$.

Problem 4.1.13 *Multiply:* $\left(\frac{a^2}{x}\right)\left(\frac{x}{a^2}\right)$.

Problem 4.1.14 *Simplify and write as a single number:*

$$\frac{6^5+6^5+6^5+6^5+6^5+6^5+6^5+6^5+6^5+6^5}{3^3+3^3+3^3+3^3+3^3}.$$

Problem 4.1.15 *If* $\frac{a^8}{b^9}\div\frac{(a^2b)^3}{b^{20}}=a^mb^n$, *then* $m=$ ________ and $n=$ ________.

4.2 Negative Exponents

One of the big misapprehensions about mathematics that we perpetrate in our classrooms is that the teacher always seems to know the answer to any problem that is discussed. This gives students the idea that there is a book somewhere with all the right answers to all of the interesting questions, and that teachers know those answers. And if one could get hold of the book, one would have everything settled. That's so unlike the true nature of mathematics. *-Leon HEINKIN*

Let $a\neq 0$ be a real number and let n be a natural number. Then

$$1=a^0=a^{n-n}=a^na^{-n}\qquad\Longrightarrow\qquad a^{-n}=\frac{1}{a^n}.$$

This gives an interpretation to negative exponents. Observe also that taking reciprocals,

$$a^{-n}=\frac{1}{a^n}\qquad\Longrightarrow\qquad\frac{1}{a^{-n}}=\frac{1}{\frac{1}{a^n}}=a^n.$$

138 Example $2^{-3}=\frac{1}{2^3}=\frac{1}{8}$.

139 Example

$$\frac{2^4}{2^9}=2^{4-9}=2^{-5}=\frac{1}{2^5}=\frac{1}{32}.$$

140 Example $\frac{2^{-3}}{3^{-2}}=\frac{3^2}{2^3}=\frac{9}{8}$.

141 Example $\left(\frac{2^4}{5^{-3}}\right)^{-2} = \frac{2^{-8}}{5^6} = \frac{1}{256 \cdot 15625} = \frac{1}{4000000}$.

142 Example Simplify, using positive exponents only:

$$\frac{x^{-6}}{y^9} \cdot \frac{y^{-8}}{x^{-12}}$$

Solution: ►

$$\frac{x^{-6}}{y^9} \cdot \frac{y^{-8}}{x^{-12}} = \frac{x^{-6}y^{-8}}{y^9x^{-12}} = x^{-6-(-12)}y^{-8-9} = x^6y^{-17} = \frac{x^6}{y^{17}}.$$

◄

143 Example Simplify, using positive exponents only:

$$\left(\frac{x^{-6}}{y^9}\right)^{-3} \cdot \left(\frac{y^{-1}}{x^{-2}}\right)^2.$$

Solution: ► *We have,*

$$\left(\frac{x^{-6}}{y^9}\right)^{-3} \cdot \left(\frac{y^{-1}}{x^{-2}}\right)^2 = \frac{x^{18}}{y^{-27}} \cdot \frac{y^{-2}}{x^{-4}} = x^{18-(-4)}y^{-2-(-27)} = x^{22}y^{25}.$$

◄

Homework

Problem 4.2.1 *Simplify, using positive exponents only.*

1. $\frac{x^{-5}}{x^4}$
2. $\frac{x^4}{x^{-5}}$
3. $x^{-5}x^4$
4. $(x^{-1}y^2z^3)^3$
5. $\frac{a^3b^{-2}}{a^{-6}b^4}$
6. $\frac{(ab)(ab^2)}{(a^{-1}b^{-2})^2(a^2b^2)}$

Problem 4.2.2 *Demonstrate that*

$$(a^{x+3}b^{y-4})(a^{2x-3}b^{-2y+5}) = a^{3x}b^{1-y}.$$

Problem 4.2.3 *Demonstrate that* $\frac{a^xb^y}{a^{2x}b^{-2y}} = a^{-x}b^{3y}$.

Problem 4.2.4 *Simplify and express with positive exponents only:* $\frac{x^6}{y^9} \div \frac{x^{-2}}{y^{-3}}$

Problem 4.2.5 *What is the exact numerical value of* $\frac{2^{20}6^{10}}{4^{11}3^{10}}$?

Problem 4.2.6 *What is the exact numerical value of* $\left(\left(\frac{2}{5}\right)^{-1} - \left(\frac{1}{2}\right)^{-1}\right)^{-2}$?

Problem 4.2.7 *What is the exact numerical value of* $\frac{(2^4)^8}{(4^8)^2}$.

4.3 Distributive Law

> ***You treat world history as a mathematician does mathematics, in which nothing but laws and formulæ exist, no reality, no good and evil, no time, no yesterday, no tomorrow, nothing but an eternal, shallow, mathematical present.*** *-Hermann HESSE*

Recall that the distributive law says that if $a, b,$ and c are real numbers, then

$$a(b+c) = ab + ac, \qquad (a+b)c = ac + bc.$$

We now apply the distributive law to multiplication of terms.

144 Example We have,

$$\begin{aligned} -3xy^2(2x^3 - 5xy^3) &= (-3xy^2)(2x^3) - (-3xy^2)(5xy^3) \\ &= -6x^4y^2 + 15x^2y^5. \end{aligned}$$

145 Example We have,

$$\begin{aligned} (-3a^2b)(-3ab^2 + 2b) - (a^2 - 3a^2b)(ab^2) &= 9a^3b^3 - 6a^2b^2 - (a^3b^2 - 3a^3b^3). \\ &= 9a^3b^3 - 6a^2b^2 + (-a^3b^2) + 3a^3b^3 \\ &= 12a^3b^3 - 6a^2b^2 - a^3b^2. \end{aligned}$$

From the distributive law we deduce that

$$(a+b)(c+d) = (a+b)c + (a+b)d = ac + bc + ad + bd.$$

146 Example We have,

$$\begin{aligned} (2x+3)(5x+7) &= (2x+3)(5x) + (2x+3)(7) \\ &= 10x^2 + 15x + 14x + 21 \\ &= 10x^2 + 29x + 21. \end{aligned}$$

Because of commutativity, we may also proceed as follows,

$$\begin{aligned} (2x+3)(5x+7) &= (2x)(5x+7) + 3(5x+7) \\ &= 10x^2 + 14x + 15x + 21 \\ &= 10x^2 + 29x + 21. \end{aligned}$$

147 Example We have,

$$\begin{aligned} (2x-3)(x-4) &= (2x-3)(x) + (2x-3)(-4) \\ &= 2x^2 - 3x - 8x + 12 \\ &= 2x^2 - 11x + 12. \end{aligned}$$

☞ *Because of commutativity, we may also proceed as follows,*

$$\begin{aligned}(2x-3)(x-4) &= (2x)(x-4)-3(x-4)\\ &= 2x^2-8x-3x+12\\ &= 2x^2-11x+12.\end{aligned}$$

We also recall the meaning of exponentiation.

148 Example We have,

$$\begin{aligned}(x-4)^2 &= (x-4)(x-4)\\ &= x(x-4)-4(x-4)\\ &= x^2-4x-4x+16\\ &= x^2-8x+16.\end{aligned}$$

149 Example To expand $(x+1)^3$, we first find

$$(x+1)^2 = (x+1)(x+1) = x(x+1)+1(x+1) = x^2+x+x+1 = x^2+2x+1.$$

We use this result and find,

$$\begin{aligned}(x+1)^3 &= (x+1)(x+1)^2\\ &= (x+1)(x^2+2x+1)\\ &= x(x^2+2x+1)+1(x^2+2x+1)\\ &= x^3+2x^2+x+x^2+2x+1\\ &= x^3+3x^2+3x+1.\end{aligned}$$

Homework

Problem 4.3.1 *Expand and simplify.*

1. $x(x^2+x+1)$
2. $2x(x^2-2x+3)$
3. $(2x+1)(2x+1)$
4. $(2x+1)(2x-1)$
5. $(2x-1)(2x-1)$

Problem 4.3.2 *Expand and simplify:*

$$(x^2+1)+2x(x^2+2)-3x^3(x^2+3).$$

Problem 4.3.3 *Expand and simplify:*

$$4x^3(x^2-2x+2)-2x^2(x^2-x^2-1).$$

Problem 4.3.4 *Multiply and collect like terms.*

1. $-2x(x+1)+3x^2(x-3x^2)$
2. $-2x(x+1)^2$
3. $a^2c(ab-ac)+ac^2(a^2-2b)$
4. $(x+2y+3z)(x-2y)$

Problem 4.3.5 *Expand the product* $(2x+2y+1)(x-y+1)$.

Problem 4.3.6 *Expand and collect like terms:*

$$(x+2)(x+3)-(x-2)(x-3).$$

Problem 4.3.7 *Expand and collect like terms:*

$$(x-1)(x^2+x+1)-(x+1)(x^2-x+1).$$

Problem 4.3.8 *Expand and simplify:*

$$(x+y-z)(x-y+2z).$$

Problem 4.3.9 *Expand the product*

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

Problem 4.3.10 *Expand the product* $(a+b+c)(x+y+z)$.

Problem 4.3.11 *Prove that the product of two even integers is even.*

Problem 4.3.12 *Prove that the product of two odd integers is odd.*

Problem 4.3.13 *Prove that the product of two integers that leave remainder* 3 *upon division by* 4 *leaves remainder* 1 *upon division by* 4.

Problem 4.3.14 *Prove that the product of any two integers leaving remainder* 2 *upon division by* 3 *leaves remainder* 1 *upon division by* 3.

Problem 4.3.15 *Prove that the product of two integers leaving remainder* 2 *upon division by* 5 *leaves remainder* 4 *upon division by* 5.

Problem 4.3.16 *Prove that the product of two integers leaving remainder* 3 *upon division by* 5 *leaves remainder* 4 *upon division by* 5.

Problem 4.3.17 *If* $x^2+x-1=0$, *find* $x^4+2x^3+x^2$.

Problem 4.3.18 *Expand and collect like terms:*

$$2x(x+1)-x^2-x(x-1).$$

Then use your result to evaluate

$$2\cdot(11112)\cdot(11113)-11112^2-(11111)\cdot(11112)$$

without a calculator.

4.4 Square of a Sum

Galileo was no idiot. Only an idiot could believe that science requires martyrdom - that may be necessary in religion, but in time a scientific result will establish itself.
-David HILBERT

In this section we study some special products that often occur in algebra.

150 Theorem (Square of a Sum) For all real numbers a, b the following identity holds

$$(a\pm b)^2 = a^2 \pm 2ab + b^2.$$

Proof: *Using the distributive law,*

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \\ &= a(a+b)+b(a+b) \\ &= a^2+ab+ba+b^2 \\ &= a^2+2ab+b^2,\end{aligned}$$

and so $(a+b)^2 = a^2+2ab+b^2$. *Putting* $-b$ *instead of* b *in this last identity,*

$$(a-b)^2 = (a+(-b))^2 = a^2+2a(-b)+(-b)^2 = a^2-2ab+b^2,$$

proving both identities. ❑

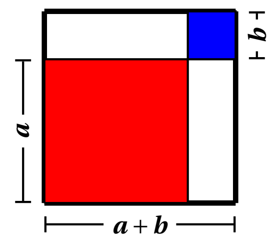


Figure 4.1: $(a+b)^2 = a^2 + 2ab + b^2$.

The identity $(a+b)^2 = a^2 + 2ab + b^2$ has a pictorial justification in terms of areas of squares, as seen in figure 4.1. The area of the large $(a+b)\times(a+b)$ square is $(a+b)^2$. In terms, this area can be decomposed into two square regions, one of area a^2 and the other of area b^2, and two rectangular regions each of area ab.

151 Example We have,

$$(2x+3y)^2 = (2x)^2 + 2(2x)(3y) + (3y)^2 = 4x^2 + 12xy + 9y^2.$$

152 Example We have,

$$(2-x^2)^2 = (2)^2 - 2(2)(x^2) + (x^2)^2 = 4 - 4x^2 + x^4.$$

The principle expounded here can be used to facilitate mental arithmetic.

153 Example One may compute mentally 52^2 as follows:

$$52^2 = (50+2)^2 = 50^2 + 2(50)(2) + 2^2 = 2500 + 200 + 4 = 2704.$$

The formula $(a+b)^2 = a^2 + 2ab + b^2 = a^2 + b^2 + 2ab$ gives the square of a sum in terms of the sum of squares and the product of the numbers. Thus knowing any two of them will determine the other.

154 Example If the sum of two numbers is 3 and their product 4, find the sum of their squares.

Solution: ► *Let the two numbers be x, y. Then $x+y=3$ and $xy=4$. Hence*

$$9 = (x+y)^2 = x^2 + 2xy + y^2 = x^2 + 8 + y^2 \implies x^2 + y^2 = 1.$$

◄

Homework

Problem 4.4.1 *Expand and simplify:* $(3-y)^2$.

Problem 4.4.2 *Expand and simplify:* $(5-2x^2)^2$.

Problem 4.4.3 *Expand and simplify:* $(2ab^2 - 3c^3d^4)^2$.

Problem 4.4.4 *Expand and simplify:* $(x+2y)^2 + 4xy$.

Problem 4.4.5 *Expand and simplify:* $(x+2y)^2 - 4xy$.

Problem 4.4.6 *Expand* $(xy-4y)^2$.

Problem 4.4.7 *Expand* $(ax+by)^2$.

Problem 4.4.8 *Expand and simplify:* $(a+2)^2 + (a-2)^2$.

Problem 4.4.9 *Expand and simplify:* $(a+2)^2 - (a-2)^2$.

Problem 4.4.10 *Expand and simplify:* $(a+2b+3c)^2$.

Problem 4.4.11 *Expand and simplify:* $(a+2b-c)^2$.

Problem 4.4.12 *If* $x+\frac{1}{x}=6$, *find* $x^2+\frac{1}{x^2}$.

Problem 4.4.13 *Find the value of* n *in*

$$(10^{2002}+25)^2-(10^{2002}-25)^2=10^n.$$

Problem 4.4.14 *Given that* $2x+3y=3$ *and* $xy=4$, *find* $4x^2+9y^2$.

Problem 4.4.15 *If the difference of two numbers is* 3 *and their product* 4, *find the sum of their squares.*

Problem 4.4.16 *Given that* $x+y=4$ *and* $xy=-3$, *find* x^2+y^2.

Problem 4.4.17 *Given that* $x+y=4$ *and* $xy=-3$, *find* x^4+y^4.

Problem 4.4.18 *Expand* $(x+y+z)^2$.

Problem 4.4.19 *Demonstrate that*

$$\begin{aligned}(x+y+z+w)^2 &= x^2+y^2+z^2+w^2\\ &\quad +2xy+2xz+2xw+2yz+2yw+2zw.\end{aligned}$$

4.5 Difference of Squares

> ***Science, being human enquiry, can hear no answer except an answer couched somehow in human tones. Primitive man stood in the mountains and shouted against a cliff; the echo brought back his own voice, and he believed in a disembodied spirit. The scientist of today stands counting out loud in the face of the unknown. Numbers come back to him - and he believes in the Great Mathematician.*** *-Richard HUGHES*

In this section we continue our study of special products.

155 Theorem (Difference of Squares) For all real numbers a, b the following identity holds

$$(a+b)(a-b)=a^2-b^2.$$

Proof: *Using the distributive law,*

$$\begin{aligned}(a+b)(a-b) &= a(a-b)+b(a-b)\\ &= a^2-ab+ba-b^2\\ &= a^2-b^2,\end{aligned}$$

proving the identity. ❑

156 Example

$$(4x-5y)(4x+5y)=16x^2-25y^2.$$

157 Example

$$(xy-5y)(xy+5y)=x^2y^2-25y^2.$$

158 Example Shew that

$$(a-b-c)(a+b+c)=a^2-b^2-c^2-2bc.$$

Solution: ► *We have*

$$\begin{aligned}(a-b-c)(a+b+c) &= (a-(b+c))(a+(b+c)) \\ &= a^2-(b+c)^2 \\ &= a^2-(b^2+2bc+c^2) \\ &= a^2-b^2-c^2-2bc,\end{aligned}$$

as we wanted to shew. ◄

159 Example

$$\begin{aligned}(x+1)^2-(x-1)^2 &= ((x+1)+(x-1))((x+1)-(x-1)) \\ &= (2x)(2) \\ &= 4x\end{aligned}$$

160 Example We have

$$\begin{aligned}(a-1)(a+1)(a^2+1)(a^4+1)(a^8+1) &= (a^2-1)(a^2+1)(a^4+1)(a^8+1) \\ &= (a^4-1)(a^4+1)(a^8+1) \\ &= (a^8-1)(a^8+1) \\ &= a^{16}-1.\end{aligned}$$

161 Example We have

$$\begin{aligned}(x^2-x+1)(x^2+x+1) &= ((x^2+1)-x)((x^2+1)+x) \\ &= (x^2+1)^2-x^2 \\ &= x^4+2x^2+1-x^2 \\ &= x^4+x^2+1.\end{aligned}$$

162 Example Prove that

$$x^2-(x+2)(x-2)=4.$$

Explain how to find the exact value of

$$(987654321)(987654321)-(987654323)(987654319),$$

mentally.

Solution: ► *We have*

$$\begin{aligned} x^2-(x+2)(x-2) &= x^2-(x^2-4) \\ &= x^2-x^2+4 \\ &= 4. \end{aligned}$$

Now put $x = 987654321$. *Then* $x+2 = 987654323$ *and* $x-2 = 987654319$, *from where*

$$(987654321)(987654321)-(987654323)(987654319) = 4$$

follows. ◄

Homework

Problem 4.5.1 *Multiply and collect like terms:*

$$(a+4)^2-(a+2)^2.$$

Problem 4.5.2 *Prove that*

$$(x-y)(x+y)(x^2+y^2) = x^4-y^4.$$

Problem 4.5.3 *Multiply and collect like terms:*

$$(a+1)^4-(a-1)^4.$$

Problem 4.5.4 *Shew, without expressly computing any power of* 2, *that*

$$(2+1)(2^2+1)(2^4+1)(2^8+1)(2^{16}+1) = 2^{32}-1.$$

Problem 4.5.5 *Find*

$$1^2-2^2+3^2-4^2+\cdots+99^2-100^2.$$

Problem 4.5.6 Without using a calculator, *what is the exact numerical value of*

$$(123456789)^2-(123456787)(123456791)\ ?$$

Problem 4.5.7 Without using a calculator, *what is the exact numerical value of* $(666\ 666\ 666)^2-(333\ 333\ 333)^2$?

4.6 Cube of a Sum

If we take in our hand any volume; of divinity or school metaphysics, for instance; let us ask, 'Does it contain any abstract reasoning concerning quantity or number?' No. 'Does it contain any experimental reasoning concerning matter of fact and existence?' No. Commit it then to the flames: for it can contain nothing but sophistry and illusion.
-David HUME

We introduce now a special product formula for the cube of a sum.

163 Theorem Let a and b be real numbers. Then

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3.$$

Proof: *Using the distributive law and the square of a sum identity,*

$$\begin{aligned} (a+b)^3 &= (a+b)^2(a+b) \\ &= (a^2+2ab+b^2)(a+b) \\ &= a^2(a+b)+2ab(a+b)+b^2(a+b) \\ &= a^3+a^2b+2a^2b+2ab^2+b^2a+b^3 \\ &= a^3+3a^2b+3ab^2+b^3, \end{aligned}$$

as claimed. ❑

164 Corollary Let a and b be real numbers. Then

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

Proof: *Replace* b *by* $-b$ *in Theorem 163, obtaining*

$$\begin{aligned}(a-b)^3 &= (a+(-b))^3 \\ &= a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3 \\ &= a^3 - 3a^2b + 3ab^2 - b^3,\end{aligned}$$

as claimed. ❑

165 Example We have,

$$(x+2y)^3 = x^3 + 3x^2(2y) + 3x(2y)^2 + (2y)^3 = x^3 + 6x^2y + 12xy^2 + 8y^3.$$

166 Example We have,

$$(3x-2y)^3 = (3x)^3 - 3(3x)^2(2y) + 3(3x)(2y)^2 - (2y)^3 = 27x^3 - 54x^2y + 36xy^2 - 8y^3.$$

Observe that

$$3ab(a+b) = 3a^2b + 3ab^2,$$

hence we may write the cube of a sum identity as

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b). \tag{4.1}$$

167 Example Given that $a+b=2$ and $ab=3$, find a^3+b^3.

Solution: ► *We have*

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b) = 2^3 - 3(3)(2) = 8 - 18 = -10.$$

◄

Homework

Problem 4.6.1 *Expand and simplify:* $(5x+1)^3$.

Problem 4.6.2 *Multiply and collect like terms:*

$$(x+1)^3 + (x+1)^2.$$

Problem 4.6.3 *Multiply and collect like terms:*

$$(x-1)^3 - (x+2)^2.$$

Problem 4.6.4 *Multiply and collect like terms:*

$$(x-2)^3 - (x-2)^2.$$

Problem 4.6.5 *Given that* $a-b=6$ *and* $ab=3$, *find* a^3-b^3.

Problem 4.6.6 *Given that* $a+2b=6$ *and* $ab=3$, *find* a^3+8b^3.

4.7 Sum and Difference of Cubes

Take from all things their number and all shall perish. *-Saint Isidore of Seville*

In this section we study our last special product.

168 Theorem (Sum and Difference of Cubes) For all real numbers a, b the following identity holds

$$(a \pm b)(a^2 \mp ab + b^2) = a^3 \pm b^3.$$

Proof: *Using the distributive law,*

$$\begin{aligned}(a+b)(a^2-ab+b^2) &= a^3-a^2b+ab^2+ba^2-ab^2+b^3\\ &= a^3+b^3,\end{aligned}$$

from where the sum of cubes identity is deduced. We obtain

$$(a-b)(a^2+ab+b^2) = a^3-b^3$$

upon replacing b by $-b$ in the above sum of cubes identity. ❑

169 Example

$$(a-2b)(a^2+2ab+4b^2) = a^3-8b^3.$$

170 Example

$$(5a+2b)(25a^2-10ab+4b^2) = 125a^3-8b^3.$$

Homework

Problem 4.7.1 *Expand and simplify:*

$$(x-2)(x^2+2x+4).$$

Problem 4.7.2 *Expand and simplify:*

$$(x+8)(x^2-8x+64).$$

Problem 4.7.3 *Given that $a-b=6$ and $ab=3$, find a^3-b^3.*

5 Division

From the intrinsic evidence of his creation, the Great Architect of the Universe now begins to appear as a pure mathematician. *-Sir James JEANS*

5.1 Term by Term Division

The science of mathematics presents the most brilliant example of how pure reason may successfully enlarge its domain without the aid of experience. *-Emmanuel KANT*

From the distributive law we deduce

$$\frac{b+c}{a} = \frac{1}{a}(b+c) = \frac{b}{a} + \frac{c}{a}.$$

171 Example $\frac{x^2-2xy}{x} = \frac{x^2}{x} - \frac{2xy}{x} = x - 2y.$

☞ *Be careful not to confuse an expression of the form* $\frac{a+b}{c}$ *with one of the form* $\frac{ab}{c}$*. We also remark that an expression of the form* $\frac{ab}{c}$ *can be evaluated in various ways, for example, one may evaluate first the product* ab *and then divide this by* c*, or, one may divide* a *by* c *and then multiply this by* b*, etc.*

172 Example We have,

$$\frac{(x^2)(-2xy)}{x} = \frac{-2x^3y}{x} = -2x^2y.$$

This may also be evaluated in the following manner,

$$\frac{(x^2)(-2xy)}{x} = \frac{x^2}{x}(-2xy) = x(-2xy) = -2x^2y,$$

as before.

173 Example $\frac{-24x^6-32x^4}{-8x^3} = \frac{-24x^6}{-8x^3} - \frac{32x^4}{-8x^3} = 3x^3 - (-4x) = 3x^3 + 4x.$

174 Example We have,

$$\frac{(-24x^6)(-32x^4)}{-8x^3} = \frac{768x^{10}}{-8x^3} = -96x^7.$$

We may also proceed as follows,

$$\frac{(-24x^6)(-32x^4)}{-8x^3} = \frac{-24x^6}{-8x^3}(-32x^4) = (3x^3)(-32x^4) = -96x^7.$$

When a negative exponent arises in a term, we prefer to express the term with positive exponents.

175 Example

$$\begin{aligned}\frac{12x^2y^3+6x^4y^2}{3x^3y^3} &= \frac{12x^2y^3}{3x^3y^3}+\frac{6x^4y^2}{3x^3y^3}\\ &= 4x^{-1}+2xy^{-1}\\ &= \frac{4}{x}+\frac{2x}{y}.\end{aligned}$$

Homework

Problem 5.1.1 *Perform the division:* $\dfrac{q^2-pq-pqr}{-q}$.

Problem 5.1.2 *Divide term by term:* $\dfrac{34x^3y^2+51x^2y^3}{17xy}$.

Problem 5.1.3 *Divide:* $\dfrac{(34x^3y^2)(51x^2y^3)}{17xy}$.

Problem 5.1.4 *Divide term by term:*

$$\frac{6aaaaxxx+4aaaxxx}{2aaxx}.$$

Problem 5.1.5 *Divide:* $\dfrac{(6aaaaxxx)(4aaaxxx)}{2aaxx}$.

Problem 5.1.6 *Divide term by term:* $\dfrac{x^2-xy-xz}{-x}$.

Problem 5.1.7 *Divide term by term:* $\dfrac{5a^2b-7ab^3}{-ab}$.

Problem 5.1.8 *Divide:* $\dfrac{(5a^2b)(-7ab^3)}{-ab}$.

Problem 5.1.9 *Divide:* $\dfrac{(x^2)(-xy)(-xz)}{-x}$.

5.2 Long Division

Medicine makes people ill, mathematics make them sad and theology makes them sinful. *-Martin LUTHER*

We must now confront the problem when the divisor consists of more than one term. The algorithm for algebraic long division resembles that for long division of natural numbers.

176 Example Find $(-x+x^2-6)\div(x+2)$.

Solution: ► *First, we rewrite* $-x+x^2-6$ *so that the exponents of each of the terms are listed in decreasing order:* x^2-x-6. *Next display dividend and divisor as follows.*

$$x+2\overline{\smash{\big)}\ x^2\ \ -x-6}$$

Now, think, by what must x *(the term with the largest degree in the divisor) must be multiplied to give* x^2 *(the term with the largest degree in the dividend? The answer is of course* $\dfrac{x^2}{x}=x$, *and so we write* x *in the quotient,*

$$\begin{array}{r}x\\ x+2\overline{\smash{\big)}\ x^2\ \ -x-6}\end{array}$$

Multiply this x *of the quotient by the divisor, obtaining* $x(x+2)=x^2+2x$ *and change all signs:*

$$\begin{array}{r}x\\ x+2\overline{\smash{\big)}\ x^2\ \ -x-6}\\ \underline{-x^2-2x}\end{array}$$

Add, and obtain

$$\begin{array}{rl}
 & x \\
x+2 \!\!\!\! & \overline{\smash{)}\, x^2 \quad -x-6} \\
 & \underline{-x^2-2x} \\
 & \qquad -3x-6
\end{array}$$

The new dividend is now $-3x-6$. *Again, think, by what must* x *(the term with the largest degree in the divisor) must be multiplied to give* $-3x$ *(the term with the largest degree in the dividend? The answer is of course* $\frac{-3x}{x}=-3$, *and so we write* -3 *in the quotient,*

$$\begin{array}{rl}
 & x-3 \\
x+2 \!\!\!\! & \overline{\smash{)}\, x^2 \quad -x-6} \\
 & \underline{-x^2-2x} \\
 & \qquad -3x-6
\end{array}$$

Multiply this -3 *of the quotient by the divisor, obtaining* $-3(x+2)=-3x-6$ *and change all signs:*

$$\begin{array}{rl}
 & x-3 \\
x+2 \!\!\!\! & \overline{\smash{)}\, x^2 \quad -x-6} \\
 & \underline{-x^2-2x} \\
 & \qquad -3x-6 \\
 & \qquad \underline{3x+6}
\end{array}$$

Add, and obtain

$$\begin{array}{rl}
 & x-3 \\
x+2 \!\!\!\! & \overline{\smash{)}\, x^2 \quad -x-6} \\
 & \underline{-x^2-2x} \\
 & \qquad -3x-6 \\
 & \qquad \underline{3x+6} \\
 & \qquad \qquad \quad 0
\end{array}$$

Since the remainder is 0, *the division ends and we conclude that*

$$(-x+x^2-6)\div(x+2)=\frac{x^2-x-6}{x+2}=x+3.$$

◄

177 Example Find $(3x^4-2x^3-9x^2+5x-6)\div(x-2)$.

Solution: ► *The terms of* $3x^4-2x^3-9x^2+5x-6$ *are already written with exponents in decreasing order. Next display dividend and divisor as follows.*

$$x-2\overline{\smash{)}\,3x^4-2x^3-9x^2+5x-6}$$

Now, think, by what must x *(the term with the largest degree in the divisor) must be multiplied to give* $3x^4$ *(the term with the largest degree in the dividend? The answer is of course* $\frac{3x^4}{x}=3x^3$, *and so we write* x *in the quotient,*

$$\begin{array}{rl}
 & \quad 3x^3 \\
x-2 \!\!\!\! & \overline{\smash{)}\,3x^4-2x^3-9x^2+5x-6}
\end{array}$$

Multiply this $3x^3$ *of the quotient by the divisor, obtaining* $3x^3(x-2)=3x^4-6x^3$ *and change all signs:*

$$\begin{array}{rl}
 & \quad 3x^3 \\
x-2 \!\!\!\! & \overline{\smash{)}\,3x^4-2x^3-9x^2+5x-6} \\
 & \underline{-3x^4+6x^3}
\end{array}$$

Add, and obtain

$$\begin{array}{r|l} & 3x^3 \\ \hline x-2) & 3x^4-2x^3-9x^2+5x-6 \\ & \underline{-3x^4+6x^3} \\ & \quad\; 4x^3-9x^2 \end{array}$$

The new dividend is now $4x^3-9x^2$. *Again, think, by what must* x *(the term with the largest degree in the divisor) must be multiplied to give* $4x^3$ *(the term with the largest degree in the dividend? The answer is of course* $\frac{4x^3}{x}=4x^2$, *and so we write* $+4x^2$ *in the quotient,*

$$\begin{array}{r|l} & 3x^3+4x^2 \\ \hline x-2) & 3x^4-2x^3-9x^2+5x-6 \\ & \underline{-3x^4+6x^3} \\ & \quad\; 4x^3-9x^2 \end{array}$$

Multiply this $+4x^2$ *of the quotient by the divisor, obtaining* $+4x^2(x-2)=4x^3-8x^2$ *and change all signs:*

$$\begin{array}{r|l} & 3x^3+4x^2 \\ \hline x-2) & 3x^4-2x^3-9x^2+5x-6 \\ & \underline{-3x^4+6x^3} \\ & \quad\; 4x^3-9x^2 \\ & \quad \underline{-4x^3+8x^2} \end{array}$$

Add, and obtain

$$\begin{array}{r|l} & 3x^3+4x^2 \\ \hline x-2) & 3x^4-2x^3-9x^2+5x-6 \\ & \underline{-3x^4+6x^3} \\ & \quad\; 4x^3-9x^2 \\ & \quad \underline{-4x^3+8x^2} \\ & \qquad\quad -x^2+5x \end{array}$$

The new dividend is now $-x^2+5x$. *Again, think, by what must* x *(the term with the largest degree in the divisor) must be multiplied to give* $-x^2$ *(the term with the largest degree in the dividend? The answer is of course* $\frac{-x^2}{x}=-x$, *and so we write* $-x$ *in the quotient,*

$$\begin{array}{r|l} & 3x^3+4x^2 \quad -x \\ \hline x-2) & 3x^4-2x^3-9x^2+5x-6 \\ & \underline{-3x^4+6x^3} \\ & \quad\; 4x^3-9x^2 \\ & \quad \underline{-4x^3+8x^2} \\ & \qquad\quad -x^2+5x \end{array}$$

Multiply this $-x$ *of the quotient by the divisor, obtaining* $-x(x-2)=-x^2+2x$ *and change all signs:*

$$\begin{array}{r|l} & 3x^3+4x^2 \quad -x \\ \hline x-2) & 3x^4-2x^3-9x^2+5x-6 \\ & \underline{-3x^4+6x^3} \\ & \quad\; 4x^3-9x^2 \\ & \quad \underline{-4x^3+8x^2} \\ & \qquad\quad -x^2+5x \\ & \qquad\quad \underline{\;\; x^2-2x} \end{array}$$

Add, and obtain

$$
\begin{array}{rr}
 & 3x^3+4x^2\quad -x \\ \cline{2-2}
x-2) & 3x^4-2x^3-9x^2+5x-6 \\
 & \underline{-3x^4+6x^3} \\
 & 4x^3-9x^2 \\
 & \underline{-4x^3+8x^2} \\
 & -x^2+5x \\
 & \underline{x^2-2x} \\
 & 3x-6
\end{array}
$$

The new dividend is now $3x-6$*. Again, think, by what must* x *(the term with the largest degree in the divisor) must be multiplied to give* $3x$ *(the term with the largest degree in the dividend? The answer is of course* $\frac{3x}{x}=3$*, and so we write* 3 *in the quotient,*

$$
\begin{array}{rr}
 & 3x^3+4x^2\quad -x+3 \\ \cline{2-2}
x-2) & 3x^4-2x^3-9x^2+5x-6 \\
 & \underline{-3x^4+6x^3} \\
 & 4x^3-9x^2 \\
 & \underline{-4x^3+8x^2} \\
 & -x^2+5x \\
 & \underline{x^2-2x} \\
 & 3x-6
\end{array}
$$

Multiply this 3 *of the quotient by the divisor, obtaining* $3(x-2)=3x-6$ *and change all signs:*

$$
\begin{array}{rr}
 & 3x^3+4x^2\quad -x+3 \\ \cline{2-2}
x-2) & 3x^4-2x^3-9x^2+5x-6 \\
 & \underline{-3x^4+6x^3} \\
 & 4x^3-9x^2 \\
 & \underline{-4x^3+8x^2} \\
 & -x^2+5x \\
 & \underline{x^2-2x} \\
 & 3x-6 \\
 & \underline{-3x+6}
\end{array}
$$

Add, and obtain

$$
\begin{array}{rr}
 & 3x^3+4x^2\quad -x+3 \\ \cline{2-2}
x-2) & 3x^4-2x^3-9x^2+5x-6 \\
 & \underline{-3x^4+6x^3} \\
 & 4x^3-9x^2 \\
 & \underline{-4x^3+8x^2} \\
 & -x^2+5x \\
 & \underline{x^2-2x} \\
 & 3x-6 \\
 & \underline{-3x+6} \\
 & 0
\end{array}
$$

Since the remainder is 0*, the division ends and we conclude that*

$$(3x^4-2x^3-9x^2+5x-6)\div(x-2)=\frac{3x^4-2x^3-9x^2+5x-6}{x-2}=3x^3+4x^2-x+3.$$

◄

We now give an example where there is a remainder.

178 Example Find $(2x^2+1) \div (x+1)$.

Solution: ► *Display dividend and divisor as follows.*

$$x+1\overline{\smash{)}\; 2x^2 \qquad +1}$$

Now, think, by what must x *(the term with the largest degree in the divisor) must be multiplied to give* $2x^2$ *(the term with the largest degree in the dividend? The answer is of course* $\frac{2x^2}{x} = 2x$, *and so we write* $2x$ *in the quotient,*

$$\begin{array}{rl} & \quad\; 2x \\ \cline{2-2} x+1) & 2x^2 \qquad +1 \end{array}$$

Multiply this $2x$ *of the quotient by the divisor, obtaining* $2x(x+1) = 2x^2+2x$ *and change all signs:*

$$\begin{array}{rl} & \quad\; 2x \\ \cline{2-2} x+1) & 2x^2 \qquad +1 \\ & -2x^2-2x \\ \cline{2-2} \end{array}$$

Add, and obtain

$$\begin{array}{rl} & \quad\; 2x \\ \cline{2-2} x+1) & 2x^2 \qquad +1 \\ & -2x^2-2x \\ \cline{2-2} & \qquad -2x+1 \end{array}$$

The new dividend is now $-2x+1$. *Again, think, by what must* x *(the term with the largest degree in the divisor) must be multiplied to give* $-2x$ *(the term with the largest degree in the dividend? The answer is of course* $\frac{-2x}{x} = -2$, *and so we write* -2 *in the quotient,*

$$\begin{array}{rl} & \quad\; 2x-2 \\ \cline{2-2} x+1) & 2x^2 \qquad +1 \\ & -2x^2-2x \\ \cline{2-2} & \qquad -2x+1 \end{array}$$

Multiply this -2 *of the quotient by the divisor, obtaining* $-2(x+1) = -2x-2$ *and change all signs:*

$$\begin{array}{rl} & \quad\; 2x-2 \\ \cline{2-2} x+1) & 2x^2 \qquad +1 \\ & -2x^2-2x \\ \cline{2-2} & \qquad -2x+1 \\ & \qquad\;\; 2x+2 \\ \cline{2-2} \end{array}$$

Add, and obtain

$$\begin{array}{rl} & \quad\; 2x-2 \\ \cline{2-2} x+1) & 2x^2 \qquad +1 \\ & -2x^2-2x \\ \cline{2-2} & \qquad -2x+1 \\ & \qquad\;\; 2x+2 \\ \cline{2-2} & \qquad\qquad\; 3 \end{array}$$

Now, the remainder is 3, *which has lower degree than the divisor* $x+1$, *hence the division ends. We write*

$$(2x^2+1) \div (x+1) = \frac{2x^2+1}{x+1} = 2x-2+\frac{3}{x+1}.$$

◄

We provide below some assorted examples, without explanations.

179 Example We have,

$$
\begin{array}{r|l}
 & x^2-3x\ \ +9 \\ \hline
x+3 & x^3 \qquad\qquad +27 \\
 & -x^3-3x^2 \\ \hline
 & \quad -3x^2 \\
 & \quad 3x^2+9x \\ \hline
 & \qquad\quad 9x+27 \\
 & \qquad\quad -9x-27 \\ \hline
 & \qquad\qquad\quad 0
\end{array}
$$

180 Example We have,

$$
\begin{array}{r|l}
 & x^3+2x^2+4x\ \ +8 \\ \hline
x-2 & x^4 \qquad\qquad\qquad -16 \\
 & -x^4+2x^3 \\ \hline
 & \quad 2x^3 \\
 & \quad -2x^3+4x^2 \\ \hline
 & \qquad\quad 4x^2 \\
 & \qquad\quad -4x^2+8x \\ \hline
 & \qquad\qquad\quad 8x-16 \\
 & \qquad\qquad\quad -8x+16 \\ \hline
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

181 Example We have,

$$
\begin{array}{r|l}
 & \qquad\qquad\quad x^2+4x+4 \\ \hline
x^2-x-2 & x^4+3x^3-2x^2\ \ +x-1 \\
 & -x^4\ \ +x^3+2x^2 \\ \hline
 & \qquad 4x^3 \qquad\quad +x \\
 & \quad -4x^3+4x^2+8x \\ \hline
 & \qquad\qquad 4x^2+9x-1 \\
 & \qquad\quad -4x^2+4x+8 \\ \hline
 & \qquad\qquad\qquad 13x+7
\end{array}
$$

and hence we write

$$\frac{x^4+3x^3-2x^2+x-1}{x^2-x-2} = x^2+4x+4+\frac{13x+7}{x^2-x-2}.$$

Homework

Problem 5.2.1 *Perform the division:* $(x^2-2x+1)\div(x-1)$.

Problem 5.2.2 *Perform the division:* $(x^2+5x+6)\div(x+2)$.

Problem 5.2.3 *Perform the division:* $(x^3+1)\div(x+1)$.

Problem 5.2.4 *Expand and collect like terms:*

$$\frac{x^3-8}{x-2}-x^2.$$

Problem 5.2.5 *Expand and collect like terms:*

$$\frac{6x^3-x^2-4x-1}{2x+1}-\frac{3x^3+3x-x^2-1}{3x-1}.$$

5.3 Factoring I

Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work. . . . A good teacher

should understand and impress on his students the view that no problem whatever is completely exhausted. One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything else. We have a natural opportunity to investigate the connections of a problem when looking back at its solution.
-George PÓLYA

182 Definition To *factor* an algebraic expression is to express it as a product.

The idea behind factoring is essentially that of using the distributive law backwards,

$$ab + ac = a(b+c),$$

where the dextral quantity is a decomposition into factors of the sinistral quantity.

We will begin by giving examples where the greatest common divisor of the terms is different from 1 and hence it may be removed. Thus an expressions with two terms will be factored as

$$a + b = \blacksquare(\square_1 + \square_2),$$

one with three terms will be factored as

$$a + b + c = \blacksquare(\square_1 + \square_2 + \square_3),$$

etc.

183 Example Factor $20a^2b^3 + 24a^3b^2$.

Solution: ► *The greatest common divisor of 20 and 24 is 4, hence it can be removed. Also for the two expressions a^2 and a^3, their greatest common divisor is the one with the least exponent, that is a^2. Similarly, the greatest common divisor of b^2 and b^3 is b^2. Thus $4a^2b^2$ is a common factor of both terms and the desired factorisation has the form,*

$$20a^2b^3 + 24a^3b^2 = 4a^2b^2(\square_1 + \square_2).$$

We need to determine $\square_1$ and $\square_2$. We have

$$20a^2b^3 = 4a^2b^2\square_1 \implies \square_1 = 5b,$$

and

$$24a^3b^2 = 4a^2b^2\square_2 \implies \square_1 = 6a.$$

The desired factorisation is thus

$$20a^2b^3 + 24a^3b^2 = 4a^2b^2(5b + 6a).$$

◄

Multiplying, $4a^2b^2(5b+6a) = 20a^2b^3 + 24a^3b^2$.

184 Example Factor $-30x^2yza^2b + 36xy^2za + 16x^3y^2z^2b$.

Solution: ► *The greatest common divisor of $-30, 36, 16$ is 2. The letters x, y, z appear in all three terms, but not so for a and b. The least power of x appearing in all three terms is 1, and same for y and z. Thus the greatest common divisor of the three terms is $2xyz$ and the desired factorisation is thus*

$$2xyz(-15xa^2b + 18ya + 8x^2yzb).$$

◄

185 Example Factor $ab + a^2b + ab^2$.

Solution: ► *The desired factorisation is clearly*

$$ab + a^2b + ab^2 = ab(1 + a + b).$$

◄

186 Example Prove that the sum of two even integers is even.

Solution: ► *Let $2a$ and $2b$ be two even integers. Then*

$$2a + 2b = 2(a + b),$$

that is, twice the integer $a + b$, and hence an even integer. ◄

187 Example Factor $14a^3b^2(x + y)^2(x - y)^3 + 20ab^3(x + y)^3(x - y)^4$.

Solution: ► *The desired factorisation is clearly*

$$14a^3b^2(x + y)^2(x - y)^3 + 20ab^3(x + y)^3(x - y)^4 = 2ab^2(x + y)^2(x - y)^3(7a^2 + 10b(x + y)(x - y)).$$

◄

188 Example Resolve into factors: $ax^2 + a + bx^2 + b$.

Solution: ► *In this case, no factor greater than 1 is common to all terms. We have, however,*

$$ax^2 + a + bx^2 + b = a(x^2 + 1) + b(x^2 + 1).$$

Notice now that $x^2 + 1$ is a common factor of both summands, and finally,

$$ax^2 + a + bx^2 + b = a(x^2 + 1) + b(x^2 + 1) = (x^2 + 1)(a + b).$$

◄

189 Example Resolve into factors: $x^3 + x^2 + x + 1$.

Solution: ► *Like in the preceding problem, we have no factor greater than 1 being common to all terms. Observe that*

$$x^3 + x^2 + x + 1 = x^2(x + 1) + 1(x + 1).$$

Notice now that $x + 1$ is a common factor of both summands, and finally,

$$x^3 + x^2 + x + 1 = x^2(x + 1) + 1(x + 1) = (x + 1)(x^2 + 1).$$

◄

190 Example Resolve into factors: $x^2 - ax + bx - ab$.

Solution: ► *We have,*

$$x^2 - ax + bx - ab = x(x - a) + b(x - a) = (x - a)(x + b).$$

◄

Sometimes it is necessary to rearrange the summands.

191 Example Resolve into factors: $12a^2-4ab-3ax^2+bx^2$.

Solution: ► *We have,*

$$12a^2-4ab-3ax^2+bx^2=(12a^2-3ax^2)-(4ab-bx^2)=3a(4a-x^2)-b(4a-x^2)=(4a-x^2)(3a-b).$$

◄

Homework

Problem 5.3.1 *Factor* $-2a^2b^3$ *from* $-4a^6b^5+6a^3b^8-12a^5b^3-2a^2b^3$.

Problem 5.3.2 *Factor* $2a^3b^3$ *from* $4a^3b^4-10a^4b^3$.

Problem 5.3.3 *Factor* $\frac{3}{4}x$ *from* $\frac{9}{16}x^2-\frac{3}{4}x$.

Problem 5.3.4 *Factor* -1 *from* $-x+2y-3z$.

Problem 5.3.5 *Prove that the sum of two odd integers is even.*

Problem 5.3.6 *Resolve into factors:* x^3-x^2.

Problem 5.3.7 *Factor*

$$125a^4b^5c^5-45a^5b^3c^4+5a^3b^2c^4-300a^4b^2c^8-10a^3b^2c^5.$$

Problem 5.3.8 *Resolve into factors:* $5x^5-10a^7x^3-15a^3x^3$.

Problem 5.3.9 *Factor* $3x^3y+4x^2y^3-6x^6-10x^4$.

Problem 5.3.10 *Factor*

$$3m^6p^4q^2-9m^5p^2qx+3m^7p^3qx+3m^4p^2q-6m^5p^4qx^2y.$$

Problem 5.3.11 *Resolve into factors:* $38a^2x^5+57a^4x^2$.

Problem 5.3.12 *Decompose into factors:* $a^2+ab+ac+bc$.

Problem 5.3.13 *Decompose into factors:* $a^2-ab+ac-bc$.

Problem 5.3.14 *Decompose into factors:* y^3-y^2+y-1.

Problem 5.3.15 *Decompose into factors:* $2x^3+3x^2+2x+3$.

Problem 5.3.16 *Decompose into factors:*

$$a^2x+abx+ac+aby+b^2y+bc.$$

5.4 Factoring II

The simplest schoolboy is now familiar with facts for which Archimedes would have sacrificed his life.
-Ernst RENAN

In this section we study factorisations of expressions of the form ax^2+bx+c, the so-called *quadratic trinomials* .

Let us start with expressions of the form x^2+bx+c. Suppose that

$$x^2+bx+c=(x+p)(x+q).$$

Then, upon multiplying,

$$x^2+bx+c=(x+p)(x+q)=x^2+(p+q)x+pq.$$

Thus $c=pq$ and $b=p+q$, that is, the constant term is the product of two numbers, and the coefficient of x is the sum of these two numbers.

192 Example To factor x^2+5x+6 we look for two numbers whose product is 6 and whose sum is 5. Clearly 2 and 3 are these two numbers and so,

$$x^2+5x+6=(x+2)(x+3).$$

You should multiply the last product to verify the equality.

193 Example To factor x^2+7x+6 we look for two numbers whose product is 6 and whose sum is 7. Clearly 1 and 6 are these two numbers and so,

$$x^2+7x+6=(x+1)(x+6).$$

You should multiply the last product to verify the equality.

194 Example To factor x^2-5x+6 we look for two numbers whose product is 6 and whose sum is -5. Clearly -2 and -3 are these two numbers and so,

$$x^2-5x+6=(x-2)(x-3).$$

195 Example To factor x^2-x-6 we look for two numbers whose product is -6 and whose sum is -1. Clearly -3 and $+2$ are these two numbers and so,

$$x^2-x-6=(x-3)(x+2).$$

196 Example To factor x^2+x-6 we look for two numbers whose product is -6 and whose sum is $+1$. Clearly $+3$ and -2 are these two numbers and so,

$$x^2+x-6=(x+3)(x-2).$$

197 Example Here are some more examples which you should inspect:

$$x^2+10x+24=(x+4)(x+6),$$
$$x^2+11x+24=(x+3)(x+8),$$
$$x^2+14x+24=(x+2)(x+12),$$
$$x^2+25x+24=(x+1)(x+24).$$

198 Example Here are some more examples which you should inspect:

$$x^2-10x+24=(x-4)(x-6),$$
$$x^2-11x+24=(x-3)(x-8),$$
$$x^2-14x+24=(x-2)(x-12),$$
$$x^2-25x+24=(x-1)(x-24).$$

199 Example Here are some more examples which you should inspect:

$$x^2+2x-24=(x-4)(x+6),$$
$$x^2-5x-24=(x+3)(x-8),$$
$$x^2+10x-24=(x-2)(x+12),$$
$$x^2-10x-24=(x+2)(x-12).$$

In the case when the coefficient of the x^2 is different from 1, more cases must be considered. The problem is more difficult, but the method is essentially the same.

200 Example By inspection,

$$5x^2+22x+8=(5x+2)(x+4),$$
$$5x^2-22x+8=(5x-2)(x-4),$$
$$5x^2-18x-8=(5x+2)(x-4),$$
$$5x^2+18x-8=(5x-2)(x+4),$$
$$5x^2+39x-8=(5x-1)(x+8).$$

201 Example By inspection,

$$20x^2+43x+21=(5x+7)(4x+3),$$
$$20x^2+47x+21=(5x+3)(4x+7),$$
$$20x^2+76x+21=(10x+3)(2x+7),$$
$$20x^2+143x+21=(20x+3)(x+7).$$

The inquiring reader may wonder whether all quadratic trinomials ax^2+bx+c factor in the manner above, that is, into linear factors all whose coefficients are integers. The answer is *no*. For example if x^2-2 does not factor into linear factors whose coefficients are integers, since there are no two integers whose product is -2 and that add up to 0. If we augment our choice for the coefficients for linear factors, then we may write

$$x^2-2=(x-\sqrt{2})(x+\sqrt{2}),$$

but we will not consider these cases in this course.

Homework

Problem 5.4.1 *Decompose into factors:* $a^2-11a+30$.

Problem 5.4.2 *Decompose into factors:* $a^2-38a+361$.

Problem 5.4.3 *Decompose into factors:*

$$a^4b^4+37a^2b^2+300.$$

Problem 5.4.4 *Decompose into factors:* $x^2-23x+132$.

Problem 5.4.5 *Decompose into factors:* x^4-29x^2+204.

Problem 5.4.6 *Decompose into factors:* $x^2+35x+216$.

Problem 5.4.7 *Resolve into factors:* $5x^2+17x+6$.

Problem 5.4.8 *Resolve into factors:* $14x^2+29x-15$.

5.5 Special Factorisations

> ***At the age of eleven, I began Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined there was anything so delicious in the world. From that moment until I was thirty-eight, mathematics was my chief interest and my chief source of happiness.*** *-Bertrand RUSSELL*

Recall the special products we studied in the Multiplication chapter. We list them here for easy reference.

Difference of Squares

$$x^2-y^2=(x-y)(x+y). \tag{5.1}$$

Difference of Cubes

$$x^3-y^3=(x-y)(x^2+xy+y^2). \tag{5.2}$$

Sum of Cubes

$$x^3+y^3=(x+y)(x^2-xy+y^2). \tag{5.3}$$

Square of a Sum

$$x^2+2xy+y^2=(x+y)^2. \tag{5.4}$$

Square of a Difference

$$x^2-2xy+y^2=(x-y)^2. \tag{5.5}$$

We will use these special factorisations and the methods of the preceding sections in order to treat more complicated factorisation problems.

202 Example To factor $x^4 - 9y^2$ observe that it is a difference of squares, and so $x^4 - 9y^2 = (x^2 - 3y)(x^2 + 3y)$.

203 Example To factor $x^3 - 1$ observe that it is a difference of cubes, and so $x^3 - 1 = (x-1)(x^2 + x + 1)$.

Sometimes we need to use more than one method.

204 Example To factor $x^3 - 4x$ factor a common factor and then the difference of squares:

$$x^3 - 4x = x(x^2 - 4) = x(x-2)(x+2).$$

205 Example To factor $x^4 - 81$, observe that there are two difference of squares:

$$x^4 - 81 = (x^2 - 9)(x^2 + 9) = (x-3)(x+3)(x^2 + 9).$$

206 Example To factor $x^6 - 1$, observe that there is a difference of squares and then a sum and a difference of cubes:

$$x^6 - 1 = (x^3 - 1)(x^3 + 1) = (x-1)(x^2 + x + 1)(x+1)(x^2 - x + 1).$$

The following method, called *Sophie Germain's trick*[1] is useful to convert some expressions into differences of squares.

207 Example We have

$$\begin{aligned} x^4 + x^2 + 1 &= x^4 + 2x^2 + 1 - x^2 \\ &= (x^2 + 1)^2 - x^2 \\ &= (x^2 + 1 - x)(x^2 + 1 + x). \end{aligned}$$

208 Example We have

$$\begin{aligned} x^4 + 4 &= x^4 + 4x^2 + 4 - 4x^2 \\ &= (x^2 + 2)^2 - 4x^2 \\ &= (x^2 + 2 - 2x)(x^2 + 2 + 2x). \end{aligned}$$

Sophie Germain's trick is often used in factoring quadratic trinomials, where it is often referred to as the technique of *completing the square*. We will give some examples of factorisations that we may also obtain with the trial an error method of the preceding section.

[1] Sophie Germain (1776–1831) was an important French mathematician of the French Revolution. She pretended to be a man in order to study Mathematics. At the time, women were not allowed to matriculate at the École Polytechnique, but she posed as a M. Leblanc in order to obtain lessons from Lagrange.

209 Example We have

$$\begin{aligned}
x^2-8x-9 &= x^2-8x+16-9-16\\
&= (x-4)^2-25\\
&= (x-4)^2-5^2\\
&= (x-4-5)(x-4+5)\\
&= (x-9)(x+1).
\end{aligned}$$

210 Example We have

$$\begin{aligned}
x^2+4x-117 &= x^2+4x+4-117-4\\
&= (x+2)^2-11^2\\
&= (x+2-11)(x+2+11)\\
&= (x-9)(x+13).
\end{aligned}$$

The techniques learned may be used to solve some purely arithmetic problems.

211 Example Given that

$$1,\ 000,\ 002,\ 000,\ 001$$

has a prime factor greater than 9000, find it.

Solution: ► *We have*

$$\begin{aligned}
1,\ 000,\ 002,\ 000,\ 001 &= 10^{12}+2\cdot 10^6+1\\
&= (10^6+1)^2\\
&= ((10^2)^3+1)^2\\
&= (10^2+1)^2((10^2)^2-10^2+1)^2\\
&= 101^2 9901^2,
\end{aligned}$$

whence the prime sought is 9901. ◄

Homework

Problem 5.5.1 *Given that* $x+2y=3$ *and* $x-2y=-1$, *find* x^2-4y^2.

Problem 5.5.2 *Resolve into factors:* x^4-16.

Problem 5.5.3 *Resolve into factors:* $(a+b)^2-c^2$

Problem 5.5.4 *Resolve into factors:* x^3-x.

Problem 5.5.5 *Find all positive primes of the form* n^3-8, *where* n *is a positive integer.*

5.6 Rational Expressions

For all their wealth of content, for all the sum of history and social institution invested in them, music, mathematics, and chess are resplendently useless (applied mathematics is a higher plumbing, a kind of music for the police band). They are metaphysically trivial, irresponsible. They refuse to relate outward, to take reality for arbiter. This is the source of their witchery. *-G. STEINER*

To reduce an algebraic fraction, we must first factor it.

212 Example Reduce the fraction $\frac{x^2-1}{x^2-2x+1}$.

Solution: ► *We have* $\frac{x^2-1}{x^2-2x+1} = \frac{(x-1)(x+1)}{(x-1)(x-1)} = \frac{x+1}{x-1}$. ◄

213 Example Express the fraction $\frac{x^2+5x+6}{x^2-5x-14}$ in lowest terms.

Solution: ► *We have* $\frac{x^2+5x+6}{x^2-5x-14} = \frac{(x+2)(x+3)}{(x+2)(x-7)} = \frac{x+3}{x-7}$. ◄

To add or subtract algebraic fractions, we first write them in common denominators. We will prefer to expand all products and collect like terms.

214 Example Add:

$$\frac{3}{x-3}+\frac{2}{x+2}.$$

Solution: ► *We have,*

$$\begin{aligned}\frac{3}{x-3}+\frac{2}{x+2} &= \frac{3(x+2)}{(x-3)(x+2)}+\frac{2(x-3)}{(x+2)(x-3)}\\ &= \frac{3x+6}{(x-3)(x+2)}+\frac{2x-6)}{(x+2)(x-3)}\\ &= \frac{5x}{(x-3)(x+2)}\\ &= \frac{5x}{x^2-x-6}.\end{aligned}$$

◄

215 Example Add:

$$\frac{x}{x^2+2x+1}+\frac{x}{x^2-1}.$$

Solution: ► *We have,*

$$\begin{aligned}\frac{x}{x^2+2x+1}+\frac{x}{x^2-1} &= \frac{x}{(x+1)^2}+\frac{x}{(x-1)(x+1)}\\ &= \frac{x(x-1)}{(x+1)^2(x-1)}+\frac{x(x+1)}{(x-1)(x+1)^2}\\ &= \frac{x^2-x+x^2+x}{(x+1)^2(x-1)}\\ &= \frac{2x^2}{x^3+x^2-x-1}\end{aligned}$$

◄

216 Example Gather all the fractions:

$$\frac{a}{b}+\frac{c}{d}-\frac{e}{f}.$$

Solution: ► *We have,*

$$\begin{aligned}\frac{a}{b}+\frac{c}{d}-\frac{e}{f} &= \frac{adf}{bdf}+\frac{cbf}{dbf}-\frac{ebd}{fbd}\\ &= \frac{adf+cbf-ebd}{fbd}.\end{aligned}$$

◄

Multiplication and division of algebraic fractions is carried out in a manner similar to the operations of arithmetical fractions.

217 Example Multiply:

$$\frac{x-1}{x+1}\cdot\frac{x+2}{x-2}.$$

Solution: ► *We have,*

$$\begin{aligned}\frac{x-1}{x+1}\cdot\frac{x+2}{x-2} &= \frac{(x-1)(x+2)}{(x+1)(x-2)}\\ &= \frac{x^2+x-2}{x^2-x-2}.\end{aligned}$$

◄

218 Example Divide:

$$\frac{x-1}{x+1}\div\frac{x+2}{x-2}.$$

Solution: ► *We have,*

$$\begin{aligned}\frac{x-1}{x+1}\div\frac{x+2}{x-2} &= \frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\\ &= \frac{(x-1)(x-2)}{(x+1)(x+2)}\\ &= \frac{x^2-3x+2}{x^2+3x+2}.\end{aligned}$$

◄

219 Example The sum of two numbers is 7 and their product $21.$ What is the sum of their reciprocals?

Solution: ► *Let x, y be the numbers. One has $x+y=7, xy=21$ whence*

$$\frac{1}{x}+\frac{1}{y}=\frac{y+x}{xy}=\frac{7}{21}=\frac{1}{3}.$$

◄

220 Example Prove that

$$\frac{1}{1+\frac{1}{x+\frac{1}{x}}}=\frac{x^2+1}{x^2+x+1}.$$

Solution: ► *Proceeding from the innermost fraction*

$$\begin{aligned}
\frac{1}{1+\dfrac{1}{x+\dfrac{1}{x}}} &= \frac{1}{1+\dfrac{1}{\dfrac{x^2}{x}+\dfrac{1}{x}}} \\
&= \frac{1}{1+\dfrac{1}{\dfrac{x^2+1}{x}}} \\
&= \frac{1}{1+\dfrac{x}{x^2+1}} \\
&= \frac{1}{\dfrac{x^2+1}{x^2+1}+\dfrac{x}{x^2+1}} \\
&= \frac{1}{\dfrac{x^2+x+1}{x^2+1}} \\
&= \frac{x^2+1}{x^2+x+1}.
\end{aligned}$$

◄

Homework

Problem 5.6.1 *Add:* $\frac{1}{x-1}+\frac{1}{x+1}$

Problem 5.6.2 *Subtract:* $\frac{1}{x-1}-\frac{1}{x+1}$

Problem 5.6.3 *Add:* $\frac{x}{x-2}+\frac{2}{x+2}$

Problem 5.6.4 *Subtract:* $\frac{x}{x-2}-\frac{2}{x+2}$

Problem 5.6.5 *Gather the fractions:* $\frac{1}{x}+\frac{1}{x-1}-\frac{2}{x+1}$

Problem 5.6.6 *Gather the fractions:* $\frac{3x}{2}+\frac{3x}{2a}-\frac{4x}{8a^2}$.

Problem 5.6.7 *Gather the fractions:* $\frac{1}{s-1}-\frac{s}{(s-1)(s+1)}$.

Problem 5.6.8 *Gather the fractions:*

$$\frac{x^2-xy}{x^2y}-\frac{y-z}{yz}-\frac{2z^2-xz}{z^2x}.$$

Problem 5.6.9 *Gather the fractions:* $\frac{2s}{s+2}-\frac{3}{s-2}$.

Problem 5.6.10 *Gather the fractions:* $\frac{2x}{a}+\frac{3x}{a^2}-\frac{4x}{a^2}$.

Problem 5.6.11 *The sum of two numbers is* 7 *and their product* 21*. What is the sum of the squares of their reciprocals?*

Part IV

Equations

6 Linear Equations in One Variable

Mathematics is an obscure field, an abstruse science, complicated and exact; yet so many have attained perfection in it that we might conclude almost anyone who seriously applied himself would achieve a measure of success. *-Cicero*

6.1 Simple Equations

The mathematician is entirely free, within the limits of his imagination, to construct what worlds he pleases. What he is to imagine is a matter for his own caprice; he is not thereby discovering the fundamental principles of the universe nor becoming acquainted with the ideas of God. If he can find, in experience, sets of entities which obey the same logical scheme as his mathematical entities, then he has applied his mathematics to the external world; he has created a branch of science.
-John William Navin SULLIVAN

221 Definition An *identity* is an assertion that is true for all values of the variables involved.

222 Example From the square of the sum formula we get $(x+1)^2 = x^2+2x+1$. Since this is valid for all real values of x, this is an identity.

223 Example The assertion $4+x=6-x$ is not an identity. For example, if $x=0$ we have $4+0=6-0$, which is patently false. For $x=1$ we have, however, $4+1=6-1$, which is true. Thus the assertion is sometimes true and sometimes false.

224 Example The assertion $x=x+1$ is never true. How could a real number be equal to itself plus 1?

225 Definition An *equation of condition* or *equation* for short, is an assertion that is true for only particular values of the variables employed.

Example 223 is an example of an equation.

226 Definition The variable whose value is required to find is called the *unknown variable*. The process of finding a value that satisfies the equation is called *solving the equation.*

In this section our main preoccupation will be linear equations in one unknown.

227 Definition A *linear equation* in the unknown x is an equation of the form $ax+b=c$, where $a\neq 0$, b, c are real numbers.

In order to solve simple equations, we will appeal to the following axioms.

228 Axiom If to equals we add equals, the sums are equal.

229 Axiom If equals are multiplied by equals, the products are equal.

In what follows we will use the symbol $\Longrightarrow$, which is read "implies."

230 Example Solve for x: $x-3=-9$

Solution: ►

$$x-3=-9 \implies x=-9+3$$
$$\implies x=-6.$$

Verification: $x-3=-6-3\overset{\checkmark}{=}-9$. ◄

231 Example Solve for x: $x+3=-9$

Solution: ►

$$x+3=-9 \implies x=-9-3$$
$$\implies x=-12.$$

Verification: $x+3=-12+3\overset{\checkmark}{=}-9$. ◄

232 Example Solve for x: $x+3a=-9a$

Solution: ►

$$x+3a=-9a \implies x=-3a-9a$$
$$\implies x=-12a.$$

Verification: $x+3a=-12a+3a\overset{\checkmark}{=}-9a$.

233 Example Solve for x: $x-a+b=2a+3b$

Solution: ►

$$x-a+b=2a+3b \implies x=2a+3b+a-b$$
$$\implies x=3a+2b.$$

Verification: $x-a+b=3a+2b-a+b\overset{\checkmark}{=}2a+3b$. ◄

234 Example Solve for x: $2x=-88$

Solution: ►

$$2x=-88 \implies x=\frac{-88}{2}$$
$$\implies x=-44.$$

◄

Verification: $2x = 2(-44) \stackrel{\checkmark}{=} -88$. ◄

235 Example Solve for x: $2ax = 4a$

Solution: ►

$$2ax = 4a \implies x = \frac{4a}{2a}$$
$$\implies x = 2.$$

Verification: $2ax = 2a(2) \stackrel{\checkmark}{=} 4a$. ◄

236 Example Solve for x: $2ax = 4a^2$

Solution: ►

$$2ax = 4a^2 \implies x = \frac{4a^2}{2a}$$
$$\implies x = 2a.$$

Verification: $2ax = 2a(2a) \stackrel{\checkmark}{=} 4a^2$. ◄

237 Example Solve for x: $\frac{x}{3} = -9$

Solution: ►

$$\frac{x}{3} = -9 \implies x = -9(3)$$
$$\implies x = -27.$$

Verification: $\frac{x}{3} = \frac{-27}{3} \stackrel{\checkmark}{=} -9$. ◄

238 Example Solve for x: $\frac{x}{a} = a^4$

Solution: ►

$$\frac{x}{a} = a^4 \implies x = a^4(a)$$
$$\implies x = a^5.$$

Verification: $\frac{x}{a} = \frac{a^5}{a} \stackrel{\checkmark}{=} a^4$. ◄

239 Example Solve for x: $\frac{ax}{b} = \frac{b}{a}$

Solution: ►

$$\frac{ax}{b} = \frac{b}{a} \implies x = \frac{b}{a} \cdot \frac{b}{a}$$
$$\implies x = \frac{b^2}{a^2}.$$

Verification: $\dfrac{ax}{b} = \dfrac{a}{b} \cdot \dfrac{b^2}{a^2} \stackrel{\checkmark}{=} \dfrac{b}{a}$. ◀

240 Example Solve for x: $\dfrac{ax}{b} = a^2b^3$

Solution: ▶

$$\begin{aligned} \frac{ax}{b} = a^2b^3 \quad &\implies \quad x = (a^2b^3)\frac{b}{a} \\ &\implies \quad x = ab^4. \end{aligned}$$

Verification: $\dfrac{ax}{b} = (ab^4)\dfrac{a}{b} \stackrel{\checkmark}{=} a^2b^3$. ◀

241 Example Solve for x: $0 = 12x$

Solution: ▶

$$\begin{aligned} 0 = 12x \quad &\implies \quad \frac{0}{12} = x \\ &\implies \quad x = 0. \end{aligned}$$

Verification: $12x = 12(0) \stackrel{\checkmark}{=} 0$. ◀

242 Example Solve for x: $3x - 5 = 28$

Solution: ▶

$$\begin{aligned} 3x - 5 = 28 \quad &\implies \quad 3x = 28 + 5 \\ &\implies \quad x = \frac{33}{3} \\ &\implies \quad x = 11. \end{aligned}$$

Verification: $3x - 5 = 3(11) - 5 = 33 - 5 \stackrel{\checkmark}{=} 28$. ◀

243 Example Solve for x: $ax - 2a = 2ab - a$

Solution: ▶

$$\begin{aligned} ax - 2a = 2ab - a \quad &\implies \quad ax = 2ab - a + 2a \\ &\implies \quad x = \frac{2ab + a}{a} \\ &\implies \quad x = 2b + 1. \end{aligned}$$

Verification: $ax - 2a = a(2b + 1) - 2a = 2ab + a - 2a \stackrel{\checkmark}{=} 2ab - a$. ◀

Homework

Problem 6.1.1 *Solve for the variable z:* $z+4=-4$.

Problem 6.1.2 *Solve for the variable z:* $z-4=-4$.

Problem 6.1.3 *Solve for the variable z:* $4z=-44$.

Problem 6.1.4 *Solve for the variable z:* $4z=-3$.

Problem 6.1.5 *Solve for the variable z:* $\frac{z}{4}=-4$.

Problem 6.1.6 *Solve for the variable z:* $\frac{2z}{3}=-22$.

Problem 6.1.7 *Solve for the variable z:* $\frac{2z}{3}+2=-22$.

Problem 6.1.8 *Solve for the variable z:* $\frac{z}{3}=a+1$.

Problem 6.1.9 *Solve for the variable z:* $5z-3=22$.

Problem 6.1.10 *Solve for the variable z:* $\frac{bz}{a+1}-b^3=ab$.

Problem 6.1.11 *Solve for the variable N:* $3N=3A$.

Problem 6.1.12 *Solve for the variable N:* $\frac{N}{A}=A^2+1$.

Problem 6.1.13 *Solve for the variable N:*

$$\frac{XN}{Y}=\frac{Y}{X}.$$

Problem 6.1.14 *Solve for the variable N:* $N-5A=7A+A^2$.

Problem 6.1.15 *Solve for the variable N:* $N+8X=-9X$.

Problem 6.1.16 *Solve for the variable N:* $\frac{N}{A+2}=A+1$.

Problem 6.1.17 *Solve for the variable N:*

$$\frac{A^2N}{B^3}=A^2B^4.$$

Problem 6.1.18 *Solve for the variable N:* $\frac{7N}{A}=\frac{14}{A^2}$.

Problem 6.1.19 *Solve for the variable N:* $N+2=A+14$.

6.2 Miscellaneous Linear Equations

If they would, for Example, praise the Beauty of a Woman, or any other Animal, they describe it by Rhombs, Circles, Parallelograms, Ellipses, and other geometrical terms . . . .

-Jonathan SWIFT

244 Example Solve for x: $\frac{x}{2}-\frac{x}{3}=\frac{x}{4}+\frac{1}{2}$

Solution: ► *The denominators are $3,4$ and 2, and their least common multiple is* $\text{lcm}(3,4,2)=12$. *We multiply both sides of the equation by this least common multiple, and so*

$$\begin{aligned} \frac{x}{2}-\frac{x}{3}=\frac{x}{4}+\frac{1}{2} \quad &\implies \quad 12\left(\frac{x}{2}-\frac{x}{3}\right)=12\left(\frac{x}{4}+\frac{1}{2}\right) \\ &\implies \quad 6x-4x=3x+6 \\ &\implies \quad 2x=3x+6 \\ &\implies \quad 2x-3x=6 \\ &\implies \quad -x=6 \\ &\implies \quad x=-6 \end{aligned}$$

◄

245 Example Solve for a: $\frac{b-a}{6}=\frac{b-a}{15}-b$.

Solution: ► *The denominators are* 6 *and* 15*, and their least common multiple is* $\operatorname{lcm}(6,15) = 30$*. We multiply both sides of the equation by this least common multiple, and so*

$$\begin{aligned}\frac{b-a}{6} = \frac{b-a}{15} - b \quad & \Longrightarrow \quad 30\left(\frac{b-a}{6}\right) = 30\left(\frac{b-a}{15} - b\right) \\ & \Longrightarrow \quad 5(b-a) = 2(b-a) - 30b \\ & \Longrightarrow \quad 5b - 5a = 2b - 2a - 30b \\ & \Longrightarrow \quad -5a + 2a = 2b - 5b - 30b \\ & \Longrightarrow \quad -3a = -33b \\ & \Longrightarrow \quad a = 11b.\end{aligned}$$

◄

246 Example Solve for x: $\dfrac{2x-1}{3} = \dfrac{1-3x}{2}$.

Solution: ► *Cross-multiplying,*

$$\begin{aligned}\frac{2x-1}{3} = \frac{1-3x}{2} \quad & \Longrightarrow \quad 2(2x-1) = 3(1-3x) \\ & \Longrightarrow \quad 4x - 2 = 3 - 9x \\ & \Longrightarrow \quad 4x + 9x = 3 + 2 \\ & \Longrightarrow \quad 13x = 5 \\ & \Longrightarrow \quad x = \frac{5}{13}.\end{aligned}$$

◄

247 Example Write the infinitely repeating decimal $0.3\overline{45} = 0.345454545\ldots$ as the quotient of two natural numbers.

Solution: ► *The trick is to obtain multiples of* $x = 0.345454545\ldots$ *so that they have the same infinite tail, and then subtract these tails, cancelling them out.*[1] *So observe that*

$$10x = 3.45454545\ldots; \qquad 1000x = 345.454545\ldots \qquad \Longrightarrow \qquad 1000x - 10x = 342 \qquad \Longrightarrow \qquad x = \frac{342}{990} = \frac{19}{55}.$$

◄

248 Example Find the fraction that represents the repeating decimal

$$4.3\overline{21} = 4.32121212121\ldots.$$

Solution: ► *Put* $x = 4.32121212121\ldots$*. Then* $10x = 43.21212121\ldots$ *and* $1000x = 4321.2121212121\ldots$*. Hence*

$$1000x - 10x = 4321.2121212121\ldots - 43.212121212121\ldots 4278.$$

[1] That this cancellation is meaningful depends on the concept of *convergence*, which will be studied in later courses.

Finally

$$x = \frac{4278}{990} = \frac{2139}{495} = \frac{713}{165}.$$

◄

249 Example Find the sum of the first hundred positive integers, that is, find

$$1+2+3+\cdots+99+100.$$

Solution: ► *Notice that this is example 5. Put*

$$x = 1+2+3+\cdots+99+100.$$

The crucial observation is that adding the sum forwards is the same as adding the sum backwards, hence

$$x = 100+99+\cdots+3+2+1.$$

Adding,

$$\begin{array}{rclclclclcl} x & = & 1 & + & 2 & + & \cdots & + & 99 & + & 100 \\ x & = & 100 & + & 99 & + & \cdots & + & 2 & + & 1 \\ \hline 2x & = & 101 & + & 101 & + & \cdots & + & 101 & + & 101 \\ & = & 100\cdot 101, & & & & & & & & \end{array}$$

whence

$$x = \frac{100\cdot 101}{2} = 50\cdot 101 = 5050.$$

◄

250 Example Consider the arithmetic progression

$$2, 7, 12, 17, \ldots.$$

Is 302 in this progression? Is 303 in this progression?

Solution: ► *Observe that we start with* 2 *and keep adding* 5*. Thus*

$$2 = 2+5\cdot 0, \quad 7 = 2+5\cdot 1, \quad 12 = 2+5\cdot 2, \quad 17 = 2+5\cdot 3, \quad \ldots$$

so the general term has the form $5n+2$*, which are the numbers leaving remainder* 2 *when divided by* 5*. Since*

$$5n+2 = 302 \implies 5n = 300 \implies n = 60,$$

is an integer, 302 *is in this progression. Since*

$$5n+2 = 303 \implies 5n = 301 \implies n = \frac{301}{5},$$

is an not integer, 303 *is not in this progression.* ◄

Some equations which are in appearance not linear, may be reduced to a linear equation. Sometimes it is necessary to do some simplification of the expressions involved before solving an equation. In the examples that follow, we leave verification to the reader.

251 Example Solve for x: $(3x+1)^2+6+18(x+1)^2 = 9x(3x-2)+65$

Solution: ►

$$\begin{aligned}(3x+1)^2+6+18(x+1)^2=9x(3x-2)+65 \quad &\implies \quad 9x^2+6x+1+6+18x^2+36x+18\\ &\qquad = 27x^2-18x+65\\ &\implies \quad 27x^2+42x+25=27x^2-18x+65\\ &\implies \quad 27x^2+42x-27x^2+18x=65-25\\ &\implies \quad 60x=40\\ &\implies \quad x=\frac{2}{3}.\end{aligned}$$

◄

252 Example Solve for x: $(2x+1)(2x+6)-7(x-2)=4(x+1)(x-1)-9x$

Solution: ►

$$\begin{aligned}(2x+1)(2x+6)-7(x-2)=4(x+1)(x-1)-9x \quad &\implies \quad 4x^2+14x+6-7x+14\\ &\qquad = 4x^2-4-9x\\ &\implies \quad 4x^2+7x+20=4x^2-9x-4\\ &\implies \quad 4x^2+7x-4x^2+9x=-4-20\\ &\implies \quad 16x=-24\\ &\implies \quad x=-\frac{3}{2}.\end{aligned}$$

◄

253 Example Solve for x: $\dfrac{2}{x+1}=\dfrac{3}{x}$.

Solution: ► *Cross-multiplying,*

$$\frac{2}{x+1}=\frac{3}{x} \implies 2x=3(x+1) \implies 2x=3x+3 \implies 2x-3x=3 \implies -x=3 \implies x=-3.$$

◄

Homework

Problem 6.2.1 *Solve for x:* $2(3x-4)-4(2-3x)=1$.

Problem 6.2.2 *Solve for x:* $x-\dfrac{x}{2}-\dfrac{x}{3}=1$.

Problem 6.2.3 *Solve for x:* $\dfrac{x-2}{2}=\dfrac{3-x}{3}$.

Problem 6.2.4 *Solve for x:* $\dfrac{x}{a}-1=2$.

Problem 6.2.5 *Solve for x:* $\dfrac{ax}{b}=a$.

Problem 6.2.6 *Solve for x:* $ax+b=c$.

Problem 6.2.7 *Solve for* x: $\dfrac{x+a}{2}=2x+1.$

Problem 6.2.8 *Solve for* x: $\dfrac{x+1}{2}-\dfrac{x+2}{3}=\dfrac{x-1}{4}.$

Problem 6.2.9 *Solve for* x: $\dfrac{a}{x}=b.$

Problem 6.2.10 *Solve for* x: $\dfrac{ab}{cx}=d.$

Problem 6.2.11 *Solve for* x: $\dfrac{3}{x-2}=1.$

Problem 6.2.12 *Solve for* x: $\dfrac{3}{x-2}=\dfrac{2}{x+3}.$

Problem 6.2.13 *Solve for* x: $2(3-x)=3x-4$

Problem 6.2.14 *Solve for* x: $(2-x)(x+3)=-x(x-4)$

Problem 6.2.15 *Solve for* x: $(x-a)b=(b-x)a.$

Problem 6.2.16 *Solve for* x: $\dfrac{x-3}{3}-\dfrac{x-2}{2}=6.$

Problem 6.2.17 *Write the infinitely repeating decimal* $0.\overline{123}=0.123123123\ldots$ *as the quotient of two positive integers.*

6.3 Word Problems

"When I use a word," Humpty Dumpty said in rather a scornful tone, "it means just what I choose it to mean - neither more nor less." *-Lewis CARROLL*

254 Example Find two numbers whose sum is 28, and whose difference is 4.

Solution: ► *Let* x *be one of the numbers, then the other number is* $28-x$. *Then we have*

$$\begin{aligned} x-(28-x)=4 &\implies 2x-28=4 \\ &\implies x=16. \end{aligned}$$

The numbers are $x=16$ *and* $28-x=28-16=12$. ◄

255 Example Divide $\$47$ between Peter, Paul, and Mary, so that Peter may have $\$10$ more than Paul, and Paul $\$8$ more than Mary.

Solution: ► *Let* p *be Paul's amount in dollars. Then Peter has* $p+10$ *dollars and Mary has* $p-8$ *dollars. Then we have*

$$\begin{aligned} p+(p+10)+(p-8)=47 &\implies 3p+2=47 \\ &\implies 3p=45 \\ &\implies p=15. \end{aligned}$$

Thus Paul has $\$15$, *Peter has* $\$25$ *and Mary has* $\$7$. ◄

256 Example The sum of three consecutive odd integers is 609. Find the numbers.

Solution: ► *Let the numbers be* $x-2, x, x+2$. *Then*

$$\begin{aligned} (x-2)+x+(x+2)=609 &\implies 3x=609 \\ &\implies x=203 \end{aligned}$$

The numbers are $x-2=201$, $x=203$, *and* $x+2=205$. ◄

257 Example A glass of beer costs 40 cents more than a loaf of bread but 50 cents less than a glass of wine. If the cost of the three items, in cents, is 730, what is the price of each item, in cents?

Solution: ► *Let* b *be the price of a glass of beer in cents. Then bread costs* $b-40$ *cents and wine costs* $b+50$ *cents. This gives*

$$b+(b-40)+(b+50)=730 \implies 3b+10=730 \implies b=240.$$

Thus beer costs 240 *cents, bread costs* 200 *cents and wine costs* 290 *cents.* ◄

258 Example Currently, the age of a father is four times the age of his son, but in 24 years from now it will only be double. Find their ages.

Solution: ► *Let* s *be the current age of the son. Then the current age of the father is* $4s$. *In* 24 *years the son will be* $s+24$ *and the father will be* $4s+24$ *and we will have*

$$4s+24=2(s+24).$$

This give

$$4s+24=2s+48 \implies 4s-2s=48-24 \implies s=12.$$

Thus the son is currently 12 *years-old and the father is currently* 48 *years-old.* ◄

Homework

Problem 6.3.1 *Six times a number increased by* 11 *is equal to* 65. *Find it.*

Problem 6.3.2 *Find a number which when multiplied by* 11 *and then diminished by* 18 *is equal to* 15.

Problem 6.3.3 *If* 3 *is added to a number, and the sum multiplied by* 12, *the result is* 84. *Find the number.*

Problem 6.3.4 *The sum of eleven consecutive integers is* 2002. *Find them.*

Problem 6.3.5 *One number exceeds another by* 3, *and their sum is* 27; *find them.*

Problem 6.3.6 *Find two numbers whose sum is* 19, *such that one shall exceed twice the other by* 1.

Problem 6.3.7 *Split \$380 among Peter, Paul and Mary, so that Paul has \$30 more than Peter, and Mary has \$20 more than Paul.*

Problem 6.3.8 *How much pure water should be added to* 100 *gram of* 60% *acid solution to make a* 20% *acid solution?*

Problem 6.3.9 *Jane's age is twice Bob's age increased by* 3. *Bill's age is Bob's age decreased by* 4. *If the sum of their ages is* 27, *how old is Bill?*

Problem 6.3.10 *The average of six numbers is* 4. *A seventh number is added and the new average increases to* 5. *What was the seventh number?*

Problem 6.3.11 *Bill currently has five times as much money as Bob. If he gives \$20 to Bob, then Bill will only have four times as much. Find their current amounts.*

Problem 6.3.12 *Find a number so that six sevenths of it exceed four fifths of it by* 2.

Problem 6.3.13 *The difference between two numbers is* 8. *If we add* 2 *to the largest we obtain* 3 *times the smaller one. Find the numbers.*

Problem 6.3.14 *Find two numbers whose difference is* 10, *and whose sum equals twice their difference.*

Problem 6.3.15 *I bought a certain amount of avocados at four for \$2; I kept a fifth of them, and then sold the rest at three for \$2. If I made a profit of \$2, how many avocados did I originally buy?*

Problem 6.3.16 *Find a number whose fourth, sixth, and eighth add up to* 13.

Problem 6.3.17 *A fifth of the larger of two consecutive integers exceeds a seventh of the smaller by* 3*. Find the integers.*

Problem 6.3.18 *I bought a certain number of oranges at three for a dollar and five sixths of that number at four for a dollar. If I sold all my oranges at sixteen for six dollars, I would make a profit of three and a half dollars. How many oranges did I buy?*

Problem 6.3.19 *A piece of equipment is bought by a factory. During its first year, the equipment depreciates a fifth of its original price. During its second year, it depreciates a sixth of its new value. If its current value is $56, 000, what was its original price?*

Problem 6.3.20 *The difference of the squares of two consecutive integers is* 665*. Find the integers.*

Problem 6.3.21 *In how many different ways can one change* 50 *cents using nickels, dimes or quarters?*

Problem 6.3.22 *Five burglars stole a purse with gold coins. The five burglars took each different amounts according to their meanness, with the meanest among the five taking the largest amount of coins and the meekest of the five taking the least amount of coins. Unfair sharing caused a fight that was brought to an end by an arbitrator. He ordered that the meanest burglar should double the shares of the other four burglars below him. Once this was accomplished, the second meanest burglar should double the shares of the other three burglars below him. Once this was accomplished, the third meanest burglar should double the shares of the two burglars below him. Once this was accomplished, the fourth meanest burglar should double the shares of the meekest burglar. After this procedure was terminated, each burglar received the same amount of money. How many coins were in a purse if the meanest of the burglars took* 240 *coins initially?*

7 Quadratic Equations in One Variable

7.1 Quadratic Equations

259 Definition A *quadratic equation* in the unknown x is an equation of the form

$$ax^2 + bx + c = 0,$$

where $a \neq 0$, b, c are real numbers.

We will circumscribe ourselves to the study of quadratic trinomials $ax^2 + bx + c$ that are amenable to the factorisations studied earlier. The crucial observation here is that if the product of two real numbers is zero, then at least one of them must be zero. In symbols,

$$ab = 0 \implies a = 0 \quad \text{or} \quad b = 0.$$

260 Example Solve the quadratic equation $x^2 - x = 0$ for x.

Solution: ► *We first factor the expression* $x^2 - x = x(x-1)$. *Now,* $x(x-1) = 0$ *is the product of two real numbers giving* 0, *hence either* $x = 0$ *or* $x - 1 = 0$. *Thus either* $x = 0$ *or* $x = 1$. *One can easily verify now that* $0^2 - 0 = 0$ *and* $1^2 - 1 = 0$. ◄

261 Example Solve the quadratic equation $x^2 + 2x - 3 = 0$ for x.

Solution: ► *Factoring,*

$$\begin{aligned} x^2 + 2x - 3 = 0 &\implies (x+3)(x-1) = 0 \\ &\implies x + 3 = 0 \quad \text{or} \quad x - 1 = 0 \\ &\implies x = -3 \quad \text{or} \quad x = 1. \end{aligned}$$

◄

262 Example Solve the quadratic equation $8x^2 - 2x = 15$ for x.

Solution: ► *First, transform the equation into* $8x^2 - 2x - 15 = 0$. *Now, factoring,*

$$\begin{aligned} 8x^2 - 2x - 15 &\implies (2x-3)(4x+5) = 0 \\ &\implies 2x - 3 = 0 \quad \text{or} \quad 4x + 5 = 0 \\ &\implies x = \frac{3}{2} \quad \text{or} \quad x = -\frac{5}{4}. \end{aligned}$$

◄

Some equations can be reduced to quadratic equations after proper massaging.

263 Example Solve for x: $\frac{x-4}{3} = \frac{4}{x}$.

Solution: ► *Cross-multiplying,*

$$\begin{aligned}\frac{x-4}{3} = \frac{4}{x} &\implies x(x-4) = 12\\ &\implies x^2 - 4x - 12 = 0\\ &\implies (x-6)(x+2) = 0\\ &\implies x = 6 \quad \text{or} \quad x = -2.\end{aligned}$$

◄

The argument that a product of real numbers is zero if and only if at least one of its factors can be used with equations of higher degree.

264 Example Solve for x: $x^3 - 4x = 0$.

Solution: ► *Factoring a common factor and then a difference of squares,*

$$\begin{aligned}x^3 - 4x = 0 &\implies x(x^2 - 4) = 0\\ &\implies x(x-2)(x+2) = 0\\ &\implies x = 0 \quad \text{or} \quad x = +2 \quad \text{or} \quad x = -2.\end{aligned}$$

◄

Homework

Problem 7.1.1 *Solve for x:* $x^2 - 4 = 0$.

Problem 7.1.2 *Solve for x:* $x^2 - x - 6 = 0$.

Problem 7.1.3 *Solve for x:* $x^2 + x - 6 = 0$.

Problem 7.1.4 *Solve for x:* $x^2 - 4x = 5$.

Problem 7.1.5 *Solve for x:* $x^2 = 1$

Problem 7.1.6 *If eggs had cost x cents less per dozen, it would have cost 3 cents less for $x+3$ eggs than if they had cost x cents more per dozen. What is x?*

Problem 7.1.7 *A group of people rents a bus for an excursion, at the price of \$2,300, with each person paying an equal share. The day of the excursion, it is found that six participants do not shew up, which increases the share of those present by \$7.50. What was the original number of people renting the bus?*

Problem 7.1.8 *Let*

$$x = 6 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{\vdots}}}}},$$

where there are an infinite number of fractions. Prove that $x^2 - 6x - 1 = 0$.

Problem 7.1.9 *Assume that there is a positive real number x such that*

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2,$$

where there is an infinite number of x's. What is the value of x?

Part V

Inequalities

8 Linear Inequalities

8.1 Intervals

Let us start this chapter by recalling some properties of the order symbols

$<$ (is less than),

$>$ (is greater than),

$\leq$ (is less than or equal to) and,

$\geq$ (is greater than or equal to).

We will simply state the useful properties for the symbol $<$, as the same properties hold for the other three symbols *mutatis mutandis*.

265 Axiom Let a, b, c be real numbers. Then

1. If $a < b$ and $b < c$, then $a < c$. That is, inequalities are *transitive*.
2. If $a < b$ and $c < d$, then $a + c < b + d$, that is, addition preserves the inequality sense.
3. If $a < b$ and $c > 0$, then $ac < bc$, that is, multiplication by a positive factor preserves the inequality sense.
4. If $a < b$ and $c < 0$, then $ac > bc$, that is, multiplication by a negative factor reverses the inequality sense.

266 Definition A *non-degenerate interval*[1] *interval* for short, is a set I of real numbers with the following property: for any two different elements x, y in I, any number a in between x and y also belongs to I.

From the definition we see that non-degenerate intervals have infinitely many elements. For example consider the interval in figure 8.12. Every number between -2 and 1 is in this interval, so in particular, -1.5 belongs to it, so does -1.51, -1.501, -1.5001, etc.

Some examples of intervals are shewn in figures 8.1 through 8.12. If we define the *length* of the interval to be the difference between its final point and its initial point, then intervals can be finite, like those in figures 8.9 through 8.12 or infinite, like those in figures 8.1 through 8.8. In either case, intervals contain an infinite number of elements.

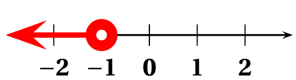


Figure 8.1: $x < -1$

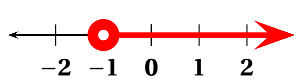


Figure 8.2: $x > -1$

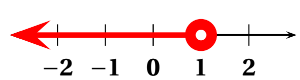


Figure 8.3: $x < 1$

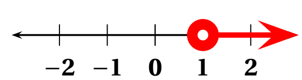


Figure 8.4: $x > 1$

[1] A *degenerate* interval is one consisting of no points or exactly one point. We will not consider such cases in these notes.

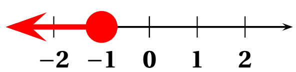


Figure 8.5: $x \leq -1$

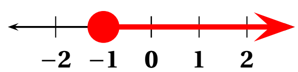


Figure 8.6: $x \geq -1$

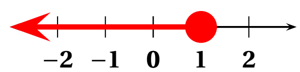


Figure 8.7: $x \leq 1$

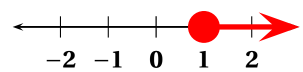


Figure 8.8: $x \geq 1$

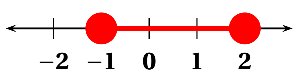


Figure 8.9: $-1 \leq x \leq 2$

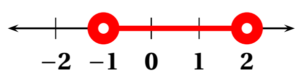


Figure 8.10: $-1 < x < 2$

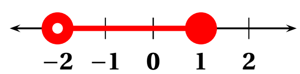


Figure 8.11: $-2 < x \leq 1$

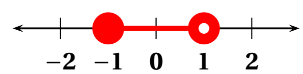


Figure 8.12: $-1 \leq x < 1$

8.2 One-Variable Linear Inequalities

We are now interested in solving linear inequalities in one variable. The procedure will closely resemble to that for solving linear equations in one variables, with the caveat that if we ever divide or multiply by a negative quantity, then the sense of the inequality changes.

267 Example Solve the inequality

$$2x - 3 < -13$$

and graph its solution set.

Solution: ► *Observe that*

$$2x - 3 < -13 \implies 2x < -13 + 3 \implies 2x < -10.$$

The next stage would be to divide by 2*, which is a positive quantity. The sense of the inequality is retained, and we gather,*

$$2x < -10 \implies x < \frac{-10}{2} \implies x < -5.$$

The solution set appears in figure 8.13. ◄

268 Example Solve the inequality

$$-2x - 3 \leq -13$$

and graph its solution set.

Solution: ► *Observe that*

$$-2x - 3 \leq -13 \implies -2x \leq -13 + 3 \implies -2x \leq -10.$$

The next stage would be to divide by -2*, which is a negative quantity. The sense of the inequality is reversed, and we gather,*

$$-2x \leq -10 \implies x \geq \frac{-10}{-2} \implies x \geq -5.$$

The solution set appears in figure 8.14. ◄

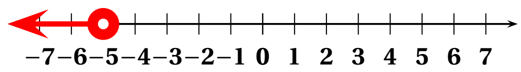


Figure 8.13: Example 267. $x < -5$.

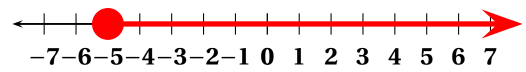


Figure 8.14: Example 268. $x \geq -5$.

Homework

Problem 8.2.1 *Find the set of all real numbers x satisfying the inequality*

$$\frac{x}{3} - \frac{x-2}{4} < 2 + x,$$

and graph its solution set.

Problem 8.2.2 *Find the set of all real numbers x satisfying the inequality*

$$4(x+1) \leq x - 5,$$

and graph its solution set.

Problem 8.2.3 *Find the set of all real numbers x satisfying the inequality*

$$\frac{x-2}{3} - \frac{1-x}{2} > 0,$$

and graph its solution set.

A Old Exam Questions

Multiple Choice Problems

Chapter II Problems

1. Which of the following expressions best represents: “the excess of a number over 3”?
(A) $\frac{x}{3}$ (B) $x-3$ (C) $3x$ (D) $\frac{3}{x}$ (E) none of these

2. Which of the following expressions best represents: “the sum of a number plus thrice another is being diminished by 3”?
(A) $\frac{x+3y}{3}$ (B) $x+3y-3$ (C) $x+3(y-3)$ (D) $x+2y-3$ (E) none of these

3. The algebraic expression $x^2-\frac{1}{x}$ can be translated as:
(A) “The square of a number is increased by the reciprocal of the number.”
(B) “The square of a number is reduced by the reciprocal of the number.”
(C) “Twice a number is increased by the reciprocal of the number.”
(D) “Twice a number is decreased by the reciprocal of the number.”
(E) none of these

4. $\frac{5^3-(-3)^3}{5^2-(-3)^2}=$
(A) $\frac{49}{2}$ (B) $\frac{3}{2}$ (C) 6 (D) $\frac{19}{2}$ (E) none of these

5. $(-100)\div(10)\div(-10)=$
(A) 1 (B) 10 (C) 100 (D) -1 (E) none of these

6. $\frac{3}{4}-\frac{11}{12}=$
(A) $\frac{1}{6}$ (B) $-\frac{1}{6}$ (C) -1 (D) $\frac{11}{16}$ (E) none of these

7. $\frac{3}{14}\div\frac{21}{8}=$
(A) $\frac{9}{16}$ (B) $\frac{4}{49}$ (C) $\frac{24}{22}$ (D) $\frac{49}{4}$ (E) none of these

8. $\frac{(1+2\cdot 3)^2-9}{(2\cdot 5-2)(2^2+1)}=$
(A) 1 (B) -1 (C) $\frac{29}{30}$ (D) $\frac{2}{3}$ (E) none of these

9. $1024\div 512\times 2=$
(A) 1 (B) 2 (C) 4 (D) 8 (E) none of these

10. $\frac{5^3-3^3}{5^2+15+3^2}=$
(A) 1 (B) $\frac{1}{2}$ (C) 2 (D) $\frac{8}{49}$ (E) none of these

11. $1-(-2)^3 =$

(A) -7 (B) 9 (C) -9 (D) 7 (E) none of these

12. You have t \$10 and w \$20 bills. How much money, in dollars, do you have?

(A) $t+w$ (B) $10t+20w$ (C) $200tw$ (D) $30+t+w$ (E) none of these

13. You have t \$10 and w \$ 20 bills. How many bank notes (bills) do you have?

(A) $t+w$ (B) $10t+20w$ (C) $200tw$ (D) $30+t+w$ (E) none of these

14. Consider the arithmetic progression $3, 14, 25, 36, \ldots$, where 3 is on the first position, 14 is on the second, etc. Which number occupies the 101st position?

(A) 101 (B) 1092 (C) 1103 (D) 1114 (E) none of these

15. $\frac{1}{2}+\frac{2}{3}-\frac{6}{7}=$

(A) $\frac{13}{42}$ (B) -3 (C) $-\frac{1}{14}$ (D) $-\frac{53}{60}$ (E) none of these

16. $-5^2+4-9-(-7)+(-9+5\cdot 3)^2=$

(A) -13 (B) -1 (C) 63 (D) 13 (E) none of these

17. Perform the calculation: $1-2+3-4+5-6+7-8+9-10$.

(A) 5 (B) 0 (C) -5 (D) 1 (E) none of these

18. Perform the calculation: $\frac{\frac{3}{4}}{\frac{5}{6}}$.

(A) $\frac{9}{10}$ (B) $\frac{10}{9}$ (C) $\frac{19}{12}$ (D) $\frac{5}{8}$ (E) none of these

19. Perform the calculation: $\frac{3}{4}+\frac{5}{6}$.

(A) $\frac{9}{10}$ (B) $\frac{10}{9}$ (C) $\frac{19}{12}$ (D) $\frac{5}{8}$ (E) none of these

Chapter III Problems

20. Collect like terms: $(x+y+2z)-(x-2y+z)$.

(A) $3y+z$ (B) $2x-y+3z$ (C) $3y+3z$ (D) $3y-z$ (E) none of these

21. Which of the following are like terms?

(A) x^6 and $6x$ (B) $2x^2$ and $-x^2$ (C) j^2kl and jk^2l (D) c and c^2 (E) none of these

22. $2(a-2b)-3(b-2a)=$

(A) $3a-b$ (B) $-5a-10b$ (C) $8a-7b$ (D) $8a+7b$ (E) none of these

23. $\left(\frac{20x-10}{5}\right)-\left(\frac{6-9x}{3}\right)=$

Ⓐ $7x$ Ⓑ $7x-4$ Ⓒ x Ⓓ $x-4$ Ⓔ none of these

24. Collect like terms: $2(a-2b)-(2a-b)$

Ⓐ $-3b$ Ⓑ $-b$ Ⓒ $4a-5b$ Ⓓ $4a-3b$ Ⓔ none of these

25. Collect like terms: $-2(x^2-1)+4(2-x-x^2)$

Ⓐ $-6x^2-4x+10$ Ⓑ $6x^2+4x+10$ Ⓒ $-6x^2+4x+10$ Ⓓ $-2x^2-4x+10$ Ⓔ none of these

26. Collect like terms: $(-2a+3b-c)+(5a-8b+c)=$

Ⓐ $7a-11b-2c$ Ⓑ $3a-5b-2c$ Ⓒ $-7a+11b-2c$ Ⓓ $3a-5b$ Ⓔ none of these

27. $-a+2b+3a-4b=$

Ⓐ $-2a+2b$ Ⓑ $2a-2b$ Ⓒ $-4a-6b$ Ⓓ $2a+2b$ Ⓔ none of these

28. $\frac{a}{3}-\frac{a}{5}=$

Ⓐ $\frac{a}{15}$ Ⓑ $-\frac{a}{2}$ Ⓒ $\frac{2a}{15}$ Ⓓ $\frac{a}{8}$ Ⓔ none of these

29. $\frac{a}{3}+\frac{a^2}{3}+\frac{a}{4}+\frac{3a^2}{4}=$

Ⓐ $\frac{4a}{7}+\frac{4a^2}{7}$ Ⓑ $\frac{7a}{12}+\frac{13a^2}{12}$ Ⓒ $\frac{a}{12}+\frac{a^2}{12}$ Ⓓ $\frac{a^6}{48}$ Ⓔ none of these

30. $(-a+2b-3c)-(-a+2b+3c)=$

Ⓐ $-6c$ Ⓑ $4b-6c$ Ⓒ $2a+4b+6c$ Ⓓ $2a-6c$ Ⓔ none of these

31. Collect like terms: $\frac{3x-6}{3}+x-1$.

Ⓐ $2x-2$ Ⓑ $2x-3$ Ⓒ $x-3$ Ⓓ $2A-B-2$ Ⓔ none of these

32. How many of the following expressions are equivalent to $5+x^2y$?

$$I: x^2y+5, \qquad II: 5+yx^2, \qquad III: 5+xy^2, \qquad IV: yx^2+5.$$

Ⓐ exactly one Ⓑ exactly two Ⓒ exactly three Ⓓ all four Ⓔ none

33. Collect like terms: $\left(\frac{-35x+20}{5}\right)+\left(\frac{12-24x}{6}\right)$.

Ⓐ $-11x+6$ Ⓑ $3x+2$ Ⓒ $3x+6$ Ⓓ $-3x+2$ Ⓔ none of these

Chapter IV Problems

34. $\frac{3^4}{3^3}=$

Ⓐ 3 Ⓑ 3^{12} Ⓒ $\frac{1}{3}$ Ⓓ 3^7 Ⓔ none of these

35. $\frac{3^3}{3^4} =$

(A) 3 (B) 3^{12} (C) $\frac{1}{3}$ (D) 3^7 (E) none of these

36. $(3^3)^4 =$

(A) 3 (B) 3^{12} (C) $\frac{1}{3}$ (D) 3^7 (E) none of these

37. $3^3 \cdot 3^4 =$

(A) 3 (B) 3^{12} (C) $\frac{1}{3}$ (D) 3^7 (E) none of these

38. $\frac{3^3}{3^{-4}} =$

(A) 3 (B) 3^{12} (C) $\frac{1}{3}$ (D) 3^7 (E) none of these

39. $\frac{1111^5 + 1111^5 + 1111^5 + 1111^5}{1111^4 + 1111^4} =$

(A) 1111 (B) 2222 (C) 4444 (D) $\frac{1}{2222}$ (E) none of these

40. Simplify and write with positive exponents only: $(ab^2)^2(a^3b^{-4})$.

(A) a^5 (B) a^5b^8 (C) $\frac{b^8}{a}$ (D) a^6 (E) none of these

41. Simplify and write with positive exponents only: $(ab^2)^2 \div (a^3b^{-4})$.

(A) a^5 (B) a^5b^8 (C) $\frac{b^8}{a}$ (D) a^6 (E) none of these

42. Perform the calculation: $\frac{2^{-3}}{3^{-2}} - \frac{3^2}{2^2}$.

(A) $\frac{9}{8}$ (B) $\frac{9}{4}$ (C) $-\frac{9}{8}$ (D) $-\frac{9}{4}$ (E) none of these

43. Perform the calculation: $\frac{1}{2^{-5}} - 5^{-2}$.

(A) 0 (B) 20 (C) $\frac{799}{25}$ (D) $\frac{2}{25}$ (E) none of these

44. Multiply and write with positive exponents only: $(x^4y^7)(x^{-2}y^{-8})$.

(A) x^6y^{15} (B) x^2y (C) $\frac{x^2}{y}$ (D) $\frac{y}{x^2}$ (E) none of these

45. Multiply and collect like terms: $(x-2)(x^2+2x+4)$.

(A) $x^3 - 8$ (B) $x^3 - 4x^2 - 4x - 8$ (C) $x^3 + 8$ (D) $x^3 - 4x^2 - 8$ (E) none of these

46. Divide and write with positive exponents only: $\dfrac{x^4y^7}{x^{-2}y^{-8}}$.

(A) x^6y^{15} (B) x^2y (C) $\dfrac{x^2}{y}$ (D) $\dfrac{y}{x^2}$ (E) none of these

47. $\dfrac{2^{-3}}{3^{-2}2^3} =$

(A) $\dfrac{9}{64}$ (B) 9 (C) $\dfrac{64}{9}$ (D) $\dfrac{1}{9}$ (E) none of these

48. $(x^{-1}y^{-2}z^3)^{-3} =$

(A) $x^4y^5z^0$ (B) $\dfrac{z^9}{x^3y^6}$ (C) $\dfrac{x^3y^6}{z^9}$ (D) $\dfrac{x^3z^9}{y^6}$ (E) none of these

49. $(x^4y^3z^2)(x^2y^4z^2) =$

(A) $\dfrac{x^2}{y}$ (B) x^2y (C) $\dfrac{y}{x^2}$ (D) $x^6y^7z^4$ (E) none of these

50. Multiply and collect like terms: $(x+2)(x-3) =$

(A) x^2-6 (B) x^2-x-6 (C) x^2+x-6 (D) x^2-5x-6 (E) none of these

51. Multiply and collect like terms: $(2x-1)(2x+1)-2(2x-1) =$

(A) $4x^2-4x$ (B) $4x^2-4x+1$ (C) $4x^2-4x-1$ (D) 1 (E) none of these

52. Multiply and collect like terms: $(a+2)(a-2)-(a-1)(a+1) =$

(A) $2a^2-3$ (B) -3 (C) -5 (D) $2a^2-5$ (E) none of these

53. Multiply and collect like terms: $(a+2)^2+(a-2)^2 =$

(A) $2a^2+8$ (B) $8a$ (C) $2a^2+4$ (D) $2a^2-4a+8$ (E) none of these

54. Multiply and collect like terms: $(a+2)^2-(a-2)^2 =$

(A) $2a^2+8$ (B) $8a$ (C) $2a^2+4$ (D) $2a^2-4a+8$ (E) none of these

55. Expand and collect like terms: $(x+2)(x^2-4x+1)$

(A) x^3-6x^2+9x-2 (B) x^3-2x^2-7x+2 (C) x^3+2x^2-9x+2 (D) x^4-1 (E) none of these

Chapter V Problems

56. Perform the division: $(x^3-8x^2)\div(-x)$.

(A) x^2-8x (B) x^2+8x (C) $-x^2+8x$ (D) $-x^2-8x$ (E) none of these

57. Factor: $2x^2-3x$.

(A) $x(2x-3)$ (B) $2x(x-3)$ (C) $x^2(2x-3)$ (D) $x(2x+3)$ (E) none of these

58. Factor: x^2+3x+2.

(A) $(x+1)(x-2)$ (B) $(x+1)(x+2)$ (C) $(x-1)(x+2)$ (D) $(x-1)(x-2)$ (E) none of these

59. Factor: x^2-3x+2.

(A) $(x+1)(x-2)$ (B) $(x+1)(x+2)$ (C) $(x-1)(x+2)$ (D) $(x-1)(x-2)$ (E) none of these

60. Factor: x^2-x-2.

(A) $(x+1)(x-2)$ (B) $(x+1)(x+2)$ (C) $(x-1)(x+2)$ (D) $(x-1)(x-2)$ (E) none of these

61. Perform the division: $(2x^3 - 3x^2 - 4x - 1) \div (2x+1)$.

(A) x^2+2x+1 (B) x^2-2x-1 (C) x^2-2x+1 (D) x^2+2x-1 (E) none of these

62. $\dfrac{6x^2+2x}{2x} =$

(A) $3x+1$ (B) $3x$ (C) $3x+2$ (D) $6x^2$ (E) none of these

63. $\dfrac{(6x^2)(2x)}{2x} =$

(A) $3x+1$ (B) $3x$ (C) $3x+2$ (D) $6x^2$ (E) none of these

64. $(9a^3b^3 - 6a^2b^4 + 3a^2b^3) \div (3a^2b^3)$

(A) $3a-2b$ (B) $3a-2b+1$ (C) $-6ab$ (D) $3a^5b^6 - 2a^4b^7 + a^4b^6$ (E) none of these

65. $(x^2+5x-14) \div (x-2) =$

(A) $x-7$ (B) $x+7$ (C) $x+2$ (D) $x+12$ (E) none of these

66. $(6x^3-5x^2+7x-2) \div (3x-1) =$

(A) $2x^2+x+2$ (B) $2x^2-x-2$ (C) $2x^2-x+2$ (D) $2x^2+x-2$ (E) none of these

67. $(x^3+27) \div (x+3) =$

(A) x^2-9x+9 (B) x^2+9x+9 (C) x^2+3x+9 (D) x^2-3x+9 (E) none of these

68. $\dfrac{x^9+x^6}{x^3} =$

(A) x^3+x^2 (B) $3x^3+2x^2$ (C) x^6+x^3 (D) x^{12} (E) none of these

69. $\dfrac{(x^9)(x^6)}{x^3} =$

(A) x^3+x^2 (B) $3x^3+2x^2$ (C) x^6+x^3 (D) x^{12} (E) none of these

70. Factor: $xy^2 - x^2y$.

(A) $xy(x-y)$ (B) $xy(y-x)$ (C) $x^2y^2(x-y)$ (D) $xy(x+y)$ (E) none of these

71. Factor: x^2-x-12.

(A) $(x-3)(x+4)$ (B) $(x-6)(x+2)$ (C) $(x+3)(x-4)$ (D) $(x-12)(x+1)$ (E) none of these

72. Factor: $x^2-12x+27$.

(A) $(x-3)(x-9)$ (B) $(x+3)(x+9)$ (C) $(x-3)(x+9)$ (D) $(x-1)(x-27)$ (E) none of these

73. Factor: $x^3 - 12x^2 + 27x$.

(A) $x(x-3)(x-9)$ (B) $x(x+3)(x+9)$ (C) $x(x-3)(x+9)$ (D) $x(x-1)(x-27)$ (E) none of these

74. Factor: $a^2 + 16a + 64$.

(A) $(a+8)^2$ (B) $(a-8)(a+8)$ (C) $(a-8)^2$ (D) $(a+16)(a+4)$ (E) none of these

75. Factor: $a^3 + 16a^2 + 64a$.

(A) $a(a+8)^2$ (B) $a(a-8)(a+8)$ (C) $a(a-8)^2$ (D) $a(a+16)(a+4)$ (E) none of these

76. Factor: $a^2 - 64$.

(A) $(a+8)^2$ (B) $(a-8)(a+8)$ (C) $(a-8)^2$ (D) $(a+16)(a+4)$ (E) none of these

77. Divide: $(6x^3 + x^2 - 11x - 6) \div (3x+2)$

(A) $2x^2 - x + 3$ (B) $2x^2 - x - 3$ (C) $2x^2 + x - 3$ (D) $2x^2 + x + 3$ (E) none of these

78. Reduce the fraction: $\dfrac{x^3 - x}{x^2 - 1}$

(A) x (B) $\dfrac{1}{x}$ (C) $x+1$ (D) $x-1$, (E) none of these

79. Add: $\dfrac{2}{x-2} + \dfrac{3}{x+3}$

(A) $\dfrac{5x}{x^2+x+6}$ (B) $\dfrac{5x}{x^2+x-6}$ (C) $\dfrac{2x+1}{x^2+x-6}$ (D) $\dfrac{12-x}{x^2+x-6}$ (E) none of these

80. Reduce the fraction: $\dfrac{a^2 - a}{a^2 + a}$.

(A) $\dfrac{a^2-1}{a^2+1}$ (B) -1 (C) $\dfrac{a-1}{a+1}$ (D) $\dfrac{a}{a-1}$ (E) none of these

81. Reduce the fraction: $\dfrac{a^2}{a^2 + a}$.

(A) $\dfrac{a}{a+1}$ (B) 1 (C) $\dfrac{1}{a+1}$ (D) $\dfrac{a}{a-1}$ (E) none of these

82. Add the fractions: $\dfrac{2}{x-3} + \dfrac{3}{x+2}$.

(A) $\dfrac{5x-5}{x^2-x-6}$ (B) $\dfrac{13+x}{x^2+x-6}$ (C) $\dfrac{13-x}{x^2-x-6}$ (D) $\dfrac{5x+5}{x^2+x-6}$ (E) none of these

83. Perform the subtraction: $\dfrac{2}{x-3} - \dfrac{3}{x+2}$.

(A) $\dfrac{5x-5}{x^2-x-6}$ (B) $\dfrac{13+x}{x^2+x-6}$ (C) $\dfrac{13-x}{x^2-x-6}$ (D) $\dfrac{5x+5}{x^2+x-6}$ (E) none of these

Chapter VI Problems

84. Solve the following equation for x: $ax = 2$.

(A) $x = \dfrac{2}{a}$ (B) $x = 2 - a$ (C) $x = \dfrac{a}{2}$ (D) $x = a$ (E) none of these

85. Solve the following equation for x: $\frac{ax}{b} = 2a$.

(A) $x = b$ (B) $x = \frac{2a}{b}$ (C) $x = 2b$ (D) $x = \frac{2}{b}$ (E) none of these

86. Solve the following equation for x: $ax + 2b = 5b$.

(A) $x = 7ab$ (B) $x = \frac{7b}{a}$ (C) $x = 3ab$ (D) $x = \frac{3b}{a}$ (E) none of these

87. Solve the following equation for x: $2x - 1 = x$.

(A) $x = 1$ (B) $x = -1$ (C) $x = \frac{1}{2}$ (D) $x = \frac{1}{3}$ (E) none of these

88. Solve the following equation for x: $1 - x = 2$.

(A) $x = 3$ (B) $x = -1$ (C) $x = -3$ (D) $x = \frac{1}{3}$ (E) none of these

89. Solve the following equation for x: $2x - 1 = 1 - 3x$.

(A) $x = \frac{5}{2}$ (B) $x = \frac{2}{5}$ (C) $x = 0$ (D) $x = -\frac{2}{5}$ (E) none of these

90. Solve the following equation for x: $2(x-1) = 3(1-3x)$.

(A) $x = \frac{5}{11}$ (B) $x = \frac{1}{7}$ (C) $x = \frac{5}{7}$ (D) $x = -\frac{11}{5}$ (E) none of these

91. Solve the following equation for x: $\frac{x-1}{3} = \frac{1-3x}{2}$.

(A) $x = \frac{5}{11}$ (B) $x = \frac{1}{7}$ (C) $x = \frac{5}{7}$ (D) $x = -\frac{11}{5}$ (E) none of these

92. Solve the following equation for x: $ax - 2b = 5b$.

(A) $x = 7ab$ (B) $x = \frac{7b}{a}$ (C) $x = 3ab$ (D) $x = -\frac{3b}{a}$ (E) none of these

93. Solve the following equation for x: $\frac{x}{2} - \frac{3}{4} = \frac{9}{8}$.

(A) $x = \frac{3}{4}$ (B) $x = \frac{1}{8}$ (C) $x = \frac{15}{8}$ (D) $x = \frac{15}{4}$ (E) none of these

94. Adam, Eve, Snake, and Jake share **\$240** between them. Adam has twice as much as Eve, Eve has twice as much as Snake, and Snake has twice as much as Jake. How much money does Eve have?

(A) **\$15** (B) **\$60** (C) **\$64** (D) **\$128** (E) none of these

95. Peter, Paul, and Mary share **\$153** between them. Peter has twice as much as Paul, and Mary has thrice as much as Peter. How much money does Peter have?

(A) **\$17** (B) **\$34** (C) **\$51** (D) **\$102** (E) none of these

96. The sum of three consecutive odd integers is **909**. Which one is the last number?

(A) **301** (B) **302** (C) **303** (D) **305** (E) none of these

97. Jane's age is twice Bob's age increased by **3**. Bill's age is Bob's age decreased by **4**. If the sum of their ages is **27**, how old is Bill?

(A) **3** (B) **7** (C) **4** (D) **17** (E) none of these

98. A jar of change contains nickels, dimes, and quarters. There are twice as many dimes as nickels, and twice as many quarters as dimes. If the jar has a total of \$8.75, how many quarters are there?

(A) **5** (B) **7** (C) **14** (D) **28** (E) none of these

99. Currently, the age of a father is four times the age of his son, but in **24** years from now it will only be double. Find the son's current age.

(A) **12** (B) **36** (C) **48** (D) **6** (E) none of these

Chapter VII Problems

100. Solve for x: $2x^2 - 3x = 0$.
(A) $x = 0$ or $x = \frac{3}{2}$ (B) $x = 0$ or $x = \frac{2}{3}$ (C) $x = 0$ or $x = -\frac{3}{2}$ (D) $x = 0$ or $x = -\frac{2}{3}$ (E) none of these

101. Solve for x: $x^2 + 3x + 2 = 0$.
(A) $x = -1$ or $x = 2$ (B) $x = -1$ or $x = -2$ (C) $x = 1$ or $x = -2$ (D) $x = 1$ or $x = 2$ (E) none of these

102. Solve for x: $x^2 - 3x + 2 = 0$.
(A) $x = -1$ or $x = 2$ (B) $x = -1$ or $x = -2$ (C) $x = 1$ or $x = -2$ (D) $x = 1$ or $x = 2$ (E) none of these

103. Solve for x: $x^2 - x - 2 = 0$.
(A) $x = -1$ or $x = 2$ (B) $x = -1$ or $x = -2$ (C) $x = 1$ or $x = -2$ (D) $x = 1$ or $x = 2$ (E) none of these

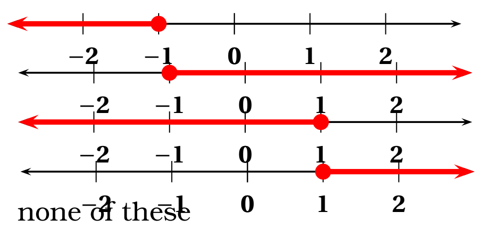
Chapter VIII Problems

104. If $2(x - 1) \geq x + 3$ then
(A) $x \geq 1$ (B) $x \geq 5$ (C) $x \leq 1$ (D) $x \leq 5$ (E) none of these

105. If $-2x > x + 3$ then
(A) $x > -1$ (B) $x > 1$ (C) $x < -1$ (D) $x < 1$ (E) none of these

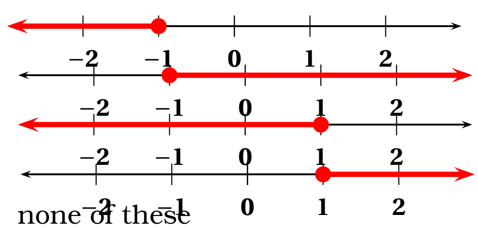
106. Which graph gives the correct solution to $2 - 5x \leq -3$?

(A)
(B)
(C)
(D)
(E) none of these



107. Which graph gives the correct solution to $2 + 5x \leq -3$?

(A)
(B)
(C)
(D)
(E) none of these



True or False Problems.

108. ☐ $(2+3)^{10} = 2^{10} + 3^{10}$.

109. ☐ $(2 \cdot 3)^{10} = 2^{10} \cdot 3^{10}$.

110. ☐ $(2^5)^2 = 2^7$.

111. ☐ It is always true that $x(2x+3) = x(2x)(3)$.

112. ☐ $(1^1 + 2^2 + 3^3 + 4^4 - 5^5)^0 + 4$ equals 5.

In each of the items below, select the choice, or choices, that is—or are—equivalent to the given algebraic expression. Denominators are presumed not to vanish.

113. $-a =$

(A) $a(-1)$ (B) $-1 \cdot a$ (C) $\dfrac{a}{-1}$ (D) $\dfrac{-a}{1}$ (E) $1-2a$

114. $xy - z =$

(A) $z - xy$ (B) $yx - z$ (C) $xy + (-z)$ (D) $x(y-z)$ (E) $-z + xy$

115. $\dfrac{a-b}{2} =$

(A) $a - b \div 2$ (B) $(a-b) \div 2$ (C) $\dfrac{a}{2} - \dfrac{b}{2}$ (D) $\dfrac{-b+a}{2}$ (E) $\dfrac{1}{2}(a-b)$

116. $10x\left(\dfrac{y}{5} + x\right) =$

(A) $\dfrac{xy}{2} + 10x^2$ (B) $2xy + 10x$ (C) $10\left(\dfrac{y}{5} + x\right)x$ (D) $5x \cdot \left(x + \dfrac{y}{5}\right) \cdot 2$ (E) $2xy + 10x^2$

117. $(a-2b)^2 =$

(A) $(a-2b)(a-2b)$ (B) $a^2 - 4b^2$ (C) $a^2 + 4b^2$ (D) $a^2 - 4ab + 4b^2$ (E) $-(2b-a)(a-2b)$

118. $\dfrac{x-y}{y-x} =$

(A) 1 (B) -1 (C) -2 (D) $(x-y)^2$ (E) $\dfrac{y-x}{x-y}$

119. $(2x^2y^2)(3xy^4) =$

(A) $6x^3y^6$ (B) $6(xy^4)^2$ (C) $6(xy^2)^3$ (D) $\dfrac{12x^9y^{12}}{2x^3y^2}$ (E) $\dfrac{12x^6y^7}{2x^3y}$

120. $x\left(x^3-1\right) + 2\left(x^3-1\right) =$

(A) $(x^3-1)x - 2(-x^3+1)$

(B) $2(x^3-1) + x(-x^3+1)$

(C) $(x+2)(x^3-1)$

(D) $x^4 - x + 2x^3 - 2$

(E) $x^4 - x + 1(x^3 - 2)$

B Answers and Hints

1.2.1 The Alexandrian mathematician Diophantus, for the introduction of syncopation (symbols to express abstract quantities and procedures on them) in Algebra.

1.2.2 The word *algebra* is Arabic, but Algebra was studiend long before the Arabs had any prominent rôle in History.

1.2.3 Ossifrage is Latin for *breaking bones*. Algebra is Arabic for *bone setting.*

1.3.1 You need to increase $\$20$, and $\$20$ is 25% of $\$80$, and so you need to increase by 25%.

1.3.2 10¢. The cork costs 10¢ and the bottle 90¢.

1.4.1 St. Augustine is using the word *mathematician* in lieu of the word *astrologer*.

1.4.2 No. When we add an even integer to another even integer the result is an even integer. Thus the sum of five even integers is even, but 25 is odd.

1.4.3 27th floor.

1.4.4 Between 9999 and 10101.

1.4.5 $\boxed{2}75\boxed{3}6-\boxed{2}56\boxed{1}=24\boxed{9}75$.

1.4.6 Observe that the first and second rows, and the second and third columns add up to 8. Thus $a=4$ works.

1.4.7 There are multiple solutions. They can be obtained by permuting the entries of one another. Here are two:

8	1	6
3	5	7
4	9	2

4	9	2
3	5	7
8	1	6

1.4.8 Nine. This problem is from [Toom].

1.4.9 This problem is from [Gard]. At 'tailor.'

1.4.10 We can represent the boy's and girl's amounts by boxes, each box having an equal amount of nuts, and there being two boxes for the boy and one for the girl:

$\boxed{B}\quad\boxed{B}\quad\boxed{G}$

Then clearly, each box must have 8 nuts. The boy has 16 and the girl 8. This problem was taken from [Toom].

2.1.1 $N-20$.

2.1.2 $N+20$.

2.1.3 $3x+10$.

2.1.4 $3(x+10)$.

2.1.5 In knitting s scarves, I need $3s$ balls of wool, and hence, at the end of the day I have $b-3s$ balls of wool. This problem is taken from [Gard].

2.1.6 On the first step you have x. On the second step you have $2x$. On the third step you have $2x+10$. On the four step you have $\frac{2x+10}{2}$. On the fifth step you have $\frac{2x+10}{2}-x$. You are asserting that $\frac{2x+10}{2}-x$ is identically equal to 5.

2.1.7 $4n; 4n+1; 4n+2; 4n+3$, where n is a natural number.

2.1.8 $6n+1$ where $n=0,1,2,3,\ldots$.

2.1.9 First observe that

$$3 = 2\times 1+1,$$

$$5 = 2\times 2+1,$$

$$7 = 2\times 3+1,$$

etc. Then it becomes clear that if the last number on the left is n, then the right hand side is $n(n+1)(2n+1)$ divided by 6. Hence we are asserting that

$$1^2+2^2+3^2+\cdots+(n-1)^2+n^2 = \frac{(n)(n+1)(2n+1)}{6}.$$

2.1.10 The general formula is

$$1+3+\cdots+(2n-1) = n^2.$$

2.1.11 You have $25q+10d$ cents at the beginning of the day, and then you lose $25a+10b$, which leaves you with

$$(25q+10d)-(25a+10b)$$

cents.

2.1.12 The cube is x^3, the square is x^2. The cube reduced by the square is x^3-x^2. In conclusion, what is left after division by 8 is

$$\frac{x^3-x^2}{8}.$$

2.1.13 The jacket costs $3h$ and the trousers cost $3h-8$. Hence he paid in total

$$h+3h+3h-8 = 7h-8.$$

2.1.14 The profit is that amount over the real price. If the real price is Γ, then $\Gamma+b$ is the real price plus the profit, that is, what the merchant sold the camel for. In other words, $\Gamma+b=a$, whence $\Gamma=a-b$.

2.1.15 He starts with m, gains x for a total of $m+x$. He then loses y, for a total of $m+x-y$. After he gains b more dollars he has a running total of $m+x-y+b$, and finally, after losing z dollars he has a grand total of $m+x-y+b-z$.

2.2.1 108.

2.2.2 1.

2.2.3 28.

2.2.4 21

2.2.5 15

2.2.6 Since $100 = 7\times 14+2$, 98 days from today will be a Thursday, and 100 days from today will be a Saturday.

2.2.7 Observe that 4 and 5 are on opposite sides, so they never meet. The largest product is $3\times 5\times 6 = 90.$

2.2.8 The first contractor binds $4500\div 30 = 150$ books per day, and the second, $4500\div 45 = 100$ books per day. Therefore, if they worked simultaneously, they could bind $150+100 = 250$ books per day, and it would take them $4500\div 250 = 18$ days to bind all books. This problem is taken from [Toom].

2.2.9 Seven.

2.2.10 9 minutes.

2.2.11 If $36 \div n$ is a natural number, then n must evenly divide 36, which means that n is a divisor of 36. Thus n can be any one of the values in $\{1,2,3,4,6,9,12,18,36\}$.

2.2.12 From the distributive law,

$$\begin{aligned}
&(666)(222)+(1)(333)+(333)(222) \\
&\quad +(666)(333)+(1)(445)+(333)(333) \\
&\quad +(666)(445)+(333)(445)+(1)(222) &= (666+333+1)(445+333+222) \\
& &= (1000)(1000) \\
& &= 1000000.
\end{aligned}$$

2.2.13 Travelling $30,000$ miles with 4 tyres is as travelling $120,000$ miles on one tyre. The average wear of each of the 5 tyres is thus $120000 \div 5 = 24000$ miles. This problem is taken from [Gard].

2.2.14 Anna answered 20 questions correctly (She could not answer less than 20, because then her score would have been less than $19 \times 4 = 76 < 77$; She could not answer more than 20, because her score would have been at least $21 \times 4 - 3 = 81.77$). To get exactly 77 points Anna had to answer exactly 3 questions wrong, which means she omitted 2 questions.

2.2.15 The trick is to use a technique analogous to the one for multiplying, but this time three-digits at a time:

$$321 \times 654 = 209934,$$

$$321 \times 987 = 316827,$$
$$745 \times 654 = 487230,$$
$$745 \times 987 = 735315.$$

Thus

$$\begin{array}{rrrr}
 & & \boxed{987} & \boxed{654} \\
\times & & \boxed{745} & \boxed{321} \\
\hline
 & & 209 & 934 \\
 & 316 & 827 & \\
 & 487 & 230 & \\
735 & 315 & & \\
\hline
736 & 119 & 266 & 934
\end{array}$$

2.2.16 There are 28 digits, since

$$4^{16}5^{25} = 2^{32}5^{25} = 2^{7}2^{25}5^{25} = 128 \times 10^{25},$$

which is the 3 digits of 128 followed by 25 zeroes.

2.2.17 The least common multiple of $4,5$ and 6 is 60, hence we want the smallest positive multiple of 60 leaving remainder 1 upon division by 7. This is easily seen to be 120.

2.2.18 Using the definition of $\oplus$,

$$1 \oplus (2 \oplus 3) = 1 \oplus (1 + 2(3)) = 1 \oplus 7 = 1 + 1(7) = 8,$$

but

$$(1 \oplus 2) \oplus 3 = (1 + 1(2)) \oplus 3 = 3 \oplus 3 = 1 + 3(3) = 10.$$

Since these two answers differ, the operation is not associative.

Now, for the operation to be commutative, we must have, in every instance, $a \oplus b = b \oplus a$. Using the fact that multiplication of numbers is commutative, we deduce

$$a \oplus b = 1 + ab = 1 + ba = b \oplus a,$$

so our operation is indeed commutative.

2.3.1 Number the steps from 0 to N, where N is the last step, and hence there are $N+1$ steps. Notice that the top and the bottom of the stairs are counted as steps. Let us determine first the number of steps. Frodo steps on steps 0, 2, 4, ..., N; Gimli steps on steps 0, 3, 6, ..., N; Legolas steps on steps 0, 4, 8, ..., N; and Aragorn steps on steps 0, 5, 10, ..., N. For this last step to be common, it must be a common multiple of 2, 3, 4, and 5, and hence

$$N = \text{lcm}(()2,3,4,5) = 60.$$

Since every step that Legolas covers is also covered by Frodo, we don't consider Legolas in our count. Frodo steps alone on the steps which are multiples of 2, but not multiples of 3 or 5. Hence he steps on the 8 steps

$$2, 14, 22, 26, 34, 38, 46, 58.$$

Gimli steps alone on the 8 steps that are multiples of 3 but not multiples of 2 nor 5:

$$3, 9, 21, 27, 33, 39, 51, 57.$$

Finally, Aragorn steps alone on the 4 steps that are multiples of 5 but not multiples of 2 or 3:

$$5, 25, 35, 55.$$

Our final count is thus $8+8+4=20$. This problem is from [Gard].

2.3.2 $\dfrac{36}{60} = \dfrac{3}{5}$ of an hour.

2.3.3 $\dfrac{17}{35}$.

2.3.4 Observe that $3990 = 210 \cdot 19$. Thus $\dfrac{102}{210} = \dfrac{102 \cdot 19}{210 \cdot 19} = \dfrac{1938}{3990}$.

2.3.5 Observe that $3 \cdot 5 \cdot 13 = 195$ is a common denominator. Now

$$\frac{2}{3} = \frac{130}{195}; \quad \frac{3}{5} = \frac{117}{195}; \quad \frac{8}{13} = \frac{120}{195},$$

whence $\dfrac{3}{5}$ is the smallest, $\dfrac{8}{13}$ is in the middle, and $\dfrac{2}{3}$ is the largest.

2.3.6 Three lines have been sung before the fourth singer starts, and after that he sings 12 more lines. So the total span of lines is 15. They start singing simultaneously from line 4, and the first singer is the first to end, in line 12. Thus $12 - 4 + 1 = 9$ lines out of 15 are sung simultaneously and the fraction sought is $\frac{9}{15} = \frac{3}{5}$.

2.4.1 Here are the solutions.

$\frac{1}{3}$	$+$	$\frac{5}{3}$	$=$	2
$+$		$-$		
$\frac{3}{4}$	$\times$	1	$=$	$\frac{3}{4}$
$=$		$=$		
$\frac{13}{12}$	$\div$	$\frac{2}{3}$	$=$	$\frac{13}{8}$

2.4.2 $\dfrac{7}{45}$.

2.4.3 $\dfrac{3}{17}$.

2.4.4 We have,

$$\frac{10+10^2}{\frac{1}{10}+\frac{1}{100}}=\frac{10^3+10^4}{10+1}=\frac{11000}{11}=1000.$$

2.4.5 1.

2.4.6 We have,

$$\frac{1}{1+\frac{1}{5}}=\frac{1}{\frac{6}{5}}=\frac{5}{6}=\frac{a}{b},$$

whence $a^2+b^2=5^2+6^2=61$.

2.4.7 $\frac{1}{100}$.

2.4.8 In one hour John does $\frac{1}{2}$ of the job. In one hour Bill does $\frac{1}{3}$. Thus in one hour they together accomplish $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$ of the job. Thus in $\frac{6}{5}$ of an hour they accomplish the job. Since $\frac{6}{5}\cdot 60=72$, it takes them 72 minutes for them to finish the job.

2.4.9 We have $1\frac{7}{8}=\frac{15}{8}$ and

$$16\div\frac{15}{8}=\frac{16}{1}\cdot\frac{8}{15}=\frac{128}{15}=8\tfrac{8}{15},$$

so she is able to wrap 8 gifts.

2.4.10 Proceeding from the innermost fraction one easily sees that

$$\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{2}}}}=\frac{1}{2-\frac{1}{2-\frac{2}{3}}}=\frac{1}{2-\frac{3}{4}}=\frac{4}{5}.$$

2.4.11 $\frac{43}{30}$

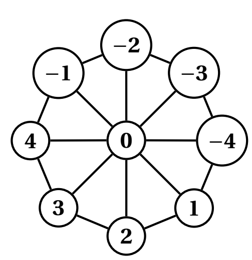
2.4.12 Of the 100 students, only one is male. He is 2% of the on-campus population. Thus the whole on-campus population consists of 50 students, so there are $100-50=50$ off-campus students.

2.4.13 Observe that $5\frac{1}{2}=\frac{11}{2}=\frac{22}{4}$. Hence $\frac{21}{4}$ miles are at the rate of $40¢$. The trip costs $\$0.85+\$.40\cdot 21=\$.85+\$8.40=\$9.25$.

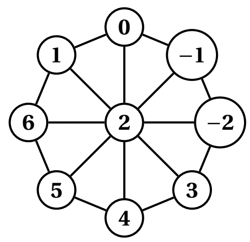
2.5.1

1. $+8$
2. -26
3. -26
4. 0
5. -303
6. 0
7. $+4$
8. -100

2.5.2 Here is one possible answer.



2.5.3 Here is one possible answer.



2.5.4 Here are two answers.

−1	×	+1	=	−1
×		×		×
−1	×	−1	=	+1
=		=		=
+1	×	−1	=	−1

−1	×	−1	=	+1
×		×		×
+1	×	−1	=	−1
=		=		=
−1	×	+1	=	−1

2.5.5 We have,

$$\begin{aligned}\frac{a^3+b^3+c^3-3abc}{a^2+b^2+c^2-ab-bc-ca} &= \frac{(2)^3+(-3)^3+(5)^3-3(2)(-3)(5)}{(2)^2+(-3)^2+(5)^2-(2)(-3)-(-3)(5)-(5)(2)}\\ &= \frac{8-27+125+90}{4+9+25+6+15-10}\\ &= \frac{196}{49}\\ &= 4.\end{aligned}$$

2.6.1 We have,

$$
\begin{aligned}
\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}-\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)^2 \\
&= \frac{-1}{2+(-3)}+\frac{2}{-3+(-1)}+\frac{-3}{-1+2}+\left(\frac{-1}{2}+\frac{2}{-3}+\frac{-3}{-1}\right)^2 \\
&= 1-\frac{1}{2}-3+\left(-\frac{1}{2}-\frac{2}{3}+3\right)^2 \\
&= -2-\frac{1}{2}+\left(-\frac{1}{2}-\frac{2}{3}+3\right)^2 \\
&= -\frac{4}{2}-\frac{1}{2}+\left(-\frac{3}{6}-\frac{4}{6}+\frac{18}{6}\right)^2 \\
&= -\frac{5}{2}-\left(\frac{11}{6}\right)^2 \\
&= -\frac{5}{2}-\frac{121}{36} \\
&= -\frac{90}{36}-\frac{121}{36} \\
&= -\frac{211}{36}
\end{aligned}
$$

2.6.2

1. $2x+3y=2\left(-\frac{2}{3}\right)+3\left(\frac{3}{5}\right)=-\frac{4}{3}+\frac{9}{5}=-\frac{20}{15}+\frac{27}{15}=\frac{7}{15}.$

2. $xy-x-y=\left(-\frac{2}{3}\cdot\frac{3}{5}\right)-\left(-\frac{2}{3}\right)-\frac{3}{5}=-\frac{2}{5}+\frac{2}{3}-\frac{3}{5}=-\frac{6}{15}+\frac{10}{15}-\frac{9}{15}=-\frac{5}{15}=-\frac{1}{3}.$

3. $x^2+y^2=\left(-\frac{2}{3}\right)^2+\left(\frac{3}{5}\right)^2=\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)+\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)=\frac{4}{9}+\frac{9}{25}=\frac{100}{225}+\frac{81}{225}=\frac{181}{225}.$

2.6.3 Here are some possible answers. In some cases, there might be more than one answer, as for example,

$1 = \frac{44}{44} = \frac{4+4}{4+4} = \frac{4^4}{4^4}$, etc.

$1 = \frac{44}{44}$	$2 = \frac{4 \cdot 4}{4+4}$
$3 = \frac{4+4+4}{4}$	$4 = 4+4 \cdot (4-4)$
$5 = \frac{4 \cdot 4+4}{4}$	$6 = 4 + \frac{4+4}{4}$
$7 = \frac{44}{4} - 4$	$8 = 4 \cdot 4 - 4 - 4$
$9 = \frac{4}{.4} - \frac{4}{4}$	$10 = \frac{44-4}{4}$
$11 = \frac{4}{.4} + \frac{4}{4}$	$12 = \frac{44+4}{4}$
$13 = 4! - \frac{44}{4}$	$14 = 4 \cdot (4 - .4) - .4$
$15 = \frac{44}{4} + 4$	$16 = 4 \cdot 4 + 4 - 4$
$17 = 4 \cdot 4 + \frac{4}{4}$	$18 = \frac{4}{.4} + 4 + 4$
$19 = 4! - 4 - \frac{4}{4}$	$20 = \frac{4}{.4} + \frac{4}{.4}$

2.6.4 To do this problem correctly, you must compare all combinations of numbers formed with three fours. Observe that

$$(4^4)^4 = 256^4 = 4294967296,$$

$$44^4 = 3748096, \quad 4^{44} = 309485009821345068724781056,$$

and that

$$\begin{aligned} 4^{(4^4)} &= 4^{256} \\ &= 13407807929942597099574024998205846127479365820592393377723561443721764030073 5 \rightarrow \\ &\rightarrow 4697680187429816690342769003185818648605085375388281194656994643364900608409 6 \end{aligned}$$

where we have broken the number so that it fits the paper.

Note: This last calculation I did with a computer programme. How does one actually check that the computer is right? At any rate, since each time we multiply a given quantity by 4 the quantity increases, we must have $4^{44} < 4^{256} = 4^{4^4}$.

2.6.5 $\frac{5}{36}$

2.6.6 $0.1111\ldots = 0.\overline{1} = \frac{1}{3}(0.333333\ldots) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

2.6.7 From the text, we know that $0.\overline{09} = \frac{1}{11}$, and hence $121(0.\overline{09}) = 121 \cdot \frac{1}{11} = 11$.

2.6.8 $\sqrt{2} + \sqrt{3} + \sqrt{5} \approx 5.38$

2.6.9 $\sqrt{2} \cdot \sqrt{3} \cdot \sqrt{5} \approx 5.48$

2.6.10 No, this is not always true. For example, $\sqrt{1} + \sqrt{3} = 1 + \sqrt{3} \approx 2.73$, but $\sqrt{1+3} = \sqrt{4} = 2$. But, is it *ever* true? For $a = 0$, $\sqrt{a} + \sqrt{b} = \sqrt{0} + \sqrt{b} = \sqrt{b}$ and $\sqrt{a+b} = \sqrt{0+b} = \sqrt{b}$, so it is true in this occasion.

3.1.1 After buying the marbles, he has $+a$ marbles. After winning b marbles, he now has $a+b$ marbles. After losing c marbles, he ends up with $a+b-c$. Conclusion: he has $a+b-c$ marbles in the end.

3.1.2

1. 0
2. $-6a$
3. $-10a$
4. $6a$
5. $-4a+4b$

3.1.3 $2a-2b+2c-2d+2a^2-2c^2$.

3.1.4

1. $4a+4b+4c$
2. $4a+2c$
3. $5a+5b+5c$
4. $a-b+c$
5. $2x^2+2$
6. x^3+x^2+1
7. $6x^3-3x^2-3x-2$
8. $-\frac{9a}{8}$

3.1.5 Since $(2x^2+x-2)+(-x^2-x+2)=x^2$, the value is $2^2=4$.

3.1.6 Yes, this is always true.

3.1.7 No, this is not always true. For example, if $a=10$, then $a+a=10+10=20$, but $2a^2=2(10)^2=2\cdot 100=200$. Observe, however, that $0+0=2\cdot 0^2$ and that $1+1=2\cdot 1^2$, so the assertion is true in these two cases. That is it *only* true in these two cases is more difficult to prove, and we will have to wait till we study quadratic equations.

3.1.8 No, this is not always true. For example, if $a=10$, then $a+a^2=10+100=110$, but $2a^3=2(10)^3=2\cdot 1000=2000$. Observe, however, that $0+0^2=2\cdot 0^3$, that $1+1^2=2\cdot 1^3$, and that $\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)^2=2\cdot\left(-\frac{1}{2}\right)^3$ so the assertion is true in these three cases. That is it *only* true in these three cases is more difficult to prove, and we will have to wait till we study quadratic equations.

3.1.9 Letting $x=1$ one gathers that

$$243=3^5=(8(1)-5)^5=A+B+C+D+E+F.$$

3.1.10 Here are a few. Many more are possible.

1. $3xy-x^2y$
2. $-x^2y+3yx$
3. $3yx-yx^2$
4. $3xy-yx^2$
5. $-yx^2-3xy$

3.2.1 Since $b-c$ inches are cut of from $a+b$, we must subtract $b-c$ from $a+b$, giving

$$(a+b)-(b-c)=a+b-b+c=a+c,$$

whence there remain $a+c$ inches.

3.2.2 Since there are x people to share the cost of D, the share of each person is $\frac{D}{x}$. If p persons disappear, there remain $x-p$ persons, and the share of each would increase to $\frac{D}{x-p}$. This means they will have to pay

$$\frac{D}{x-p}-\frac{D}{x}$$

more.

3.2.3 $2b+2c+2c^2+2d^2$

3.2.4 $-a+5b-c$

3.2.5

1. $-2a+2c$
2. $3a+3b+3c$
3. $a+3b+c$
4. x^3-3x^2+2x-3
5. $-2x^3-x^2+3x-2$
6. $9.3a-3.1b$
7. $\frac{11b-4}{a}$
8. $10-\frac{8}{a}$
9. $10\clubsuit$

3.2.6 We have

$$\frac{a}{3}-\frac{a^2}{5}+\frac{a}{2}+\frac{5a^2}{6}=\frac{a}{3}+\frac{a}{2}-\frac{a^2}{5}+\frac{5a^2}{6}=\frac{2a}{6}+\frac{3a}{6}-\frac{6a^2}{30}+\frac{25a^2}{30}=\frac{5a}{6}+\frac{19a^2}{30}=\frac{25a}{30}+\frac{19a^2}{30}=\frac{25a+19a^2}{30}$$

3.2.7

1. $-2x-2y$
2. $3x-5$
3. $4x^2+2x-5$

3.2.8 $x-y$ by commutativity is $-y+x$, and so 1 matches with B. By the subtraction rule, $x-(-y)=x+(+y)=x+y$, and so 2 matches with C. Finally, using the distributive law and the commutative law, $-(x-y)=-x+y=y-x$, and so 3 matches with A.

3.2.9 You start with A, your uncle Bob gives you A more so that you now have $A+A=2A$. Your Aunt Rita gives you 10 more, so that now you have $2A+10$. You pay B dollars in fines, hence you are left with $2A+10-B$. You spent 12 in fuel, so now you have

$$2A+10-B-12=2A-B-2,$$

in total.

3.2.10

1. $(2-4)x-5(-6)x=-2x+30x=28x$
2. $5-2(2x)=5-4x$
3. $(5-2)(1-2^2)^3x=(3)(1-4)x=(3)(-3)^3x=(3)(-27)x=-81x$

3.2.11

1. $(6t)+x=6t+x$, so the parentheses are redundant.
2. In $6(t+x)$, the parentheses are needed.
3. In $(ab)^2$, the parentheses are needed.
4. In $(ab-c)^2$, the parentheses are needed.

3.2.12

1. In $\frac{a-2b}{c}=\frac{a}{c}+\boxed{}$, the $\boxed{}$ is $-\frac{2b}{c}$
2. In $\frac{t+3}{t-3}=(t+3)\boxed{}$, the $\boxed{}$ is $\frac{1}{t-3}$.

3.2.13

1. In $1-t=\boxed{}\,t+1$, the $\boxed{}$ is -1 or simply, $-$.

2. In $3x + y - t = \boxed{} + 3x$ the $\boxed{}$ is $y - t$.

3.2.14 The opposite sought is

$$-(4 - y + 2) = -4 + y - 2.$$

3.2.15 The additive inverse sought is

$$-\left(\frac{-a}{-(b-c)}\right) = -\left(\frac{a}{(b-c)}\right) = -\frac{a}{b-c}.$$

4.1.1 $-8xyz$

4.1.2 $24abc$

4.1.3 $\frac{4096}{625}$

4.1.4 $\frac{4}{625}$

4.1.5 The product equals

$$\frac{4(4^5)}{3(3^5)} \cdot \frac{6(6^5)}{2(2^5)} = 4(4^5) = 2^{12},$$

so $n = 12$.

4.1.6 We have

$$3^{2001} + 3^{2002} + 3^{2003} = 3^{2001}(1 + 3 + 3^2) = (13)3^{2001},$$

whence $a = 13$.

4.1.7 $(a^x b^y)\left(\frac{b^{2x}}{a^{-y}}\right) = a^{x-(-y)} b^{y+2x} = a^{x+y} b^{y+2x}$

4.1.8 x

4.1.9 $\frac{1}{x^3}$

4.1.10 $s - 1$

4.1.11 $\frac{1}{(s-1)^3}$

4.1.12 $\frac{a^4}{x^2}$

4.1.13 1

4.1.14 We have,

$$\begin{aligned}
\frac{6^5 + 6^5 + 6^5 + 6^5 + 6^5 + 6^5 + 6^5 + 6^5 + 6^5 + 6^5}{3^3 + 3^3 + 3^3 + 3^3 + 3^3} &= \frac{10 \cdot 6^5}{5 \cdot 3^3} \\
&= \frac{2 \cdot 6^5}{3^3} \\
&= \frac{2 \cdot 2^5 \cdot 3^5}{3^3} \\
&= 2 \cdot 2^5 \cdot 3^2 \\
&= 2 \cdot 32 \cdot 9 \\
&= 576.
\end{aligned}$$

4.1.15 We have,

$$\begin{aligned}\frac{a^8}{b^9} \div \frac{(a^2b)^3}{b^{20}} &= \frac{a^8}{b^9} \cdot \frac{b^{20}}{(a^2b)^3}\\ &= \frac{a^8}{b^9} \cdot \frac{b^{20}}{a^6b^3}\\ &= \frac{a^8b^{20}}{a^6b^{12}}\\ &= a^{8-6}b^{20-12}\\ &= a^2b^8.\end{aligned}$$

So $m = 2$ and $n = 11$.

4.2.1

1. $\frac{1}{x^9}$
2. x^9
3. $\frac{1}{x}$
4. $\frac{y^6z^9}{x^3}$
5. $\frac{a^9}{b^6}$
6. a^2b^5

4.2.2 We have,

$$(a^{x+3}b^{y-4})(a^{2x-3}b^{-2y+5}) = a^{x+3+2x-3}b^{y-4-2y+5} = a^{3x}b^{1-y}.$$

4.2 We have,

$$\frac{a^xb^y}{a^{2x}b^{-2y}} = a^{x-2x}b^{y-(-2y)} = a^{-x}b^{3y}.$$

4.2.4 $\frac{x^8}{y^{12}}$

4.2.5 $2^8 = 256$

4.2.6 4

4.2.7 $\frac{(2^4)^8}{(4^8)^2} = \frac{2^{32}}{((2^2)^8)^2} = \frac{2^{32}}{2^{32}} = 1.$

4.3.1

1. $x^3 + x^2 + x$
2. $2x^3 - 4x^2 + 6x$
3. $4x^2 + 4x + 1$
4. $4x^2 - 1$
5. $4x^2 - 4x + 1$

4.3.2 $x^2 + 1 - 7x^3 + 4x - 3x^5$

4.3.3 $4x^5 - 8x^4 + 8x^3 + 2x^2$

4.3.4

1. $-9x^4 + 3x^3 - 2x^2 - 2x$
2. $-2x^3 - 4x^2 - 2x$

3. $a^3cb - 2ac^2b$
4. $x^2 - 4y^2 + 3zx - 6zy$

4.3.5 We have,

$$\begin{aligned}(2x+2y+1)(x-y+1) &= (2x+2y+1)(x)+(2x+2y+1)(-y)+(2x+2y+1)(1)\\ &= 2x^2+2xy+x-2xy-2y^2-y+2x+2y+1\\ &= 2x^2+3x-2y^2+y+1.\end{aligned}$$

4.3.6 $10x$.

4.3.7 -2.

4.3.8 $x^2 + zx - y^2 + 3zy - 2z^2$

4.3.9 We have,

$$\begin{aligned}(a+b+c)(a^2+b^2+c^2-ab-bc-ca) &= a^3+ab^2+ac^2-a^2b-abc-ca^2\\ &\quad +ba^2+b^3+bc^2-ab^2-b^2c-bca\\ &\quad +a^2c+b^2c+c^3-abc-bc^2-c^2a\\ &= a^3+b^3+c^3-3abc\end{aligned}$$

4.3.10 From the distributive law we deduce that

$$(a+b+c)(x+y+z) = ax+ay+az+bx+by+bz+cx+cy+cz.$$

4.3.11 Let $2a$ be one of the integers and let $2b$ be the other integer. Then $(2a)(2b) = 2(2ab)$, is twice the integer $2ab$ and hence it is even.

4.3.12 Let $2a+1$ be one of the integers and let $2b+1$ be the other integer. Then

$$\begin{aligned}(2a+1)(2b+1) &= 4ab+2a+2b+1\\ &= 2(2ab)+2a+2b+1\\ &= 2(2ab+a+b)+1\\ &= 2m+1,\end{aligned}$$

where $m = 2ab+a+b$. Since m is an integer, the equality $(2a+1)(2b+1) = 2m+1$ shews that the product $(2a+1)(2b+1)$ leaves remainder 1 upon remainder by 2, that is, the product is odd.

4.3.13 Let $4a+3$ be one of the integers and let $4b+3$ be the other integer. Then

$$\begin{aligned}(4a+3)(4b+3) &= 16ab+12a+12b+9\\ &= 16ab+12a+12b+8+1\\ &= 4(4ab)+4(3a)+4(3b)+4(2)+1\\ &= 4(4ab+3a+3b+2)+1\\ &= 4m+1,\end{aligned}$$

where $m = 4ab+3a+3b+1$. Since m is an integer, the equality $(4a+3)(4b+3) = 4m+1$ shews that the product $(4a+3)(4b+3)$ leaves remainder 1 upon remainder by 4.

4.3.14 Let $3a+2$ and $3b+2$ be two numbers leaving remainder 2 upon division by 3. Then

$$(3a+2)(3b+2) = 9ab+6a+6b+4 = 3(3ab+2a+2b+1)+1,$$

which is a number leaving remainder 1 upon division by 3.

4.3.15 Let $5a+2$ and $5b+2$ be two numbers leaving remainder 2 upon division by 5. Then

$$(5a+2)(5b+2) = 25ab+10a+10b+4 = 5(5ab+2a+2b)+4,$$

which is a number leaving remainder 4 upon division by 5.

4.3.16 Let $5a+3$ and $5b+3$ be two numbers leaving remainder 3 upon division by 5. Then

$$(5a+3)(5b+3) = 25ab+15a+15b+9 = 5(5ab+2a+2b+1)+4,$$

which is a number leaving remainder 4 upon division by 5.

4.3.17 Observe that $x^2+x=1$. Hence $x^4+x^3=x^2$ and $x^3+x^2=x$. Thus

$$x^4+2x^3+x^2 = x^4+x^3+x^3+x^2 = x^2+x = 1.$$

4.3.18 We have

$$\begin{aligned} 2x(x+1)-x^2-x(x-1) &= 2x^2+2x-x^2-x^2+x \\ &= 3x. \end{aligned}$$

Putting $x=11112$ one obtains from the preceding part that

$$2\cdot(11112)\cdot(11113)-11112^2-(11111)\cdot(11112) = 3\cdot 11112 = 33336.$$

4.4.1 $9-6y+y^2$

4.4.2 $25-20x^2+4x^4$

4.4.3 $4a^2b^4-12ab^2c^3d^4+9c^6d^8$

4.4.4 $x^2+8xy+4y^2$

4.4.5 x^2+4y^2

4.4.6 We have,

$$(xy-4y)^2 = (xy)^2-2(xy)(4y)+(4y)^2 = x^2y^2-8xy^2+16y^2.$$

4.4.7 We have,

$$(ax+by)^2 = (ax)^2+2(ax)(by)+(by)^2 = a^2x^2+2abxy+b^2y^2.$$

4.4.8 $2a^2+8$

4.4.9 $8a$

4.4.10 $a^2+4ab+6ac+4b^2+12bc+9c^2$

4.4.11 $a^2+4ab-2ac+4b^2-4bc+c^2$.

4.4.12 Observe that

$$36 = \left(x+\frac{1}{x}\right)^2 = x^2+2+\frac{1}{x^2} \implies x^2+\frac{1}{x^2} = 36-2 = 34.$$

4.4.13 We have,

$$\begin{aligned} (10^{2002}+25)^2-(10^{2002}-25)^2 &= 10^{4004}+2\cdot 25\cdot 10^{2002}+25^2 \\ &\quad -(10^{4004}-2\cdot 25\cdot 10^{2002}+25^2) \\ &= 4\cdot 25\cdot 10^{2002} \\ &= 100\cdot 10^{2002} = 10^{2004}, \end{aligned}$$

whence $n=2004$.

4.4.14 We have

$$\begin{aligned} 4x^2+9y^2 &= 4x^2+12xy+9y^2-12xy \\ &= (2x+3y)^2-12xy \\ &= 3^2-12(4) \\ &= 9-48 \\ &= -39. \end{aligned}$$

4.4.15 Let the two numbers be x, y. Then $x-y=3$ and $xy=4$. Hence

$$9=(x-y)^2=x^2-2xy+y^2=x^2-8+y^2 \implies x^2+y^2=17.$$

4.4.16 We have

$$16=(x+y)^2=x^2+2xy+y^2=x^2-6+y^2 \implies x^2+y^2=22.$$

4.4.17 From problem 4.4.16, $x^2+y^2=22.$ Observe that $2x^2y^2=2(xy)^2=2(-3)^2=18$. Hence

$$484=(x^2+y^2)^2=x^4+2x^2y^2+y^4=x^4+18+y^4 \implies x^4+y^4=466.$$

4.4.18 We have,

$$\begin{aligned} (x+y+z)^2 &= ((x+y)+z)^2 \\ &= (x+y)^2+2(x+y)z+z^2 \\ &= x^2+2xy+y^2+2xz+2yz+z^2 \\ &= x^2+y^2+z^2+2xy+2yz+2zx. \end{aligned}$$

4.4.19 We have

$$\begin{aligned} (x+y+z+w)^2 &= (x+y)^2+2(x+y)(z+w)+(z+w)^2 \\ &= x^2+2xy+y^2+2xz+2xw+2yz+2yw+z^2+2zw+w^2 \\ &= =x^2+y^2+z^2+w^2+2xy+2xz+2xw+2yz+2yw+2zw. \end{aligned}$$

4.5.1 We have

$$\begin{aligned} (a+4)^2-(a+2)^2 &= ((a+4)-(a+2))((a+4)+(a+2)) \\ &= (2)(2a+6) \\ &= 4a+12. \end{aligned}$$

We may also use the square of a sum identity:

$$\begin{aligned} (a+4)^2-(a+2)^2 &= (a^2+8a+16)-(a^2+4a+4) \\ &= 4a+12, \end{aligned}$$

like before.

4.5.2 We have,

$$\begin{aligned}(x-y)(x+y)(x^2+y^2) &= (x^2-y^2)(x^2+y^2)\\ &= x^4-y^4.\end{aligned}$$

4.5.3 We have

$$\begin{aligned}(a+1)^4-(a-1)^4 &= ((a+1)^2-(a-1)^2)((a+1)^2+(a-1)^2)\\ &= ((a+1)-(a-1))((a+1)+(a-1))((a+1)^2+(a-1)^2)\\ &= (2)(2a)((a^2+2a+1)+(a^2-2a+1))\\ &= 4a(2a^2+2)\\ &= 8a^3+8a.\end{aligned}$$

4.5.4 Multiplying by $1=2-1$ does not alter the product, and so,

$$\begin{aligned}(2-1)(2+1)(2^2+1)(2^4+1)(2^8+1)(2^{16}+1) &= (2^2-1)(2^2+1)(2^4+1)(2^8+1)(2^{16}+1)\\ &= (2^4-1)(2^4+1)(2^8+1)(2^{16}+1)\\ &= (2^8-1)(2^8+1)(2^{16}+1)\\ &= (2^{16}-1)(2^{16}+1)\\ &= 2^{32}-1\end{aligned}$$

4.5.5 We have,

$$\begin{aligned}(1^2-2^2)+(3^2-4^2)+\cdots+(99^2-100^2) &= (1-2)(1+2)+(3-4)(3+4)+\cdots+(99-100)(99+100)\\ &= -1(1+2)-1(3+4)+\cdots+-1(99+100)\\ &= -(1+2+3+\cdots+100)\\ &= -5050,\end{aligned}$$

using the result of Example 5.

4.5.6 Put $x=123456789$. Then

$$(123456789)^2-(123456787)(123456791)=x^2-(x-2)(x+2)=x^2-(x^2-4)=4.$$

4.5.7 Using $x^2-y^2=(x-y)(x+y)$,

$$\begin{aligned}(666\ 666\ 666)^2-(333\ 333\ 333)^2 &= (666\ 666\ 666-333\ 333\ 333)(666\ 666\ 666+333\ 333\ 333)\\ &= (333\ 333\ 333)(999\ 999\ 999)\\ &= (333\ 333\ 333)(10^9-1)\\ &= 333\ 333\ 333\ 000\ 000\ 000-333\ 333\ 333\\ &= 333\ 333\ 332\ 666\ 666\ 667\end{aligned}$$

4.6.1 $125x^3+75x^2+15x+1$

4.6.2 x^3+4x^2+5x+2

4.6.3 x^3-4x^2-x-5

4.6.4 $x^3-7x^2+16x-12$

4.6.5 We have,

$$216=(a-b)^3=a^3-3a^2b+3ab^2-b^3=a^3-b^3-3ab(a-b)=a^3-b^3-3(3)(6) \implies a^3-b^3=270.$$

4.6.6 We have,

$$216=(a+2b)^3=a^3+6a^2b+12ab^2+8b^3=a^3+8b^3+6ab(a+2b)=a^3+8b^3+6(3)(6) \implies a^3+8b^3=108.$$

4.7.1 x^3-8

4.7.2 x^3+512

4.7.3 Notice that this is problem 4.6.5. We will solve it here in a different way. We have

$$a^3-b^3=(a-b)(a^2+ab+b^2)=6(a^2+3+b^2),$$

so the problem bois down to finding a^2+b^2. Now,

$$36=(a-b)^2=a^2-2ab+b^2=a^2-6+b^2 \implies a^2+b^2=42,$$

whence

$$a^3-b^3=6(a^2+3+b^2)=6(42+3)=270,$$

as was obtained in the solution of problem 4.6.5.

5.1.1

$$\begin{aligned}\frac{q^2-pq-pqr}{-q} &= \frac{q^2}{-q}-\frac{pq}{-q}-\frac{-pqr}{-q}\\ &= -q+p+pr\\ &= -q+p+pr.\end{aligned}$$

5.1.2 $2x^2y+3xy^2$

5.1.3 $102x^4y^4$

5.1.4 $3a^2x+2ax$

5.1.5 $12a^5x^4$

5.1.6 $-x+y+z$

5.1.7 $-5a+7b^2$

5.1.8 $35a^2b^3$

5.1.9 $-x^3yz$

5.2.1 $x-1$

5.2.2 $x+3$

5.2.3 x^2-x+1

5.2.4 $2x+4.$

5.2.5 $2x^2-2x-2.$

5.3.1 We have
$$-4a^6b^5+6a^3b^8-12a^5b^3-2a^2b^3=-2a^2b^3(2a^4b^2-3ab^5+6a^3+1).$$

5.3.2 $4a^3b^4-10a^4b^3=2a^3b^3(2b-5a)$.

5.3.3 $\frac{9}{16}x^2-\frac{3}{4}x=\frac{3}{4}x\left(\frac{3}{4}x-1\right)$.

5.3.4 $-x+2y-3z=-1(x-2y+3z)$.

5.3.5 Let $2a+1$ and $2b+1$ be odd integers. Then
$$2a+1+2b+1=2a+2b+2=2(a+b+1),$$
that is, twice the integer $a+b+1$, and hence even.

5.3.6 $x^2(x-1)$

5.3.7 $5a^3b^2c^4(25cb^3a-9ba^2+1-60c^4a-2c)$

5.3.8 $5x^3(x^2-2a^7-3a^3)$

5.3.9 $x^2(3xy+4y^3-6x^4-10x^2)$

5.3.10 $3m^4p^2q(m^2p^2q-2p^2x^2ym+m^3px-3mx+1)$

5.3.11 $19a^2x^2(2x^3+3a^2)$

5.3.12 $(a+c)(a+b)$

5.3.13 $(a+c)(a-b)$

5.3.14 $(y-1)(1+y^2)$

5.3.15 $(2x+3)(x^2+1)$

5.3.16 $(a+b)(xa+c+by)$

5.4.1 $(a-5)(a-6)$

5.4.2 $(a-19)^2$

5.4.3 $(a^2b^2+25)(a^2b^2+12)$

5.4.4 $(x-11)(x-12)$

5.4.5 $(x^2-12)(x^2-17)$

5.4.6 $(x+27)(x+8)$

5.4.7 $(5x+2)(x+3)$

5.4.8 $(7x-3)(2x+5)$

5.5.1 We have
$$x^2-4y^2=(x+2y)(x-2y)=(3)(-1)=-3.$$

5.5.2 $(x-2)(x+2)(x^2+4)$

5.5.3 $(a+b-c)(a+b+c)$

5.5.4 $x(x-1)(x+1)$

5.5.5 Since $n^3-8=(n-2)(n^2+2n+2)$ and since $n-2<n^2+2n+4$, we must have $n-2=1 \implies n=3$ and $n^3-8=19$.

5.6.1 $\frac{2x}{x^2-1}$

5.6.2 $\frac{2}{x^2-1}$

5.6.3 $\frac{x^2+4x-4}{x^2-4}$

5.6.4 $\frac{x^2+4}{x^2-4}$

5.6.5 $\frac{3x-1}{x^3-x}$

5.6.6 $\frac{3a^2x+3ax-x}{2a^2}$

5.6.7 $\frac{1}{s^2-1}$

5.6.8 $\frac{2x-3y}{xy}$

5.6.9 $\frac{2s^2-7s-6}{s^2-4}$

5.6.10 $\frac{x(2a-1)}{a^2}$

5.6.11 We have,

$$x+y=7 \implies x^2+2xy+y^2=49 \implies x^2+y^2=49-2xy=49-2(21)=7.$$

Observe also that $x^2y^2=(xy)^2=21^2$. Hence

$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{y^2+y^2}{x^2y^2}=\frac{7}{21^2}=\frac{1}{63}.$$

6.1.1 $z=-8$

6.1.2 $z=0$

6.1.3 $z=-11$

6.1.4 $z=-\frac{3}{4}$

6.1.5 $z=-16$

6.1.6 $z=-33$

6.1.7 $z=-36$

6.1.8 $z=3a+3$

6.1.9 $z=5$

6.1.10 $z=(b^2+a)(a+1)=b^2a+b^2+a^2+a$

6.1.11 $N=A$

6.1.12 $N=A^3+A$

6.1.13 $N=\frac{Y^2}{X^2}$

6.1.14 $N=12A+A^2$

6.1.15 $N=-17X$

6.1.16 $N=A^2+3A+2$

6.1.17 $N = B^7$

6.1.18 $N = \frac{2}{A}$

6.1.19 $N = A + 12$

6.2.1 $x = \frac{17}{18}$

6.2.2 $x = 6$

6.2.3 $x = \frac{12}{5}$

6.2.4 $x = 3a$

6.2.5 $x = b$

6.2.6 $x = \frac{c-b}{a}$

6.2.7 $x = \frac{a-2}{3}$

6.2.8 $x = 1$

6.2.9 $x = \frac{a}{b}$

6.2.10 $x = \frac{ab}{cd}$

6.2.11 $x = 5$

6.2.12 $x = -13$

6.2.13 $x = 2$

6.2.14 $x = \frac{6}{5}$

6.2.15 We have,

$$(x-a)b = (b-x)a \implies bx - ab = ab - ax \implies bx = -ax \implies (a+b)x = 0 \implies x = 0.$$

6.2.17 If $x = 0.123123123\ldots$ then $1000x = 0.123123123\ldots$ giving $1000x - x = 123$, since the tails cancel out. This results in $x = \frac{123}{999} = \frac{41}{333}$.

6.3.1 If x is the number, then

$$6x + 11 = 65 \implies 6x = 54 \implies x = 9,$$

so the number is 9.

6.3.2 If x is the number, then

$$11x - 18 = 15 \implies 11x = 33 \implies x = 3,$$

so the number is 3.

6.3.3 If x is the number, then

$$12(x+3) = 84 \implies 12x + 36 = 84 \implies 12x = 48 \implies x = 4,$$

so the number is 4.

6.3.4 Let

$$x-5, x-4, x-3, x-2, x-1, x, x+1, x+2, x+3, x+4, x+5,$$

be the eleven consecutive integers. Then

$$x-5+x-4+x-3+x-2+x-1+x+x+1+x+2+x+3+x+4+x+5 = 2002 \implies 11x = 2002 \implies x = 182.$$

The numbers are

$$177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187.$$

6.3.5 If the two numbers be x and $x+3$, then

$$x+x+3=27 \implies x=12,$$

so the numbers are 12 and 15.

6.3.6 If x is one number, the other is $19-x$. If x exceeds twice $19-x$ by 1, then

$$x-2(19-x)=1 \implies 3x-38=1 \implies 3x=39 \implies x=13.$$

Thus one number is 13 and the other is 6.

6.3.7 If Paul has x dollars, then Mary has $x+20$ and Peter has $x-30$. Therefore

$$x+x+20+x-30=380 \implies 3x-10=380 \implies 3x=390 \implies x=130.$$

Hence Paul has \$130, Mary \$150 and Peter \$100.

6.3.8 The amount of acid does not change. Hence

$$0.60=0.20(100+x) \implies x=200,$$

thus 200 grams should be added.

6.3.9 If Bob's age is x, then Bill's age is $x-4$ and Jane's $2x+3$. Therefore

$$2x+3+x-4+x=27 \implies 4x-1=27 \implies 4x=28 \implies x=7.$$

Thus Bob is 7, Bill is 3, and Jane is 17.

6.3.10 The sum of the original 6 numbers is $S=6\cdot 4=24.$ If the 7th number is x, then

$$\frac{24+x}{7}=5,$$

whence $x=11.$

6.3.11 If Bob currently has b dollars, then Bill has $5b$ dollars. After giving \$20 to Bob, Bill now has $5b-20$ and Bob now has $b+20$. We are given that

$$5b-20=4(b+20) \implies 5b-20=4b+80 \implies b=100.$$

So currently, Bill has \$500 and Bob has \$100.

6.3.12 If x is the number then

$$\frac{6x}{7}-\frac{4x}{5}=2.$$

Multiplying both sides of this equation by 35 we obtain

$$35\left(\frac{6x}{7}-\frac{4x}{5}\right)=2(35) \implies 30x-28x=70 \implies 2x=70 \implies x=35.$$

6.3.13 If one number is x, the larger is $x+8$. Thus

$$x+8+2=3x \implies x+10=3x \implies 10=2x \implies 5=x,$$

so the smallest number is 5 and the larger is 13.

6.3.14 If one of the numbers is x, the other is $x+10$. Thus

$$x+x+10=2(10) \implies 2x+10=20 \implies 2x=10 \implies x=5.$$

The smaller number is 5 and the larger is 15.

6.3.15 If x is the amount originally bought, then I spent $2\left(\frac{x}{4}\right)=\frac{x}{2}$ dollars. Since I kept $\frac{x}{5}$, I must have sold $\frac{4x}{5}$ of them, making $2\left(\frac{\frac{4x}{5}}{3}\right)=\frac{8x}{15}$ dollars on this sale. My net gain is thus

$$\frac{8x}{15}-\frac{x}{2}=2.$$

Multiplying both sides by 30, we have

$$30\left(\frac{8x}{15}-\frac{x}{2}\right)=2(30) \implies 16x-15x=60.$$

I originally bought sixty avocados.

6.3.16 If x is the number, then

$$\frac{x}{4}+\frac{x}{6}+\frac{x}{8}=13.$$

Multiplying both sides by 24, we have

$$24\left(\frac{x}{4}+\frac{x}{6}+\frac{x}{8}\right)=13(24) \implies 6x+4x+3x=312 \implies 13x=312 \implies x=24,$$

whence the number is 24.

6.3.17 Let x and $x+1$ be the integers. Then

$$\frac{x+1}{5}-\frac{x}{7}=3 \implies 35\left(\frac{x+1}{5}-\frac{x}{7}\right)=3(35) \implies 3(x+1)-5x=105 \implies 3-2x=105 \implies 2x=102 \implies x=51.$$

The integers are 51 and 52.

6.3.18 Let x be total amount of oranges bought at three for a dollar. On these I spent $\frac{1}{3}\cdot x\cdot 1=\frac{x}{3}$ dollars. I bought $\frac{5x}{6}$ oranges at four for a dollar, thus spending $\frac{1}{4}\cdot\left(\frac{5x}{6}\right)\cdot 1=\frac{5x}{24}$ dollars. Notice that I have bought a total of $x+\frac{5x}{6}=\frac{11x}{6}$ oranges. If I sell all of them at sixteen for six dollars, I make $6\cdot\frac{11x}{6}\cdot\frac{1}{16}=\frac{11x}{16}$ dollars. Thus

$$\frac{11x}{16}-\left(\frac{x}{3}+\frac{5x}{24}\right)=\frac{7}{2} \implies x=24.$$

Hence I bought $\left(\frac{11}{6}\right)24=44$ oranges.

6.3.19 Let x be its original price. After a year, its new price will $x-\frac{x}{5}=\frac{4x}{5}$. After another year, its new price will be $\frac{4x}{5}-\frac{1}{6}\cdot\frac{4x}{5}=\frac{2x}{3}$. Hence we must have

$$\frac{2x}{3}=56000 \implies x=56000\cdot\frac{3}{2}=84000.$$

Thus the original price was \$84,000.

6.3.20 Let the integers be n and $n+1$. Then

$$(n+1)^2-n^2=665 \implies 2n+1=665 \implies n=332.$$

Thus the integers are 332 and 333.

6.3.21 There are ten different ways. We want the number of solutions of

$$5x+10y+25z=50,$$

that is, of

$$x+2y+5z=10,$$

with integer $0 \le x \le 10$, $0 \le y \le 5$, $0 \le z \le 2$. The table below exhausts all ten possibilities.

z	y	x
2	0	0
1	2	1
1	1	3
1	0	5
0	5	0
0	4	2
0	3	4
0	2	6
0	1	8
0	0	10

6.3.22 Let there be a coins in the purse. There are five stages. The fifth stage is when all the burglars have the same amount of money. Let

$$a_k, b_k, c_k, d_k, e_k$$

be the amount of money that each burglar has, decreasing lexicographically, with the a's denoting the amount of the meanest burglar and e_k denoting the amount of the meekest burglar. Observe that for all k we have

$$a_k + b_k + c_k + d_k + e_k = a.$$

On stage five we are given that

$$a_5 = b_5 = c_5 = d_5 = e_5 = \frac{a}{5}.$$

On stage four

$$a_4 = b_4 = c_4 = \frac{a}{5}, \quad d_4 = \frac{a}{5} + \frac{a}{10} = \frac{3a}{10} \quad e_4 = \frac{a}{10}.$$

On stage three we have

$$a_3 = b_3 = \frac{a}{5}, \qquad c_3 = \frac{a}{5} + \frac{3a}{20} + \frac{a}{20} = \frac{2a}{5} \quad d_3 = \frac{3a}{20} \quad e_3 = \frac{a}{20}.$$

On stage two we have

$$a_2 = \frac{a}{5}, \quad b_2 = \frac{a}{5} + \frac{a}{5} + \frac{3a}{40} + \frac{a}{40} = \frac{a}{2}, \quad c_2 = \frac{a}{5} \quad d_2 = \frac{3a}{40} \quad e_2 = \frac{a}{40}.$$

On stage one we have

$$a_1 = \frac{a}{5} + \frac{a}{4} + \frac{a}{10} + \frac{3a}{80} + \frac{a}{80} = \frac{3a}{5}, \quad b_1 = \frac{a}{4}, \quad c_1 = \frac{a}{10} \quad d_1 = \frac{3a}{80} \quad e_1 = \frac{a}{80}.$$

Since $a_1 = 240$, we deduce $\frac{3a}{5} = 240 \implies a = 400$.

Check: On the first stage the distribution is (from meanest to meekest):

$$240, \quad 100, \quad 40, \quad 15, \quad 5.$$

On the second stage we have

$$80, \quad 200, \quad 80 \quad 30, \quad 10.$$

On the third stage we have

$$80, \quad 80, \quad 160, \quad 60, \quad 20.$$

On the fourth stage we have

$$80, \quad 80, \quad 80, \quad 120, \quad 40.$$

On the fifth stage we have

$$80, \quad 80, \quad 80, \quad 80, \quad 80.$$

7.1.1 $x = 2$ or $x = -2$

7.1.2 $x = 3$ or $x = -2$

7.1.3 $x = -3$ or $x = 2$

7.1.4 $x = 5$ or $x = -1$

7.1.5 $x = 1$ or $x = -1$

7.1.6 If eggs had cost x cents less per dozen, I would have saved $\frac{x}{12}$ cents per egg. If eggs had cost x cents more per dozen, I would have lost $\frac{x}{12}$ cents per egg. The difference between these two prices is $2(\frac{x}{12})$ per egg. Thus if I buy $x+3$ eggs and the total difference is 3 cents, I can write

$$(x+3)2\left(\frac{x}{12}\right) = 3,$$

which is to say $x^2 - 3x - 18 = (x-3)(x+6) = 0$. Since x has to be a positive number, $x = 3$.

7.1.7 Let x be the original number of people renting the bus. Each person must originally pay $\frac{2300}{x}$ dollars. After six people do not shew up there are $x - 6$ people, and each must pay $\frac{2300}{x} + 7.50$ dollars. We need

$$(x-6)\left(\frac{2300}{x} + 7.50\right) = 2300.$$

Rearranging,

$$(x-6)(2300+7.5x) = 2300x \implies 7.5x^2 - 45x - 13800 = 0 \implies 75x^2 - 450x - 138000 = 0 \implies 75(x+40)(x-46) = 0 \implies x \in \{-40, 46\}.$$

Since x must be positive, there were 46 people on the bus originally.

7.1.8 One has $x = 6 + \frac{1}{x}$ or $x^2 - 6x - 1 = 0.$ Later on you will learn how to solve this equation and see that $x = 3 + \sqrt{10}$. There is some deep mathematics going on here. That the infinite expression makes sense depends on the concept of *convergence*, which will be dealt with in Calculus courses.

7.1.9 We have $x^2 = 2,$ and since x is positive, $x = \sqrt{2}$. Like problem 7.1.8, this problem is deep and depends on the concept of convergence.

8.2.1 $x > -\frac{18}{11}$.

8.2.2 $x \leq -3$.

8.2.3 $x > \frac{7}{5}$.

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