Maths 165

David A. Santos
dsantos@ccp.edu

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Contents

Preface

These notes started the fall of 2004, when I taught Maths 165, Differential Calculus, at Community College of Philadelphia.

The students at that course were Andrea BATEMAN, Kelly BLOCKER, Alexandra LOUIS, Cindy LY, Thoraya SABER, Stephanilee MAHONEY, Brian McCLINTON, Jessica MENDEZ, Labaron PALMER, Leonela TROKA, and Samneak SAK. I would like to thank them for making me a better teacher with their continuous input and questions.

The main goal of these notes is to initiate students in the study of Calculus. Chapter 1 introduces most of the notation used throughout the notes. The central problem is: given a (simple) formula relating two quantities, how can we graphically represent this relationship? This problem is partially answered in Chapter 2, where we derive formulæ for lines, semicircles, parabolas and hyperbolas by means of the distance formula, without the necessity of the machinery of derivatives. Their curves and equations provide then meaningful examples for functions, which are introduced in Chapter 3. Once the basic operations and transformations of functions are presented, and the basic vocabulary for the graph of a function is given, the insufficiency of the methods of Chapter 2 leads us to look at the graphs of functions through the methods of Calculus. The strong derivative of a function is presented in Chapter 4, where theorems regarding its influence on the graph of a function are proven. In Chapter 4 we introduce polynomial functions, the Sum Rule, Product Rule, and the Chain Rule. A few results from the Theory of Equations are proved, via the introduction of Taylor polynomials. In Chapter 5 we introduce rational functions and a few algebraic functions. The Quotient Rule is proved in this Chapter.

I have profitted from conversations with José Mason and Alain Schremmer regarding approaches to teaching this course.

David A. Santos

Please send comments to dsantos@ccp.edu

Things to do

Need to

- Write a chapter on exponential and logarithmic functions.
- Write a chapter on goniometric functions.
- Weave Taylor and McLaurin
- Write a proof of Rolle's Theorem for polynomials, using Taylor polynomials and the Bolzano's Theorem.
- Write a section on Lagrange Interpolation.
- Write a section on Partial Fractions.
- Rewrite the big oh appendix.

Legal Notice

To the Student

These notes are provided for your benefit as an attempt to organise the salient points of the course. They are a *very terse* account of the main ideas of the course, and are to be used mostly to refer to central definitions and theorems. The number of examples is minimal, and here you will not find exercises. The *motivation* or informal ideas of looking at a certain topic, the ideas linking a topic with another, the worked-out examples, etc., are given in class. Hence these notes are not a substitute to lectures: **you must always attend to lectures**. The order of the notes may not necessarily be the order followed in the class.

There is a certain algebraic fluency that is necessary for a course at this level. These algebraic prerequisites would be difficult to codify here, as they vary depending on class response and the topic lectured. If at any stage you stumble in Algebra, seek help! I am here to help you!

Tutoring can sometimes help, but bear in mind that whoever tutors you may not be familiar with my conventions. Again, I am here to help! On the same vein, other books may help, but the approach presented here is at times unorthodox and finding alternative sources might be difficult.

Here are more recommendations:

- Read a section before class discussion, in particular, read the definitions.
- Class provides the informal discussion, and you will profit from the comments of your classmates, as well as gain confidence by providing your insights and interpretations of a topic. **Don't be absent!**
- I encourage you to form study groups and to discuss the assignments. Discuss among yourselves and help each other but don't be *parasites!* Plagiarising your classmates' answers will only lead you to disaster!
- Once the lecture of a particular topic has been given, take a fresh look at the notes of the lecture topic.
- Try to understand a single example well, rather than ill-digest multiple examples.
- Start working on the distributed homework ahead of time.
- **Ask questions during the lecture.** There are two main types of questions that you are likely to ask.
  1. *Questions of Correction: Is that a minus sign there?* If you think that, for example, I have missed out a minus sign or wrote P where it should have been Q,[1] then by all means, ask. No one likes to carry an error till line XLV because the audience failed to point out an error on line I. Don't wait till the end of the class to point out an error. Do it when there is still time to correct it!
  2. *Questions of Understanding: I don't get it!* Admitting that you do not understand something is an act requiring utmost courage. But if you don't, it is likely that many others in the audience also don't. On the same vein, if you feel you can explain a point to an inquiring classmate, I will allow you time in the lecture to do so. The best way to ask a question is something like: "How did you get from the second step to the third step?" or "What does it mean to complete the square?" Asseverations like "I don't understand" do not help me answer your queries. If I consider that you are asking the same questions too many times, it may be that you need extra help, in which case we will settle what to do outside the lecture.
- Don't fall behind! The sequence of topics is closely interrelated, with one topic leading to another.
- You will need square-grid paper, a ruler (preferably a T-square), some needle thread, and a compass.
- The use of calculators is allowed, especially in the occasional lengthy calculations. However, when graphing, you will need to provide algebraic/analytic/geometric support of your arguments. The questions on assignments and exams will be posed in such a way that it will be of no advantage to have a graphing calculator.
- Presentation is critical. Clearly outline your ideas. When writing solutions, outline major steps and write in complete sentences. As a guide, you may try to emulate the style presented in the scant examples furnished in these notes.

[1] My doctoral adviser used to say "I said A, I wrote B, I meant C and it should have been D!

Chapter 1

Numbers

This chapter introduces essential notation and terminology that will be used throughout these notes.

1.1 The Real Line

1 Definition We will mean by a *set* a collection of well defined members or *elements*. A *subset* is a sub-collection of a set. We denote that B is a subset of A by the notation $B \subset A$.

2 Definition Let A be a set. If a belongs to the set A, then we write $a \in A$, read "a is an element of A." If a does not belong to the set A, we write $a \notin A$, read "a is not an element of A."

We denote the set of *natural numbers* $\{0,1,2,\ldots\}$[1] by the symbol $\mathbb{N}$. The natural numbers allow us to count things, and they have the property that addition and multiplication is closed within them: that is, if we add or multiply two natural numbers, we stay within the natural numbers. Observe that this is not true for subtraction and division, since, for example, neither $2-7$ nor $2 \div 7$ are natural numbers. We say that then that the natural numbers enjoy *closure* within multiplication and addition.

By appending the opposite (additive inverse) of every member of $\mathbb{N}$ to $\mathbb{N}$ we obtain the set

$$\mathbb{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}^{2}$$

of *integers*. The closure of multiplication and addition is retained by this extension and now we also have closure under subtraction and we have also the notion of *positivity*. This last property allows us to divide the integers into the *strictly positive*, the *strictly negative* or zero, and hence introduces an *ordering* in the rational numbers by defining $a \leq b$ if and only if $b-a$ is positive.

Enter now in the picture the *rational numbers*, commonly called *fractions*, which we denote by the symbol $\mathbb{Q}$.[3] They are the numbers of the form $\dfrac{a}{b}$ with $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, $b \neq 0$, that is, the division of two integers, with the divisor distinct from zero. Observe that every rational number $\dfrac{a}{b}$ is a solution to the equation (with x as the unknown) $bx - a = 0$. It can be shewn that the rational numbers are precisely those numbers whose decimal representation either is finite (e.g., 0.123) or is periodic (e.g., $0.\overline{123} = 0.123123123\ldots$). Notice that every integer is a rational number, since $\dfrac{a}{1} = a$, for any $a \in \mathbb{Z}$. Upon reaching $\mathbb{Q}$ we have formed a system of numbers having closure for the four arithmetical operations of addition, subtraction, multiplication, or division.[4]

Are there numbers which are not rational numbers? Up until the Pythagoreans[5], the ancient Greeks thought that all numbers were the ratio of two integers. It was then discovered that the length of the hypothenuse of a right triangle having both legs

[1] We follow common European usage and include 0 among the natural numbers.

[2] $\mathbb{Z}$ for the German word *Zählen*, meaning *number*.

[3] $\mathbb{Q}$ for *quotients*.

[4] "Reeling and Writhing, of course, to begin with," the Mock Turtle replied, "and the different branches of Arithmetic–Ambition, Distraction, Uglification, and Derision."

[5] Pythagoras lived approximately from 582 to 500 BC. A legend says that the fact that $\sqrt{2}$ was irrational was secret carefully guarded by the Pythagoreans. One of them betrayed this secret, and hence was assassinated by being drowned from a ship.

of unit length—which is $\sqrt{2}$ in modern notation—could not be represented as the ratio of two integers, that is, that $\sqrt{2}$ is *irrational*[6]. Appending the irrational numbers to the rational numbers we obtain the *real numbers* $\mathbb{R}$.

Observe that $\sqrt{2}$ is a solution to the equation $x^2 - 2 = 0$. A further example is $\sqrt[3]{5}$, which is a solution to the equation $x^5 - 3 = 0$. A more difficult example to visualise is $\sqrt[3]{\sqrt{2}+1}$, which is a solution to $x^6 - 2x^3 - 1 = 0$.[7] Any number which is a solution of an equation of the form $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ is called an *algebraic number*.

A number u is an *upper bound* for a set of numbers A if for all $a \in A$ we have $a \le u$. The smallest such upper bound is called the *supremum* of the set A. Similarly, a number l is a *lower bound* for a set of numbers B if for all $b \in B$ we have $l \le b$. The largest such lower bound is called the *infimum* of the set B. The real numbers have the following property, which we enunciate as an axiom.

3 Axiom (Completeness of $\mathbb{R}$) Any set of real numbers which is bounded above has a supremum. Any set of real numbers which is bounded below has a infimum.

Observe that the rational numbers are not complete. For example, there is no largest rational number in the set

$$\{x \in \mathbb{Q} : x^2 < 2\}$$

since $\sqrt{2}$ is irrational and for any good rational approximation to $\sqrt{2}$ we can always find a better one.

Are there real numbers which are not algebraic? It wasn't till the XIXth century when it was discovered that there were irrational numbers which were not algebraic. These irrational numbers are called *transcendental numbers*. It was later shewn that numbers like π and e are transcendental. In fact, in the XIXth century George Cantor proved that even though $\mathbb{N}$ and $\mathbb{R}$ are both infinite sets, their infinities are in a way "different" because they cannot be put into a one-to-one correspondence.

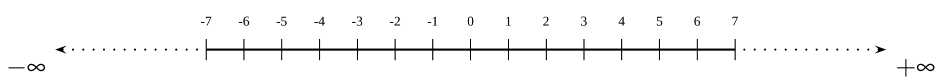

Figure 1.1: The Real Line.

Geometrically, each real number can be viewed as a point on a straight line. We make the convention that we orient the real line with 0 as the origin, the positive numbers increasing towards the right from 0 and the negative numbers decreasing towards the left of 0, as in figure 1.1. We append the object $+\infty$, which is larger than any real number, and the object $-\infty$, which is smaller than any real number. Letting $x \in \mathbb{R}$, we make the following conventions.

$$(+\infty) + (+\infty) = +\infty \tag{1.1}$$

$$(-\infty) + (-\infty) = -\infty \tag{1.2}$$

$$x + (+\infty) = +\infty \tag{1.3}$$

$$x + (-\infty) = -\infty \tag{1.4}$$

[6] An irrational number is thus one that cannot be written as the quotient of two integers $\frac{a}{b}$ with $b \neq 0$.

[7] To see this, work backwards: if $x = \sqrt[3]{\sqrt{2}+1}$, then $x^3 = \sqrt{2}+1$, which gives $(x^3-1)^2 = 2$, which is $x^6 - 2x^3 - 1 = 0$.

$$x(+\infty) = +\infty \quad \text{if } x > 0 \tag{1.5}$$

$$x(+\infty) = -\infty \quad \text{if } x < 0 \tag{1.6}$$

$$x(-\infty) = -\infty \quad \text{if } x > 0 \tag{1.7}$$

$$x(-\infty) = +\infty \quad \text{if } x < 0 \tag{1.8}$$

$$\frac{x}{\pm\infty} = 0 \tag{1.9}$$

Observe that we leave the following undefined:

$$\frac{\pm\infty}{\pm\infty}, \quad (+\infty)+(-\infty), \quad 0(\pm\infty).$$

The square of every real number x is positive[8], that is, for all real numbers x we have $x^2 \geq 0$. Introducing the object $i = \sqrt{-1}$—whose square satisfies $i^2 = -1$, a negative number—and considering the numbers of the form $a+bi$, with a and b real numbers, we obtain the *complex numbers* $\mathbb{C}$.

In summary we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

1.2 Intervals

4 Definition An *interval* I is a subset of the real numbers with the following property: if $s \in I$ and $t \in I$, and if $s < x < t$, then $x \in I$. In other words, intervals are those subsets of real numbers with the property that every number between two elements is also contained in the set. Since there are infinitely many decimals between two different real numbers, intervals with distinct endpoints contain infinitely many members. Table 1.1 shews the various types of intervals.

Observe that we indicate that the endpoints are included by means of shading the dots at the endpoints and that the endpoints are excluded by not shading the dots at the endpoints. [9]

1.3 Neighbourhood of a point

Before stating the main definition of this section, let us consider the concept of "nearness." What does it mean for one point to be "near" another point? We could argue that 1 is near to 0, but, for some purposes, this distance could be "far." We could certainly see that 0.5 is closer to 0 than 1 is, but then again, for some purposes, even this distance could be "far." Mentioning a specific number "near" 0, like 1 or 0.5 fails in what we desire for "nearness" because mentioning a specific point immediately gives a "static" quality to "nearness": once you mention a specific point, you could mention infinitely many more points which are closer than the point you mentioned. The points in the sequence

$$0.1, \qquad 0.01, \qquad 0.001, \qquad 0.0001, \qquad \ldots$$

get closer and closer to 0 with an arbitrary precision. Notice that this sequence approaches 0 through values > 0. This arbitrary precision is what will be the gist of our concept of "nearness." "Nearness" is dynamic: it involves the ability of getting closer to a point with any desired degree of accuracy. It is not static.

Again, the points in the sequence

$$-\frac{1}{2}, \qquad -\frac{1}{4}, \qquad -\frac{1}{8}, \qquad -\frac{1}{16}, \qquad \ldots$$

[8] We use the word *positive* to indicate a quantity ≥ 0, and use the term *strictly positive* for a quantity > 0. Similarly with *negative* (≤ 0) and *strictly negative* (< 0).

[9] It may seem like a silly analogy, but think that in $[a;b]$ the brackets are "arms" "hugging" a and b, but in $]a;b[$ the "arms" are repulsed. "Hugging" is thus equivalent to including the endpoint, and "repulsing" is equivalent to excluding the endpoint.

Interval Notation	**Set Notation**	**Graphical Representation**
$[a;b]$	$\{x \in \mathbb{R} : a \leq x \leq b\}$[10]	
$]a;b[$	$\{x \in \mathbb{R} : a < x < b\}$	
$[a;b[$	$\{x \in \mathbb{R} : a \leq x < b\}$	
$]a;b]$	$\{x \in \mathbb{R} : a < x \leq b\}$	
$]a;+\infty[$	$\{x \in \mathbb{R} : x > a\}$	
$[a;+\infty[$	$\{x \in \mathbb{R} : x \geq a\}$	
$]-\infty;b[$	$\{x \in \mathbb{R} : x < b\}$	
$]-\infty;b]$	$\{x \in \mathbb{R} : x \leq b\}$	
$]-\infty;+\infty[$	$\mathbb{R}$	

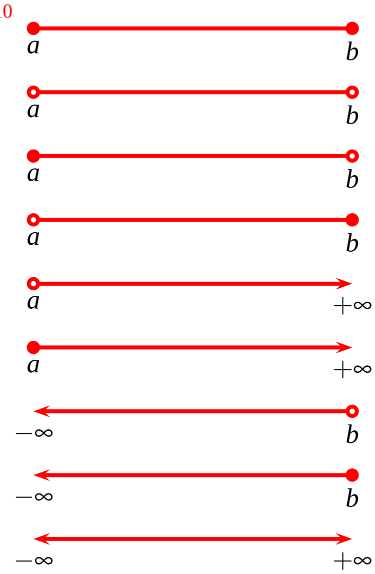

Table 1.1: Intervals.

are arbitrarily close to 0, but they "approach" 0 from the left. Once again, the sequence

$$+\frac{1}{2}, \quad -\frac{1}{3}, \quad +\frac{1}{4}, \quad -\frac{1}{5}, \quad \ldots$$

approaches 0 from both above and below. After this long preamble, we may formulate our first definition.

5 Definition The notation $x \to a$, read "x tends to a," means that x is very close, with an arbitrary degree of precision, to a. Here x can approach a through values smaller or larger than a. We write $x \to a+$ (read "x tends to a from the right") to mean that x approaches a through values larger than a and we write $x \to a-$ (read "x tends to a from the left") we mean that x approaches a through values smaller than a.

6 Definition A *neighbourhood of a point a* is an interval containing a.

Notice that the definition of neighbourhood does not rule out the possibility that a may be an endpoint of the the interval. Our interests will be mostly on arbitrarily small neighbourhoods of a point. Schematically we have a diagram like figure 1.2.

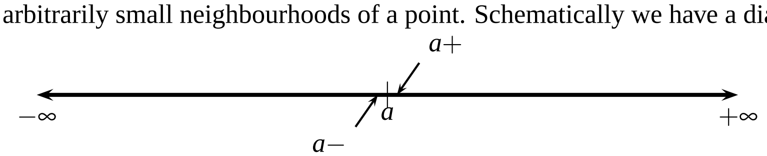

Figure 1.2: A neighbourhood of a.

1.4 Miscellaneous Notation

We will often use the symbol $\iff$ for "if and only if", and the symbol $\implies$, "implies." The symbol $\approx$ means *approximately*. From time to time we use the set theoretic notation below.

7 Definition The *union* of two sets A and B, is the set

$$A \cup B = \{x : (x \in A) \text{ or } (x \in B)\}.$$

This is read "A union B." See figure 1.3.

The *intersection* of two sets A and B, is

$$A \cap B = \{x : (x \in A) \text{ and } (x \in B)\}.$$

This is read "A intersection B." See figure 1.4.

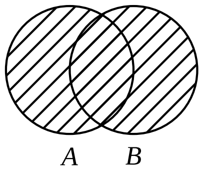

Figure 1.3: $A \cup B$

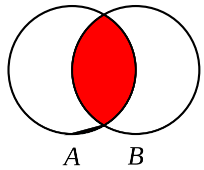

Figure 1.4: $A \cap B$

8 Example If $A = [-10; 2]$, $B =]-\infty; 1[$, then

$$A \cap B = [-10; 1[, \quad A \cup B =]-\infty; 2].$$

Chapter 2

Distance and Curves on the Plane

The main objective of this chapter is to introduce the distance formula for two points on the plane, and by means of this distance formula, the linking of certain equations with certain curves on the plane. Thus the main object of these notes, that of relating a graph to a formula, is partially answered.

2.1 Distance on the Real Line

9 Definition Let $x \in \mathbb{R}$. The *absolute value of* x—denoted by $|x|$—is defined by

$$\boxed{|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}}$$

The absolute value of a real number is thus the distance of that real number to 0, and hence $|x-y|$ is the distance between x and y on the real line. Below are some properties of the absolute value. Here x, y, t are all real numbers.

$$-|x| \leq x \leq |x|. \tag{2.1}$$

$$|x-y| = |y-x| \tag{2.2}$$

$$\sqrt{x^2} = |x| \tag{2.3}$$

$$|x|^2 = |x^2| = x^2 \tag{2.4}$$

$$|x| \leq t \iff -t \leq x \leq t \quad (t \geq 0) \tag{2.5}$$

$$|x| \geq t \iff x \leq -t \text{ or } x \geq t \quad (t \geq 0) \tag{2.6}$$

$$|x+y| \leq |x| + |y| \tag{2.7}$$

$$||x| - |y|| \leq |x-y| \tag{2.8}$$

2.2 Distance on the Real Plane

We now turn our attention to the plane, which we denote by the symbol $\mathbb{R}^2$.

Consider two points $A = (x_1, y_1), B = (x_2, y_2)$ on the Cartesian plane, as in figure 2.1. Dropping perpendicular lines to C, as in the figure, we can find their Euclidean distance AB with the aid of the Pythagorean Theorem. For

$$AB^2 = AC^2 + BC^2,$$

translates into

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This motivates the following definition.

10 Definition Let $(x_1, y_1), (x_2, y_2)$ be points on the Cartesian plane. The *Euclidean distance* between them is given by

$$\boxed{\mathbf{d}\langle (x_1, y_1), (x_2, y_2) \rangle = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.} \tag{2.9}$$

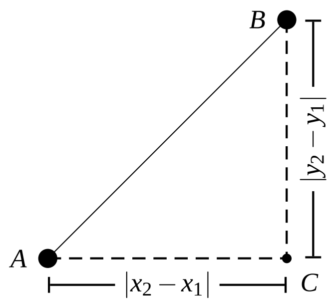

Figure 2.1: Distance between two points.

11 Example Find the Euclidean distance between $(-1, 2)$ and $(-3, 8)$.

Solution:

$$\mathbf{d}\langle (-1, 2), (-3, -8) \rangle = \sqrt{(-1 - (-3))^2 + (2 - 8)^2} = \sqrt{40} = 2\sqrt{10} \approx 6.32.$$

2.3 Circles and Semicircles

We now study our first curve on the plane: the circle. We will see that the *equation* of a circle on the plane is a consequence of the distance formula 2.9.

Here is a way to draw a circle on sand: using a string, tie it to what you wish to be the centre of the circle. Tighten up the string now and trace the path followed by the other extreme of the string. You now have a circle, whose radius is the length of the string. Notice then that every point on the circumference is at a fixed distance from the centre. This motivates the following.

12 Theorem A circle on the plane with radius R and centre (x_0, y_0) has equation

$$(x - x_0)^2 + (y - y_0)^2 = R^2, \tag{2.10}$$

called the *canonical equation* for a circle of radius R and centre (x_0, y_0). Conversely, the graph any equation of the form 2.10 is a circle.

Proof: *The point* (x,y) *belongs to circle of radius* R *and centre* (x_0,y_0)

$$\begin{aligned}
\iff \quad & \mathbf{d}\langle(x,y),(x_0,y_0)\rangle &= \quad & R \\
\iff \quad & \sqrt{(x-x_0)^2+(y-y_0)^2} &= \quad & R \quad , \\
\iff \quad & (x-x_0)^2+(y-y_0)^2 &= \quad & R^2
\end{aligned}$$

giving the desired result. ❑

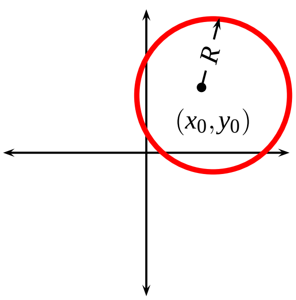

Figure 2.2: A circle with centre (x_0,y_0) and radius R.

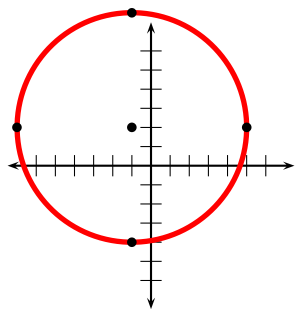

Figure 2.3: Example 13.

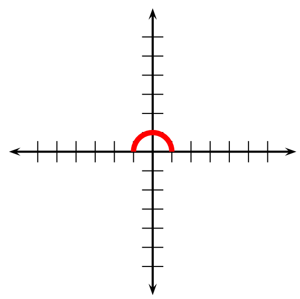

Figure 2.4: Example 14.

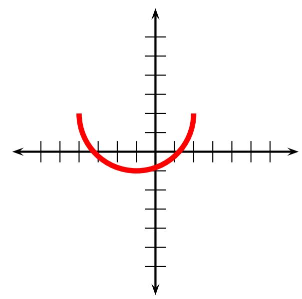

Figure 2.5: Example 15.

13 Example The equation of the circle with centre $(-1,2)$ and radius 6 is $(x+1)^2+(y-2)^2=36$. Observe that the points $(-1\pm 6,2)$ and $(-1,2\pm 6)$ are on the circle. Thus $(-7,2)$ is the left-most point on the circle, $(5,2)$ is the right-most, $(-1,-4)$ is the lower-most, and $(-1,8)$ is the upper-most. The circle is shewn in figure 2.3.

Solving for y in $(x-x_0)^2+(y-y_0)^2=R^2$, we obtain

$$y = y_0 \pm \sqrt{R^2-(x-x_0)^2}.$$

The choice of the $+$ sign gives the upper half of the circle (the upper semicircle) and the $-$ sign gives the lower semicircle.

14 Example Sketch the curve $y=\sqrt{1-x^2}$

Solution: Squaring, $y^2=1-x^2$. Hence $x^2+y^2=1$. This is the equation of a circle with centre at $(0,0)$ and radius 1. The original equation describes the upper semicircle (since $y\geq 0$). The graph is shewn in figure 2.4.

15 Example Sketch the curve $y=2-\sqrt{8-x^2-2x}$

Solution: We have $y-2=-\sqrt{8-x^2-2x}$. Squaring, $(y-2)^2=8-x^2-2x$. Hence, by completing squares,

$$x^2+2x+1+(y-2)^2=9 \implies (x+1)^2+(y-2)^2=9.$$

This is the equation of a circle with centre at $(-1,2)$ and radius 3. The original equation describes the lower semicircle (since $y\leq 2$). The graph is shewn in figure 2.5.

2.4 Lines

16 Definition Let a and b be real number constants. A *vertical line* on the plane is a set of the form

$$\{(x,y): x=a\}.$$

Similarly, a *horizontal line* on the plane is a set of the form

$$\{(x,y) : y = b\}.$$

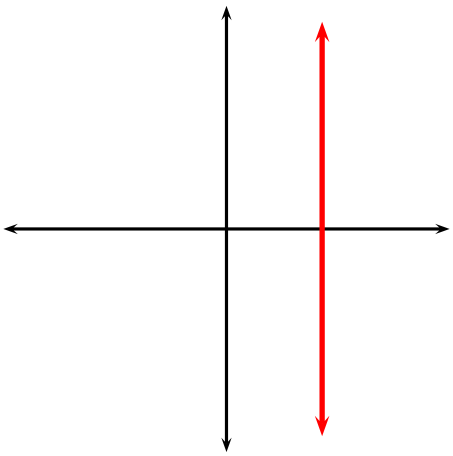

Figure 2.6: A vertical line.

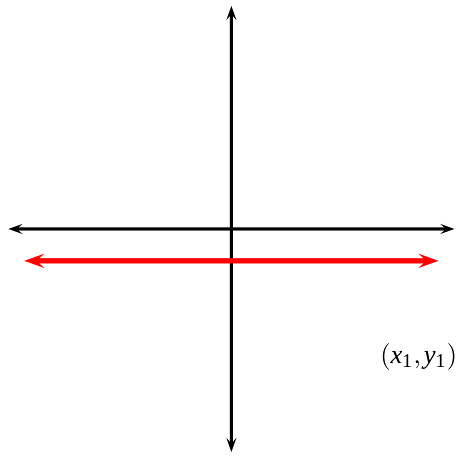

Figure 2.7: A horizontal line.

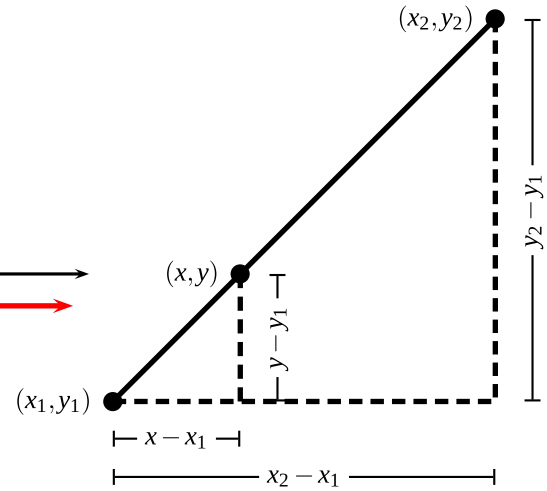

Figure 2.8: Theorem 17.

17 Theorem The equation of any non-vertical line on the plane can be written in the form $y = mx + k$, where m and k are real number constants. Conversely, any equation of the form $y = ax + b$, where a, b are fixed real numbers has as a line as a graph.

Proof: *If the line is parallel to the x-axis, that is, if it is horizontal, then it is of the form $y = b$, where b is a constant and so we may take $m = 0$ and $k = b$. Consider now a line non-parallel to any of the axes, as in figure 2.8, and let (x,y), (x_1,y_1), (x_2,y_2) be three given points on the line. By similar triangles we have*

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1},$$

which, upon rearrangement, gives

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x - x_1 \left(\frac{y_2 - y_1}{x_2 - x_1}\right) + y_1,$$

and so we may take

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad k = -x_1 \left(\frac{y_2 - y_1}{x_2 - x_1}\right) + y_1.$$

Conversely, consider real numbers $x_1 < x_2 < x_3$, and let $P = (x_1, ax_1 + b)$, $Q = (x_2, ax_2 + b)$, and $R = (x_3, ax_3 + b)$ be on the graph of the equation $y = ax + b$. We will shew that

$$\mathbf{d}\langle P, Q\rangle + \mathbf{d}\langle Q, R\rangle = \mathbf{d}\langle P, R\rangle.$$

Since the points P, Q, R are arbitrary, this means that any three points on the graph of the equation $y = ax + b$ are collinear, and so this graph is a line. Then

$$\begin{aligned}
\mathbf{d}\langle P, Q\rangle &= \sqrt{(x_2 - x_1)^2 + (ax_2 - ax_1)^2} = |x_2 - x_1|\sqrt{1 + a^2} = (x_2 - x_1)\sqrt{1 + a^2},\\
\mathbf{d}\langle Q, R\rangle &= \sqrt{(x_3 - x_2)^2 + (ax_3 - ax_2)^2} = |x_3 - x_2|\sqrt{1 + a^2} = (x_3 - x_2)\sqrt{1 + a^2},\\
\mathbf{d}\langle P, Q\rangle &= \sqrt{(x_3 - x_1)^2 + (ax_3 - ax_1)^2} = |x_3 - x_1|\sqrt{1 + a^2} = (x_3 - x_1)\sqrt{1 + a^2},
\end{aligned}$$

from where

$$\mathbf{d}\langle P, Q\rangle + \mathbf{d}\langle Q, R\rangle = \mathbf{d}\langle P, R\rangle$$

follows. This means that the points P, Q, and R lie on a straight line, which finishes the proof of the theorem. ❏

The quantity $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ *in Theorem 17 is the* slope of the line joining (x_1, y_1) and (x_2, y_2). *Since* $y = m(0) + k$, *the quantity* k *is the* y-intercept of the line joining (x_1, y_1) and (x_2, y_2).

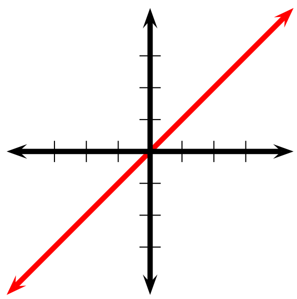

Figure 2.9: Example 18.

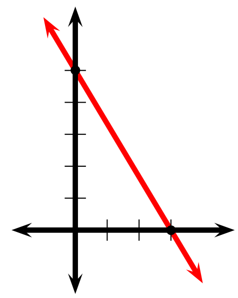

Figure 2.10: Example 19.

18 Example By Theorem 17, the equation $y = x$ represents a line with slope 1 and passing through the origin. Since $y = x$, the line makes a 45° angle with the x-axis, and bisects quadrants I and III. See figure 2.9

19 Example A line passes through $(-3, 10)$ and $(6, -5)$. Find its equation and draw it.

Solution: The equation is of the form $y = mx + k$. We must find the slope and the y-intercept. To find m we compute the ratio

$$m = \frac{10 - (-5)}{-3 - 6} = -\frac{5}{3}.$$

Thus the equation is of the form $y = -\frac{5}{3}x + k$ and we must now determine k. To do so, we substitute either point, say the first, into $y = -\frac{5}{3}x + k$ obtaining $10 = -\frac{5}{3}(-3) + k$, whence $k = 5$. The equation sought is thus $y = -\frac{5}{3}x + 5$. To draw the graph, first locate the y-intercept (at $(0, 5)$). Since the slope is $-\frac{5}{3}$, move five units down (to $(0, 0)$) and three to the right (to $(3, 0)$). Connect now the points $(0, 5)$ and $(3, 0)$. The graph appears in figure 2.10.

2.4.1 Parallel and Perpendicular Lines

The material here will be needed for example 82 and so it is optional if this example is omitted.

20 Definition Two lines are parallel if they have the same slope.

21 Example Find the equation of the line passing through $(4, 0)$ and parallel to the line joining $(-1, 2)$ and $(2, -4)$.

Solution: First we compute the slope of the line joining $(-1, 2)$ and $(2, -4)$:

$$m = \frac{2 - (-4)}{-1 - 2} = -2.$$

The line we seek is of the form $y = -2x + k$. We now compute the y-intercept, using the fact that the line must pass through $(4, 0)$. This entails solving $0 = -2(4) + k$, whence $k = 8$. The equation sought is finally $y = -2x + 8$.

22 Theorem Let $y = mx + k$ be a line non-parallel to the axes. If the line $y = m_1 x + k_1$ is perpendicular to $y = mx + k$ then $m_1 = -\frac{1}{m}$.

Proof: *Refer to figure 2.11. Since we may translate lines without affecting the angle between them, we assume without loss of generality that both $y = mx + k$ and $y = m_1x + k_1$ pass through the origin, giving thus $k = k_1 = 0$. Now, the line $y = mx$ meets the vertical line $x = 1$ at $(1, m)$ and the line $y = m_1x$ meets this same vertical line at $(1, m_1)$ (see figure 2.11). By the Pythagorean Theorem*

$$(m - m_1)^2 = (1 + m^2) + (1 + m_1^2).$$

Upon simplifying we gather that $mm_1 = -1$, which proves the assertion. ❑

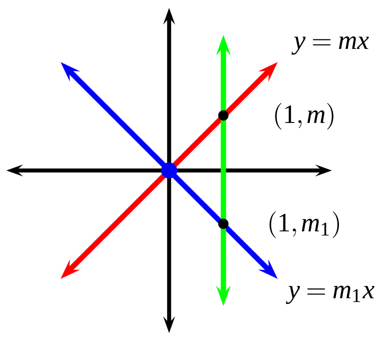

Figure 2.11: Theorem 22.

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23 Example Find the equation of the line passing through $(4, 0)$ and perpendicular to the line joining $(-1, 2)$ and $(2, -4)$.

Solution: By the preceding problem, the slope of the line joining $(-1, 2)$ and $(2, -4)$ is -2. The slope of the perpendicular line is

$$m_1 = -\frac{1}{m} = \frac{1}{2}.$$

The equation sought has the form $y = \frac{x}{2} + k$. We find the y-intercept by solving $0 = \frac{4}{2} + k$, whence $k = -2$. The equation of the perpendicular line is thus $y = \frac{x}{2} - 2$.

24 Example For a given real number t, associate the straight line L_t with the equation

$$L_t : \quad (4 - t)y = (t + 2)x + 6t.$$

❶ Determine t so that the L_t be parallel to the x-axis and determine the equation of the resulting line.

❷ Determine t so that the L_t be parallel to the y-axis and determine the equation of the resulting line.

❸ Determine t so that the L_t be parallel to the line $-5y = 3x - 1$.

❹ Determine t so that the L_t be perpendicular to the line $-5y = 3x - 1$.

❺ Is there a point (a, b) belonging to every line L_t regardless of the value of t?

Solution:

❶ We need $t + 2 = 0 \implies t = -2$. In this case

$$(4 - (-2))y = -12 \implies y = -2.$$

❷ We need $4 - t = 0 \implies t = 4$. In this case

$$0 = (4 + 2)x + 24 \implies x = -4.$$

❸ The slope of L_t is

$$\frac{t+2}{4-t},$$

and the slope of the line $-5y = 3x - 1$ is $-\frac{3}{5}$. Therefore we need

$$\frac{t+2}{4-t} = -\frac{3}{5} \implies -3(4-t) = 5(t+2) \implies t = -11.$$

❹ In this case we need

$$\frac{t+2}{4-t} = \frac{5}{3} \implies 5(4-t) = 3(t+2) \implies t = \frac{7}{4}.$$

❺ Yes. From above, the obvious candidate is $(-4,-2)$. To verify this observe that

$$(4-t)(-2) = (t+2)(-4) + 6t,$$

regardless of the value of t.

2.5 Parabolas

25 Definition A *parabola* is the collection of all the points on the plane whose distance from a fixed point F (called the *focus* of the parabola) is equal to the distance to a fixed line L (called the *directrix* of the parabola). See figure 2.12, where $FD = DP$.

We can draw a parabola as follows. Cut a piece of thread as long as the trunk of T-square (see figure 2.13). Tie one end to the end of the trunk of the T-square and tie the other end to the focus, say, using a peg. Slide the crosspiece of the T-square along the directrix, while maintaining the thread tight against the ruler with a pencil.

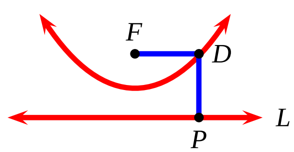

Figure 2.12: Definition of a parabola.

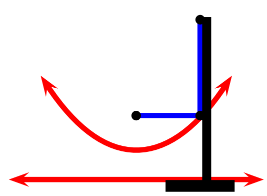

Figure 2.13: Drawing a parabola.

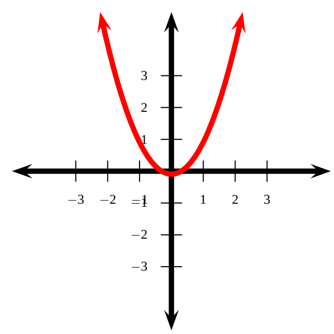

Figure 2.14: Example 27.

26 Theorem Let $d > 0$ be a real number. The equation of a parabola with focus at $(0,d)$ and directrix $y = -d$ is $y = \frac{x^2}{4d}$.

Proof: *Let (x,y) be an arbitrary point on the parabola. Then the distance of (x,y) to the line $y = -d$ is $|y+d|$. The distance of (x,y) to the point $(0,d)$ is $\sqrt{x^2+(y-d)^2}$. We have*

$$\begin{aligned}
|y+d| = \sqrt{x^2+(y-d)^2} \quad &\implies \quad (|y+d|)^2 = x^2 + (y-d)^2 \\
&\implies \quad y^2 + 2yd + d^2 = x^2 + y^2 - 2yd + d^2 \\
&\implies \quad 4dy = x^2 \\
&\implies \quad y = \frac{x^2}{4d},
\end{aligned}$$

as wanted. ❑

Observe that the midpoint of the perpendicular line segment from the focus to the directrix is on the parabola. We call this point the vertex. *For the parabola* $y = \frac{x^2}{4d}$ *of Theorem 26, the vertex is clearly* $(0,0)$.

27 Example Draw the parabola $y = x^2$.

Solution: From Theorem 26, we want $\frac{1}{4d} = 1$, that is, $d = \frac{1}{4}$. Following Theorem 26, we locate the focus at $(0, \frac{1}{4})$ and the directrix at $y = -\frac{1}{4}$ and use a T-square with these references. The vertex of the parabola is at $(0,0)$. The graph is in figure 2.14.

2.6 Hyperbolas

28 Definition A *hyperbola* is the collection of all the points on the plane whose absolute value of the difference of the distances from two distinct fixed points F_1 and F_2 (called the *foci*[1] of the hyperbola) is a positive constant. See figure 2.15, where $|F_1D - F_2D| = |F_1D' - F_2D'|$.

We can draw a hyperbola as follows. Put tacks on F_1 and F_2 and measure the distance F_1F_2. Attach piece of thread to one end of the ruler, and the other to F_2, while letting the other end of the ruler to pivot around F_1. The lengths of the ruler and the thread must satisfy

$$\text{length of the ruler} - \text{length of the thread} < F_1F_2.$$

Hold the pencil against the side of the rule and tighten the thread, as in figure 2.16.

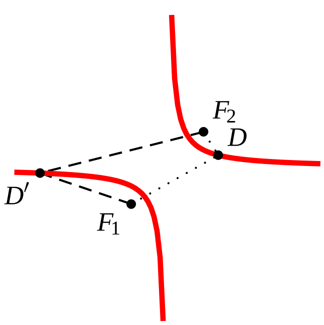

Figure 2.15: Definition of a hyperbola.

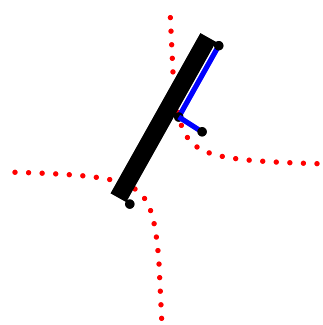

Figure 2.16: Drawing a hyperbola.

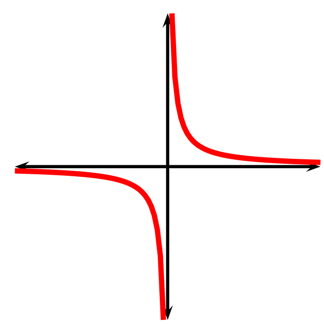

Figure 2.17: The hyperbola $y = \frac{1}{x}$.

29 Theorem Let $c > 0$ be a real number. The hyperbola with foci at $F_1 = (-c, -c)$ and $F_2 = (c, c)$, and whose absolute value of the difference of the distances from its points to the foci is $2c$ has equation $xy = \frac{c^2}{2}$.

[1] *Foci* is the plural of *focus*.

Proof: *Let (x,y) be an arbitrary point on the hyperbola. Then*

$$\begin{aligned}
&|\mathbf{d}\langle(x,y),(-c,-c)\rangle - \mathbf{d}\langle(x,y),(c,c)\rangle| = 2c \\
\iff & \left|\sqrt{(x+c)^2+(y+c)^2} - \sqrt{(x-c)^2+(y-c)^2}\right| = 2c \\
\iff & (x+c)^2+(y+c)^2+(x-c)^2+(y-c)^2 - 2\sqrt{(x+c)^2+(y+c)^2}\cdot\sqrt{(x-c)^2+(y-c)^2} = 4c^2 \\
\iff & 2x^2+2y^2 = 2\sqrt{(x^2+y^2+2c^2)+(2xc+2yc)}\cdot\sqrt{(x^2+y^2+2c^2)-(2xc+2yc)} \\
\iff & 2x^2+2y^2 = 2\sqrt{(x^2+y^2+2c^2)^2-(2xc+2yc)^2} \\
\iff & (2x^2+2y^2)^2 = 4\left((x^2+y^2+2c^2)^2-(2xc+2yc)^2\right) \\
\iff & 4x^4+8x^2y^2+4y^4 = 4((x^4+y^4+4c^4+2x^2y^2+4y^2c^2+4x^2c^2)-(4x^2c^2+8xyc^2+4y^2c^2)) \\
\iff & xy = \frac{c^2}{2},
\end{aligned}$$

where we have used the identities

$$(A+B+C)^2 = A^2+B^2+C^2+2AB+2AC+2BC \qquad \text{and} \qquad \sqrt{A-B}\cdot\sqrt{A+B} = \sqrt{A^2-B^2}.$$

❑

Observe that the points $\left(-\frac{c}{\sqrt{2}}, -\frac{c}{\sqrt{2}}\right)$ *and* $\left(\frac{c}{\sqrt{2}}, \frac{c}{\sqrt{2}}\right)$ *are on the hyperbola* $xy = \frac{c^2}{2}$. *We call these points the* vertices[2] *of the hyperbola* $xy = \frac{c^2}{2}$.

30 Example To draw the hyperbola $y = \frac{1}{x}$ we proceed as follows. According to Theorem 29, its two foci are at $(-\sqrt{2}, -\sqrt{2})$ and $(\sqrt{2}, \sqrt{2})$. Put

$$\text{length of the ruler} - \text{length of the thread} = 2\sqrt{2}.$$

By alternately pivoting about these points using the procedure above, we get the picture in figure 4.11.

[2] *Vertices* is the plural of *vertex*.

Chapter 3

Functions

This chapter introduces the central concept of a function. We concentrate on real-valued functions whose domains are subsets of the real numbers. We will use the curves obtained in the last chapter as examples to see how various transformations affect the graph of a function.

3.1 Functions

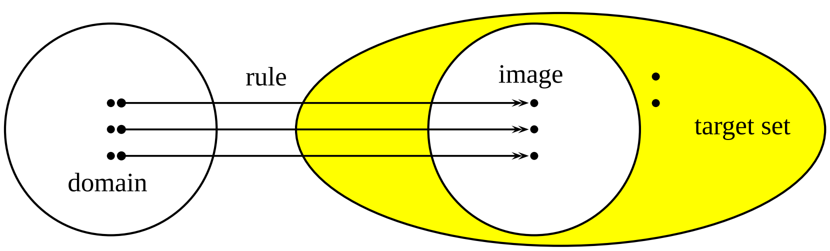

Figure 3.1: The main ingredients of a function.

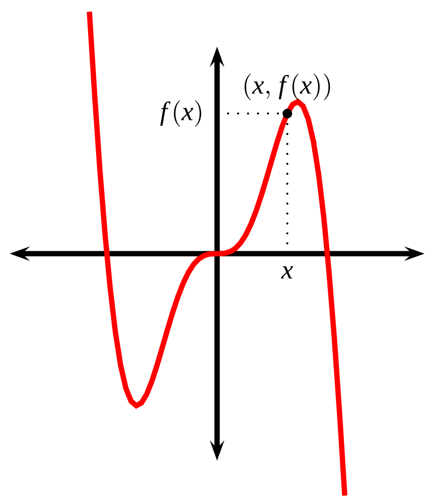

Figure 3.2: The graph of a function.

31 Definition By a *function* $f : \mathbf{Dom}(f) \to \mathbf{Target}(f)$ we mean the collection of the following ingredients:

❶ a *name* for the function. Usually we use the letter f.

❷ a set of real number inputs—usually an interval or a finite union of intervals—called the *domain* of the function. The domain of f is denoted by $\mathbf{Dom}(f)$.

❸ an *input parameter* , also called *independent variable* or *dummy variable*. We usually denote a typical input by the letter x.

❹ a set of possible real number outputs—usually an interval or a finite union of intervals—of the function, called the *target set* of the function. The target set of f is denoted by $\mathbf{Target}(f)$.

❺ an *assignment rule* or *formula*, assigning to **every input** a **unique** output. This assignment rule for f is usually denoted by $x \mapsto f(x)$. The output of x under f is also referred to as the *image of x under f*, and is denoted by $f(x)$.

The notation[1]

$$f: \begin{array}{ccc} \mathbf{Dom}\,(f) & \to & \mathbf{Target}\,(f) \\ x & \mapsto & f(x) \end{array}$$

read "the function f, with domain $\mathbf{Dom}\,(f)$, target set $\mathbf{Target}\,(f)$, and assignment rule f mapping x to $f(x)$" conveys all the above ingredients. See figure 3.1.

32 Definition The *graph* of a function $f: \begin{array}{ccc} \mathbf{Dom}\,(f) & \to & \mathbb{R} \\ x & \mapsto & f(x) \end{array}$ is the set $\{(x,y) \in \mathbb{R}^2 : y = f(x)\}$ on the plane. For ellipsis, we usually say *the graph of* f, or *the graph* $y = f(x)$ or the *the curve* $y = f(x)$. See figure 3.2.

From now on, unless otherwise stated, we will take $\mathbb{R}$ as the target set of all the functions below.

It must be emphasised that the uniqueness of the image of an element of the domain is crucial. For example, the diagram in figure 3.3 *does not* represent a function. The element 1 in the domain is assigned to more than one element of the target set. Also important in the definition of a function is the fact that *all the elements* of the domain must be operated on. For example, the diagram in 3.4 *does not* represent a function. The element 3 in the domain is not assigned to any element of the target set. Also, by the definition of the graph of a function, the x-axis contains the set of inputs and y-axis has the set of outputs. Therefore, if a vertical line crosses two or more points of a graph, the graph does not represent a function. See figures 3.5 and 3.6.

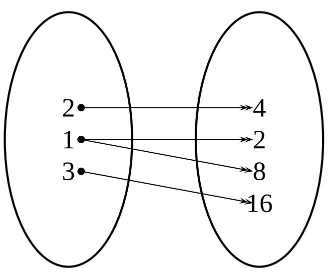

Figure 3.3: Not a function.

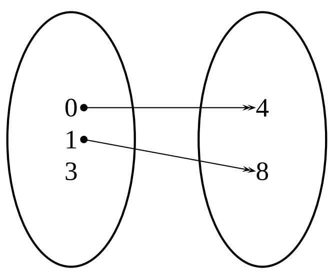

Figure 3.4: Not a function.

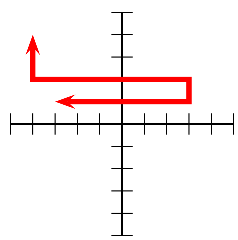

Figure 3.5: Not a function.

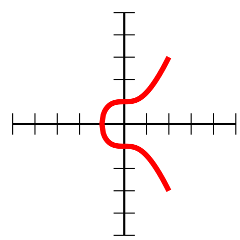

Figure 3.6: Not a function.

33 Example (The Identity Function) Consider the function

$$\mathbf{Id}\ : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array}.$$

This function assigns to every real its own value. Thus $\mathbf{Id}\,(-1) = -1$, $\mathbf{Id}\,(0) = 0$, $\mathbf{Id}\,(4) = 4$, etc. By Theorem 17, the graph of identity function is a straight line, and it is given in figure 3.7.

34 Example (The Square Function) Consider the function

$$\mathbf{Id}^{\,2} : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}.$$

[1] Notice the difference in the arrows. The straight arrow $\longrightarrow$ is used to mean that a certain set is associated with another set, whereas the arrow $\mapsto$ (read "maps to") is used to denote that an input becomes a certain output.

This function assigns to every real its square. Thus $\mathbf{Id}^2(-1) = 1$, $\mathbf{Id}^2(0) = 0$, $\mathbf{Id}^2(2) = 4$, etc. By Theorem 26, the graph of the square function is given in figure 3.8.

For ellipsis, we usually refer to the identity function $\mathbf{Id} : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array}$ *as "the function* **Id** *" or "the function* $x \mapsto x$*." Similarly, in situations when the domain of a function is not in question, we will simply give the assignment rule or the name of the function. So we will speak of "the function* f*" or "the function* $x \mapsto f(x)$*," e.g., "the function* $\mathbf{Id}^2$*" or "the function* $x \mapsto x^2$*."*

35 Example Consider the function[2]

$$f : \begin{array}{ccc} [-1;1] & \to & \mathbb{R} \\ x & \mapsto & \sqrt{1-x^2} \end{array} .$$

Then $f(-1) = 0$, $f(0) = 1$, $f\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2} \approx .866$, etc. By Example 14, the graph of f is the upper unit semicircle, which is shewn in figure 3.9.

36 Example (The Reciprocal function) Consider the function[3]

$$g : \begin{array}{ccc} \mathbb{R} \setminus \{0\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{1}{x} \end{array} .$$

Then $g(-1) = -1$, $g(1) = 1$, $g\left(\frac{1}{2}\right) = 2$, etc. By Example 30, the graph of g is the hyperbola shewn in figure 3.10.

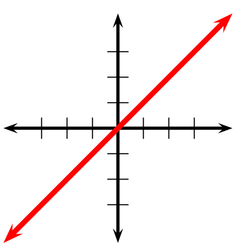

Figure 3.7: **Id**

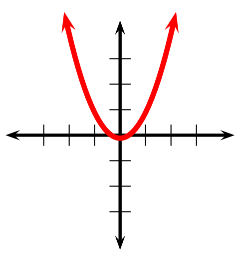

Figure 3.8: $\mathbf{Id}^2$

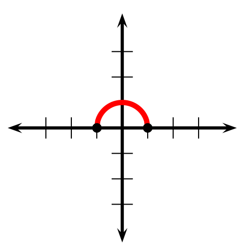

Figure 3.9: $x \mapsto \sqrt{1-x^2}$

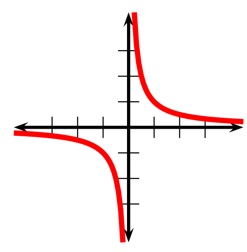

Figure 3.10: $x \mapsto \frac{1}{x}$

3.2 Piecewise Functions

Sometimes the assignment rule of a function varies from interval to interval. We call any such function a *piecewise function*.

37 Example A function f is only defined for $x \in [-4;4]$, and it is made of straight lines, as in figure 3.11. Find a piecewise formula for f.

Solution: The first line segment $\mathscr{L}_1$ has slope

$$\text{slope } \mathscr{L}_1 = \frac{1-(-3)}{-1-(-4)} = \frac{4}{3},$$

[2] Since we are concentrating exclusively on real-valued functions, the formula for f only makes sense in the interval $[-1;1]$.

[3] g only makes sense when $x \neq 0$.

and so the equation of the line containing this line segment is of the form $y = \frac{4}{3}x + k_1$. Since $(-1,1)$ is on the line, $1 = -\frac{4}{3} + k_1 \implies k_1 = \frac{7}{3}$, so this line segment is contained in the line $y = \frac{4}{3}x + \frac{7}{3}$. The second line segment $\mathscr{L}_2$ has slope

$$\text{slope } \mathscr{L}_2 = \frac{1-1}{2-(-1)} = 0,$$

and so this line segment is contained in the line $y = 1$. Finally, the third line segment $\mathscr{L}_3$ has slope

$$\text{slope } \mathscr{L}_3 = \frac{-5-1}{4-2} = -3,$$

and so this line segment is part of the line of the form $y = -3x + k_2$. Since $(1,2)$ is on the line, we have $2 = -3 + k_2 \implies k_2 = 5$, and so the line segment is contained on the line $y = -3x + 5$. Upon assembling all this we see that the piecewise function required is

$$f(x) = \begin{cases} \frac{4}{3}x + \frac{7}{3} & \text{if } x \in [-4;-1] \\ 1 & \text{if } x \in [-1;2] \\ -3x+5 & \text{if } x \in [2;4] \end{cases}$$

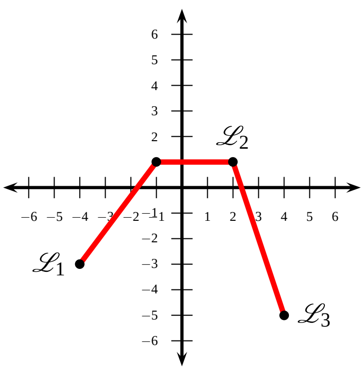

Figure 3.11: Example 37.

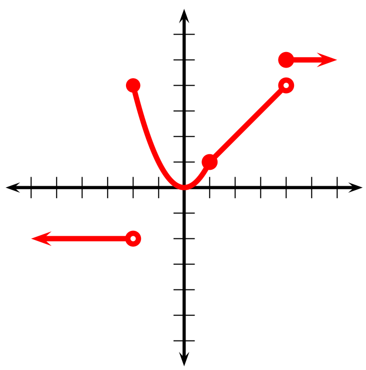

Figure 3.12: Example 38.

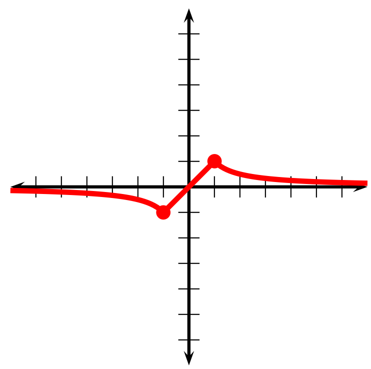

Figure 3.13: Example 39.

Sometimes the pieces in a piecewise function do not connect at a particular point, let us say at $x = a$. Then we write $f(a-)$ for the value that $f(x)$ would have if we used the assignment rule for values very close to a but smaller than a, and $f(a+)$ for the value that $f(x)$ would have if we used the assignment rule for values very close to a but larger than a.

38 Example The function $f : \mathbb{R} \to \mathbb{R}$ is piecewise defined by

$$f(x) = \begin{cases} -2 & \text{if } x \in]-\infty;-2[\\ x^2 & \text{if } x \in [-2;1] \\ x & \text{if } x \in]1;4[\\ 5 & \text{if } x \in [4;+\infty[\end{cases}$$

Its graph appears in figure 3.12. We have, for example,

1. $f(-3) = -2$
2. $f(-2-) = -2$
3. $f(-2) = (-2)^2 = 4$
4. $f(-2+) = (-2)^2 = 4$
5. $f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$
6. $f(1-) = (1)^2 = 1$
7. $f(1) = (1)^2 = 1$
8. $f(1+) = 1$
9. $f(2) = 2$
10. $f(4-) = 4$
11. $f(4) = 5$
12. $f(4+) = 5$

39 Example The function $g : \mathbb{R} \to \mathbb{R}$ is piecewise defined by

$$g(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \in]-\infty; -1[\\ x & \text{if } x \in [-1; 1] \\ \dfrac{1}{x} & \text{if } x \in]1; +\infty[\end{cases}$$

Its graph appears in figure 3.13. We have, for example,

1. $g(-\infty) = 0$, using 1.9
2. $g(-1-) = \dfrac{1}{-1} = -1$
3. $g(-1) = -1$
4. $g(-1+) = -1$
5. $g(0) = 0$
6. $g(1+) = \dfrac{1}{1} = 1$
7. $g(1) = 1$
8. $g(1+) = 1$
9. $g(+\infty) = 0$, using 1.9

3.3 Translations

40 Theorem Let f be a function and let v and h be real numbers. If (x_0, y_0) is on the graph of f, then $(x_0, y_0 + v)$ is on the graph of g, where $g(x) = f(x) + v$, and if (x_1, y_1) is on the graph of f, then $(x_1 - h, y_1)$ is on the graph of j, where $j(x) = f(x + h)$.

Proof: *Let $\Gamma_f, \Gamma_g, \Gamma_j$ denote the graphs of f, g, j respectively.*

$$(x_0, y_0) \in \Gamma_f \iff y_0 = f(x_0) \iff y_0 + v = f(x_0) + v \iff y_0 + v = g(x_0) \iff (x_0, y_0 + v) \in \Gamma_g.$$

Similarly,

$$(x_1, y_1) \in \Gamma_f \iff y_1 = f(x_1) \iff y_1 = f(x_1 - h + h) \iff y_1 = j(x_1 - h) \iff (x_1 - h, y_1) \in \Gamma_j.$$

❑

41 Definition Let f be a function and let v and h be real numbers. We say that the curve $y = f(x) + v$ is a *vertical translation* of the curve $y = f(x)$. If $v > 0$ the translation is v up, and if $v < 0$, it is v units down. Similarly, we say that the curve $y = f(x + h)$ is a *horizontal translation* of the curve $y = f(x)$. If $h > 0$, the translation is h units left, and if $h < 0$, then the translation is h units right.

42 Example If $f(x) = x^2$, then figures 3.14, 3.15 and 3.16 shew vertical translations 3 units up and 3 units down, respectively. Figures 3.17, 3.18, and 3.19, respectively shew a horizontal translation 3 units right, 3 units left, and a simultaneous translation 3 units left and down.

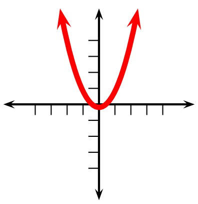

Figure 3.14: $y = f(x) = x^2$

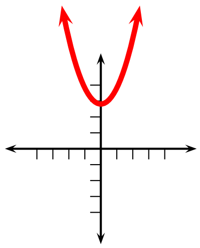

Figure 3.15: $y = f(x) + 3 = x^2 + 3$

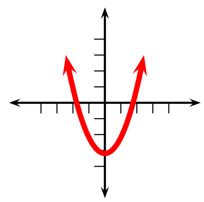

Figure 3.16: $y = f(x) - 3 = x^2 - 3$

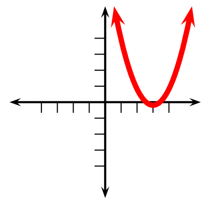

Figure 3.17: $y = f(x-3) = (x-3)^2$

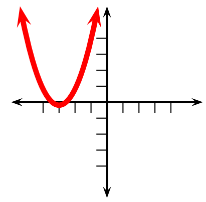

Figure 3.18: $y = f(x+3) = (x+3)^2$

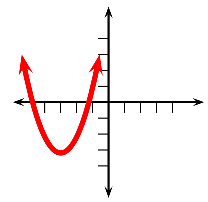

Figure 3.19: $y = f(x+3) - 3 = (x+3)^2 - 3$

43 Example If $g(x) = x$ (figure 3.20), then figures , 3.21 and 3.22 shew vertical translations 3 units up and 3 units down, respectively. Notice than in this case $g(x+t) = x+t = g(x)+t$, so a vertical translation by t units has exactly the same graph as a horizontal translation t units.

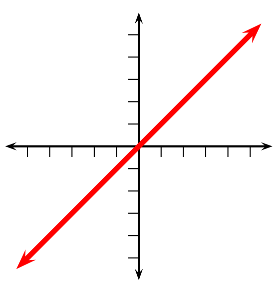

Figure 3.20: $y = g(x) = x$

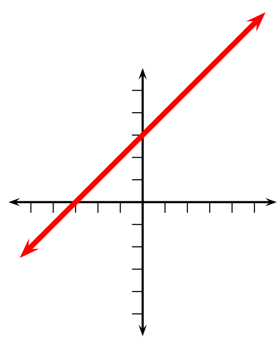

Figure 3.21: $y = g(x) + 3 = x + 3$

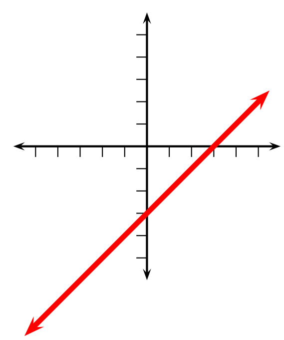

Figure 3.22: $y = g(x) - 3 = x - 3$

44 Definition Given a function f we write $f(-\infty)$ for the value that f may eventually approach for large (in absolute value) and negative inputs and $f(+\infty)$ for the value that f may eventually approach for large (in absolute value) and positive input. The line $y = b$ is a (horizontal) *asymptote* for the function f if either

$$f(-\infty) = b \qquad \text{or} \qquad f(+\infty) = b.$$

45 Definition Let $k > 0$ be an integer. A function f has a *pole of order* k at the point $x = a$ if $\lim_{x\to a}(x-a)^{k-1} f(x) = \pm\infty$ but $\lim_{x\to a}(x-a)^k f(x)$ is finite. Some authors prefer to use the term *vertical asymptote*, rather than pole.

46 Example Since $xf(x) = 1$, $f(0-) = -\infty$, $f(0+) = +\infty$ for $f : \begin{array}{ccc} \mathbb{R}\setminus\{0\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{1}{x} \end{array}$, f has a pole of order 1 at $x = 0$.

47 Example Figures 3.23 through 3.25 exhibit various transformations of $y = a(x) = \frac{1}{x}$. Notice how the poles and the asymptotes move with the various transformations.

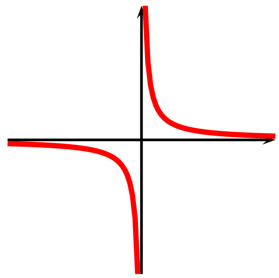

Figure 3.23: $x \mapsto \frac{1}{x}$

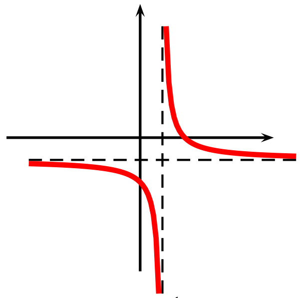

Figure 3.24: $x \mapsto \frac{1}{x-1} - 1$

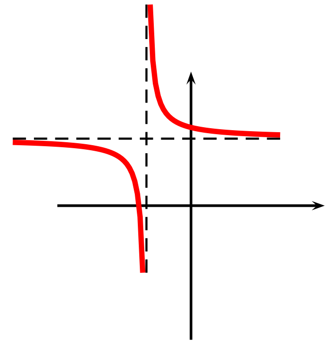

Figure 3.25: $x \mapsto \frac{1}{x+2} + 3$

3.4 Distortions

48 Theorem Let f be a function and let $V \neq 0$ and $H \neq 0$ be real numbers. If (x_0, y_0) is on the graph of f, then (x_0, Vy_0) is on the graph of g, where $g(x) = Vf(x)$, and if (x_1, y_1) is on the graph of f, then $\left(\frac{x_1}{H}, y_1\right)$ is on the graph of j, where $j(x) = f(Hx)$.

Proof: *Let $\Gamma_f, \Gamma_g, \Gamma_j$ denote the graphs of f, g, j respectively.*

$$(x_0, y_0) \in \Gamma_f \iff y_0 = f(x_0) \iff Vy_0 = Vf(x_0) \iff Vy_0 = g(x_0) \iff (x_0, Vy_0) \in \Gamma_g.$$

Similarly,

$$(x_1, y_1) \in \Gamma_f \iff y_1 = f(x_1) \iff y_1 = f\left(\frac{x_1}{H} \cdot H\right) \iff y_1 = j\left(\frac{x_1}{H}\right) \iff \left(\frac{x_1}{H}, y_1\right) \in \Gamma_j.$$

❑

49 Definition Let $V > 0$, $H > 0$, and let f be a function. The curve $y = Vf(x)$ is called a *vertical distortion* of the curve $y = f(x)$. The graph of $y = Vf(x)$ is a *vertical dilation* of the graph of $y = f(x)$ if $V > 1$ and a *vertical contraction* if $0 < V < 1$. The curve $y = f(Hx)$ is called a *horizontal distortion* of the curve $y = f(x)$ The graph of $y = f(Hx)$ is a *horizontal dilation* of the graph of $y = f(x)$ if $0 < H < 1$ and a *horizontal contraction* if $H > 1$.

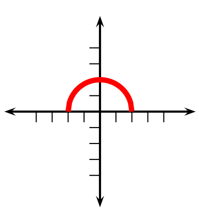

Figure 3.26: $y = a(x) = \sqrt{4-x^2}$

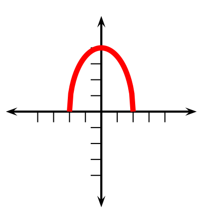

Figure 3.27: $y = 2a(x) = 2\sqrt{4-x^2}$

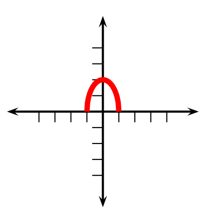

Figure 3.28: $y = a(2x) = \sqrt{4-4x^2}$

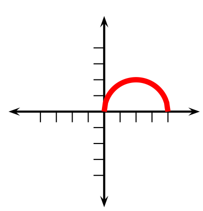

Figure 3.29: $y = a(x-2) = \sqrt{-x^2+4x}$

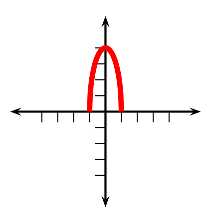

Figure 3.30: $y = 2a(2x) = 2\sqrt{4-4x^2}$

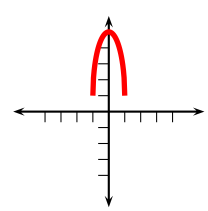

Figure 3.31: $y = 2a(2x) + 1 = 2\sqrt{4-4x^2} + 1$

50 Example Let $a(x) = \sqrt{4-x^2}$. If $y = \sqrt{4-x^2}$, then $x^2 + y^2 = 4$, which is a circle with centre at $(0,0)$ and radius 2 by virtue of 2.10. Hence $y = a(x) = \sqrt{4-x^2}$ is the upper semicircle of this circle. Figures 3.26 through 3.31 shew various transformations of this curve.

51 Example Draw the graph of the curve $y = 2x^2 - 4x + 1$.

Solution: We complete squares.

$$\begin{aligned} y = 2x^2 - 4x + 1 \quad & \iff \quad \frac{y}{2} = x^2 - 2x + \frac{1}{2} \\ & \iff \quad \frac{y}{2} + 1 = x^2 - 2x + 1 + \frac{1}{2} \\ & \iff \quad \frac{y}{2} + 1 = (x-1)^2 + \frac{1}{2} \\ & \iff \quad \frac{y}{2} = (x-1)^2 - \frac{1}{2} \\ & \iff \quad y = 2(x-1)^2 - 1, \end{aligned}$$

whence to obtain the graph of $y = 2x^2 - 4x + 1$ we (i) translate $y = x^2$ one unit right, (ii) dilate the above graph by factor of two, (iii) translate the above graph one unit down. This succession is seen in figures 3.32 through 3.34.

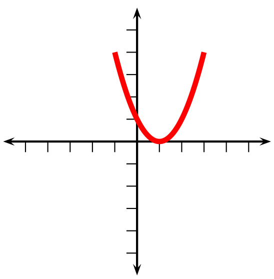

Figure 3.32: $y = (x-1)^2$

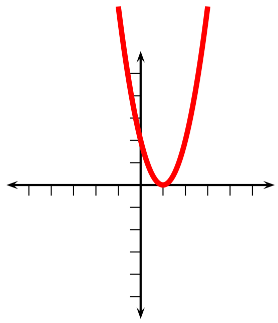

Figure 3.33: $y = 2(x-1)^2$

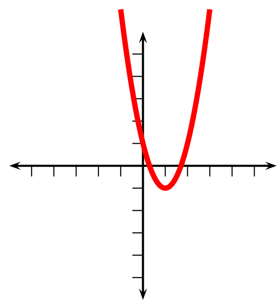

Figure 3.34: $y = 2(x-1)^2 - 1$

3.5 Reflexions

52 Theorem Let f be a function If (x_0, y_0) is on the graph of f, then $(x_0, -y_0)$ is on the graph of g, where $g(x) = -f(x)$, and if (x_1, y_1) is on the graph of f, then $(-x_1, y_1)$ is on the graph of j, where $j(x) = f(-x)$.

Proof: *Let $\Gamma_f, \Gamma_g, \Gamma_j$ denote the graphs of f, g, j respectively.*

$$(x_0, y_0) \in \Gamma_f \iff y_0 = f(x_0) \iff -y_0 = -f(x_0) \iff -y_0 = g(x_0) \iff (x_0, -y_0) \in \Gamma_g.$$

Similarly,

$$(x_1, y_1) \in \Gamma_f \iff y_1 = f(x_1) \iff y_1 = f(-(-x_1)) \iff y_1 = j(-x_1) \iff (-x_1, y_1) \in \Gamma_j.$$

❑

53 Definition Let f be a function. The curve $y = -f(x)$ is said to be the *reflexion of f about the x-axis* and the curve $y = f(-x)$ is said to be the *reflexion of f about the y-axis*.

54 Example Figures 3.35 through 3.38 shew various reflexions about the axes.

55 Theorem Let f be a function If (x_0, y_0) is on the graph of f, then $(x_0, |y_0|)$ is on the graph of g, where $g(x) = |f(x)|$.

Proof: *Let Γ_f, Γ_g denote the graphs of f, g, respectively.*

$$(x_0, y_0) \in \Gamma_f \implies y_0 = f(x_0) \implies |y_0| = |f(x_0)| \implies |y_0| = g(x_0) \implies (x_0, |y_0|) \in \Gamma_g.$$

❑

56 Example Figures 3.39 and 3.40 display $y = x$ and $y = |x|$ respectively. Figures 3.41 and 3.42 shew $y = x^2 - 1$ and $y = |x^2 - 1|$ respectively. Figures 3.43 through 3.46 exhibit various transformations of $x \mapsto \dfrac{1}{x}$.

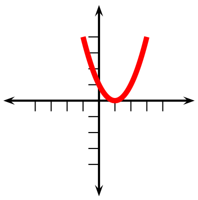

Figure 3.35: $y = d(x) = (x-1)^2$

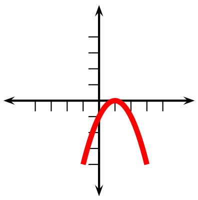

Figure 3.36: $y = -d(x) = -(x-1)^2$

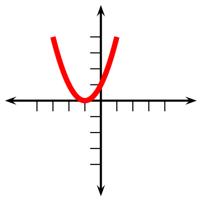

Figure 3.37: $y = d(-x) = (-x-1)^2$

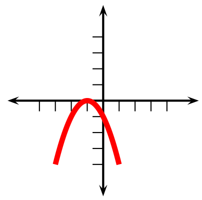

Figure 3.38: $y = -d(-x) = -(-x-1)^2$

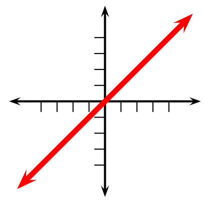

Figure 3.39: $y = f(x) = x$

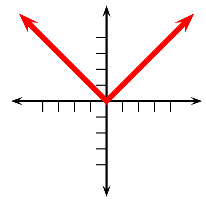

Figure 3.40: $y = |f(x)| = |x|$

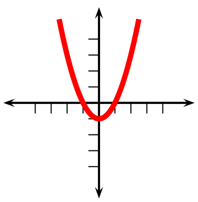

Figure 3.41: $y = g(x) = x^2 - 1$

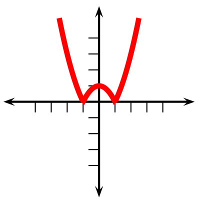

Figure 3.42: $y = |g(x)| = |x^2 - 1|$

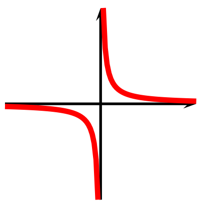

Figure 3.43: $x \mapsto \dfrac{1}{x}$

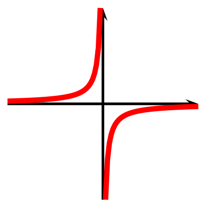

Figure 3.44: $x \mapsto -\dfrac{1}{x}$

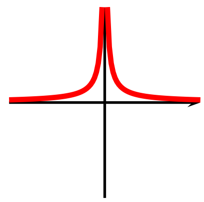

Figure 3.45: $x \mapsto \left|\dfrac{1}{x}\right|$

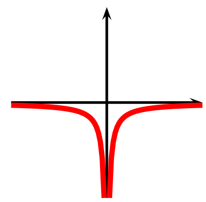

Figure 3.46: $x \mapsto -\left|\dfrac{1}{x}\right|$

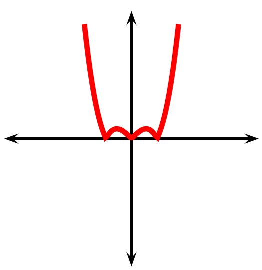

Figure 3.47: Example 58. The graph of an even function.

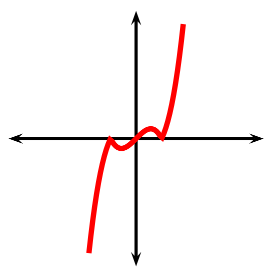

Figure 3.48: Example 58. The graph of an odd function.

3.6 Symmetry

57 Definition A function f is *even* if for all x it is verified that $f(x) = f(-x)$, that is, if the portion of the graph for $x < 0$ is a mirror reflexion of the part of the graph for $x > 0$. This means that the graph of f is symmetric about the y-axis. A function g is *odd* if for all x it is verified that $g(-x) = -g(x)$, in other words, g is odd if it is symmetric about the origin. This implies that the portion of the graph appearing in quadrant I is a 180° rotation of the portion of the graph appearing in quadrant III, and the portion of the graph appearing in quadrant II is a 180° rotation of the portion of the graph appearing in quadrant IV.

58 Example The curve in figure 3.47 is even. The curve in figure 3.48 is odd.

59 Theorem Let f be a function. Then both $x \mapsto f(|x|)$ and $x \mapsto f(-|x|)$ are even functions.

Proof: *Put* $a(x) = f(|x|)$. *Then* $a(-x) = f(|-x|) = f(|x|) = a(x)$, *whence* $x \mapsto a(x)$ *is even. Similarly, if* $b(x) = f(-|x|)$, *then* $b(-x) = f(-|-x|) = f(-|x|) = b(x)$ *proving that* $x \mapsto b(x)$ *is even.* ❑

Notice that $f(x) = f(|x|)$ for $x > 0$. Since $x \mapsto f(|x|)$ is even, the graph of $x \mapsto f(|x|)$ is thus obtained by erasing the portion of the graph of $x \mapsto f(x)$ for $x < 0$ and reflecting the part for $x > 0$. Similarly, since $f(x) = f(-|x|)$ for $x < 0$, the graph of $x \mapsto f(-|x|)$ is obtained by erasing the portion of the graph of $x \mapsto f(x)$ for $x > 0$ and reflecting the part for $x < 0$.

60 Example Figures 3.49 through 3.52 exhibit various transformations of $x \mapsto (x-1)^2 - 3$.

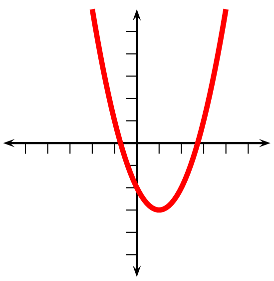

Figure 3.49: $y = f(x) = (x-1)^2 - 3$

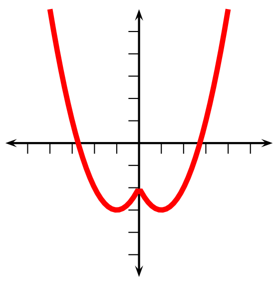

Figure 3.50: $y = f(|x|) = (|x|-1)^2 - 3$

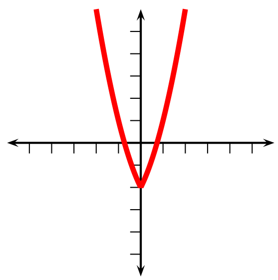

Figure 3.51: $y = f(-|x|) = (-|x|-1)^2 - 3$

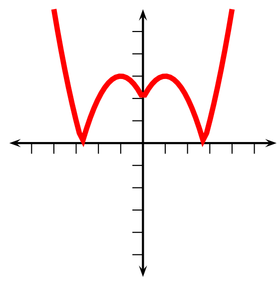

Figure 3.52: $y = |f(|x|)| = |(|x|-1)^2 - 3|$

61 Example Figures 6.7 through 6.9 shew a few transformations of $x \mapsto \dfrac{1}{x-1} - 1$.

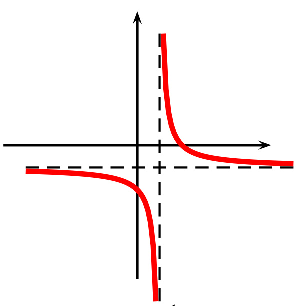

Figure 3.53: $x \mapsto \dfrac{1}{x-1} - 1$

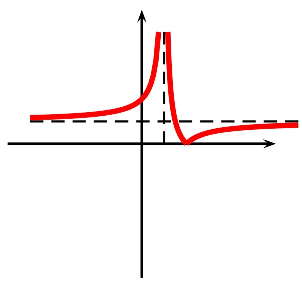

Figure 3.54: $x \mapsto \left|\dfrac{1}{x-1} - 1\right|$

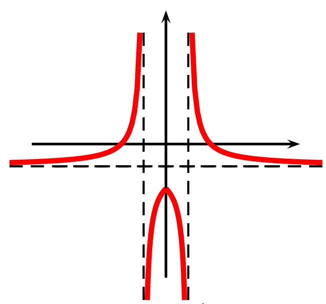

Figure 3.55: $x \mapsto \dfrac{1}{|x|-1} - 1$

3.7 Algebra of Functions

62 Definition Let f and g be two functions and let the point x be in the intersection of their domains. Then $f+g$ is their sum, defined at each point x by

$$(f+g)(x) = f(x) + g(x).$$

The difference $f - g$ is defined by

$$(f-g)(x) = f(x) - g(x),$$

and their product fg is defined by

$$(fg)(x) = f(x) \cdot g(x).$$

Furthermore, if $g(x) \neq 0$, then their quotient is defined as

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

The composition $f \circ g$ ("f composed with g") is defined at the point x by

$$(f \circ g)(x) = f(g(x)).$$

63 Example Figure 3.56 shews two functions $x \mapsto f(x) = x+1$ and $x \mapsto g(x) = x-1$. Figure 3.57 shews their sum $x \mapsto 2x$, a line, figure 3.58 shews the difference $x \mapsto (f-g)(x) = 2$, a horizontal line, and figure 3.59 shews their product $x \mapsto x^2 - 1$, a parabola. We also have $x \mapsto \left(\dfrac{g}{f}\right)(x) = \dfrac{x-1}{x+1} = 1 - \dfrac{2}{x+1}$, a hyperbola with pole at $x = -1$ and asymptote at $y = 1$ (figure 3.60); $x \mapsto \left(\dfrac{f}{g}\right)(x) = \dfrac{x+1}{x-1} = 1 + \dfrac{2}{x-1}$, a hyperbola with pole at $x = 1$ and asymptote at $y = 1$ (figure 3.61); $(f \circ g) = \mathbf{Id}$ (figure 3.62); and $x \mapsto (f \circ f)(x) = x+2$ (figure 3.63).

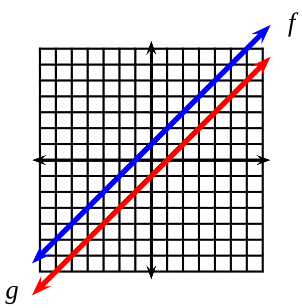

Figure 3.56: $f(x) = x+1$ and $g(x) = x-1$

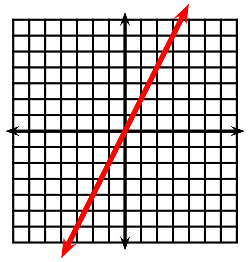
Figure 3.57: $x \mapsto (f+g)(x) = 2x$

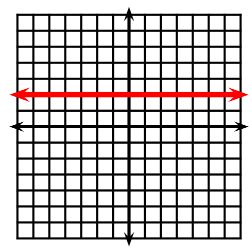
Figure 3.58: $x \mapsto (f-g)(x) = 2$

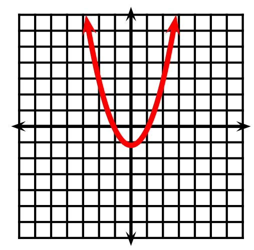
Figure 3.59: $x \mapsto (fg)(x) = x^2 - 1$

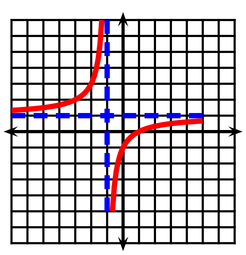
Figure 3.60: $\left(\frac{g}{f}\right)(x) = 1 - \frac{2}{x+1}$

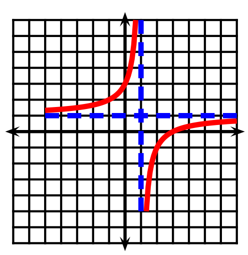
Figure 3.61: $\left(\frac{f}{g}\right)(x) = 1 + \frac{2}{x-1}$

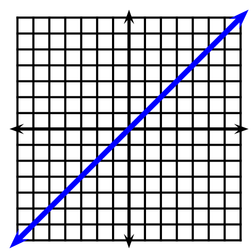
Figure 3.62: $(f \circ g)(x) = x$

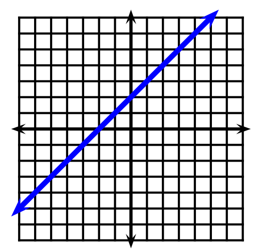
Figure 3.63: $(f \circ f)(x) = x+2$

3.8 Behaviour of the Graphs of Functions

So far we have limited our study of functions to those families of functions whose graphs are known to us: lines, parabolas, hyperbolas, or semicircles. Through some arguments involving symmetry we have been able to extend this collection to compositions of the above listed functions with the absolute value function. We would now like to increase our repertoire of functions that we can graph. For that we need the machinery of Calculus, which will be developed in the subsequent chapters. This section introduces the basic definitions of the essential features that we will be interested in when we examine the graphs of more functions.

64 Definition A function f is said to be *continuous* at the point $x = a$ if $f(a-) = f(a) = f(a+)$. It is continuous on the interval I if it is continuous on every point of I.

Heuristically speaking, a continuous function is one whose graph has no "breaks."

65 Example Given that

$$f(x) = \begin{cases} 6+x & \text{if } x \in]-\infty;-2] \\ 3x^2+xa & \text{if } x \in]-2;+\infty[\end{cases}$$

is continuous, find a.

Solution: Since $f(-2-) = f(-2) = 6-2 = 4$ and $f(-2+) = 3(-2)^2 - 2a = 12-2a$ we need

$$f(-2-) = f(-2+) \implies 4 = 12-2a \implies a = 4.$$

The graph of f is given in figure 3.64.

66 Example Given that

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & \text{if } x \neq 1 \\ a & \text{if } x = 1 \end{cases}$$

is continuous, find a.

Solution: For $x \neq 1$ we have $f(x) = \dfrac{x^2-1}{x-1} = x+1$. Since $f(1-) = 2$ and $f(1+) = 2$ we need $a = f(1) = 2$. The graph of f is given in figure 3.65.

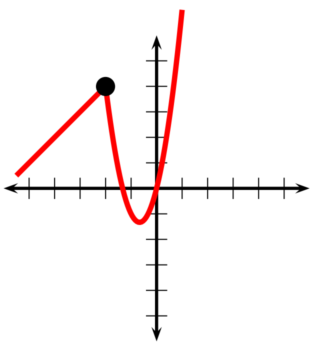

Figure 3.64: Example 65.

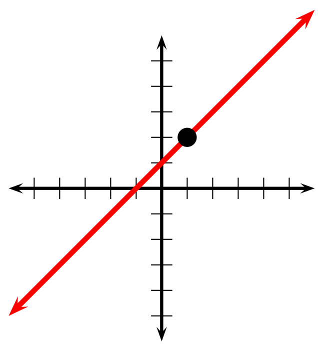

Figure 3.65: Example 66.

We will accept the following results without proof.

67 Theorem (Bolzano's Intermediate Value Theorem) If f is continuous on the interval $[a;b]$ and f and there are two different values in this interval for which f changes sign, then f is vanishes somewhere in this interval, that is, there is $c \in [a;b]$ such that $f(c) = 0$.

68 Corollary If f is continuous on the interval $[a;b]$ with $f(a) \neq f(b)$ then f assumes every value between $f(a)$ and $f(b)$, that is, for d with $\min(f(a), f(b)) \leq d \leq \max(f(a), f(b))$ there is $c \in [a;b]$ such that $f(c) = d$.

69 Theorem (Weierstrass's Theorem) A continuous function on a finite closed interval $[a;b]$ assumes a maximum value and a minimum value.

70 Definition A function f is said to be *increasing* (respectively, *strictly increasing*) if $a < b \implies f(a) \leq f(b)$ (respectively, $a < b \implies f(a) < f(b)$). A function g is said to be *decreasing* (respectively, *strictly decreasing*) if $a < b \implies g(a) \leq g(b)$ (respectively, $a < b \implies g(a) < g(b)$). A function is *monotonic* if it is either (strictly) increasing or decreasing. By the *intervals of monotonicity of a function* we mean the intervals where the function might be (strictly) increasing or decreasing.

If the function f is (strictly) increasing, its opposite $-f$ is (strictly) decreasing, and viceversa.

The following theorem is immediate.

71 Theorem A function f is (strictly) increasing if for all $a < b$ for which it is defined

$$\frac{f(b) - f(a)}{b - a} \geq 0 \qquad (\text{respectively}, \frac{f(b) - f(a)}{b - a} > 0).$$

Similarly, a function g is (strictly) decreasing if for all $a < b$ for which it is defined

$$\frac{g(b) - g(a)}{b - a} \leq 0 \qquad (\text{respectively}, \frac{g(b) - g(a)}{b - a} < 0).$$

72 Example Prove that an affine function $x \mapsto mx + k$ is strictly increasing if $m > 0$ and strictly decreasing if $m < 0$.

Solution: This is geometrically obvious. To prove it analytically, put $f(x) = mx + k$ and observe that

$$\frac{f(b) - f(a)}{b - a} = \frac{(mb + k) - (ma + k)}{b - a} = m.$$

Now apply Theorem 71.

73 Example Prove that $x \mapsto x^2$ is strictly increasing if $x > 0$ and strictly decreasing if $x < 0$.

Solution: This is geometrically obvious. To prove it analytically, put $t(x) = x^2$ observe that

$$\frac{t(b) - t(a)}{b - a} = \frac{b^2 - a^2}{b - a} = \frac{(b - a)(b + a)}{b - a} = b + a.$$

This quantity is strictly negative or strictly positive depending on whether $a < b < 0$ or $0 < a < b$. We now apply Theorem 71. We summarise this information by means of the table

x	$-\infty$	0	$+\infty$
$f(x) = x^2$	$\searrow$	0	$\nearrow$

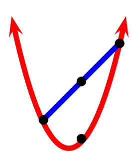

Figure 3.66: Example 75. A convex curve

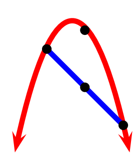

Figure 3.67: Example 75. A concave curve.

74 Definition A function f is said to be (midpoint strictly) *convex* if given $a < b$ we have

$$f\left(\frac{a+b}{2}\right) < \frac{f(a) + f(b)}{2}.$$

A function g is said to be (midpoint strictly) *concave* if given $a < b$ we have

$$g\left(\frac{a+b}{2}\right) > \frac{g(a) + g(b)}{2}.$$

By the *intervals of convexity (concavity) of a function* we mean the intervals where the function is convex (concave). An *inflexion point* is a point where a graph changes convexity.

Geometrically speaking, a convex function is one such that if two distinct points on its graph are taken and the straight line joining these two points drawn, then the midpoint of that straight line is above the graph. In other words, the graph of the function bends upwards. Notice that if f is convex, then its opposite $-f$ is concave.

75 Example Prove that the square function $x \mapsto x^2$ is convex.

Solution: Put $f(x) = x^2$. We need to prove that

$$f\left(\frac{a+b}{2}\right) = \left(\frac{a+b}{2}\right)^2 = \frac{a^2+2ab+b^2}{4}$$

is strictly smaller than

$$\frac{f(a)+f(b)}{2} = \frac{a^2+b^2}{2}.$$

This would occur if

$$\frac{a^2+2ab+b^2}{4} < \frac{a^2+b^2}{2},$$

that is

$$\frac{a^2-2ab+b^2}{4} > 0.$$

But since we always have

$$\frac{a^2-2ab+b^2}{4} = \frac{(a-b)^2}{4} > 0,$$

and the above steps are reversible, the assertion is proved. Incidentally, this also proves that $x \mapsto -x^2$ is concave. See figures 3.66 and 3.67.

76 Definition Let f be a function. If f is defined at $x = 0$, then $(0, f(0))$ is its *y-intercept*. The points $(x, 0)$ on the x-axis for which $f(x) = 0$, if any, are the *x-intercepts* of f.

77 Definition A *zero* or *root* of a function f is a solution to the equation $f(x) = 0$.

Chapter 4

The Strong Derivative of a Function

In this chapter we introduce the concept of the strong derivative. We do not give formulæ—with the exception of example 82 where the formula is obtained through a geometric argument—here to calculate derivatives, we scatter those throughout the text.

4.1 The Strong Derivative

Given a finite number of points, we can find infinitely many curves passing through them. See for example figure 4.1, where we see three very different curves f, g, h each simultaneously passing through the points A, B, C. Thus plotting a few points of the graph of a function can give a misleading picture.

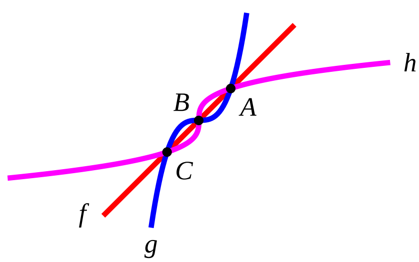

Figure 4.1: A few points do not a graph determine.

By the same token, given a formula, the plotting of a few points does not give the salient features of a graph. For example, let us say that we wanted to graph $y = 4x - x^3$. In figures 4.2 through 4.5 we have chosen a few selected points on the curve and interpolated between them through lines. But relying on this method does not give proof that the graph will not have more turns or bends, say, or that it will grow indefinitely for values of x of large magnitude.

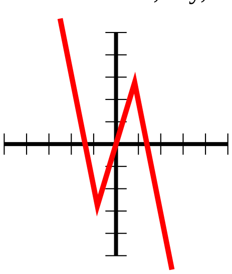

Figure 4.2: Four plot points.

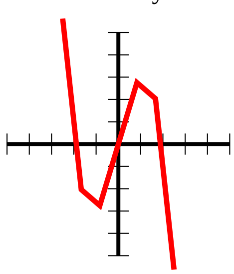

Figure 4.3: Seven plot points.

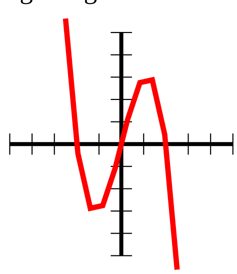

Figure 4.4: Ten plot points.

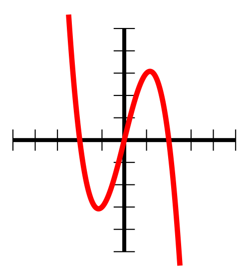

Figure 4.5: One thousand plot points.

But for all its faults, the progression of shapes in figures 4.2 through 4.5 suggests that a "reasonable" graph can be approximated by a series of straight lines. By a "reasonable" graph we mean one that does not have many sharp turns, does not oscillate wildly, does not have many jumps or many asymptotes, and that it is mostly continuous and "smooth." Admittedly, these concepts are vague, but we will gain more insight into them as we progress.

How do we choose the lines to approximate a given "reasonable" curve? Given a function f consider the point $(a, f(a))$ on the graph of the function. What happens around this point? If we approached a through values $x < a$ and joined the line with endpoints $(x, f(x))$ and $(a, f(a))$, we would obtain a secant line like that of figure 4.6. If we approached a through values $x > a$ and joined the line with endpoints $(a, f(a))$ and $(x, f(x))$, we would obtain a secant line like that of figure 4.7. Eventually, on getting closer to $(a, f(a))$ we obtain a line just barely grazing the curve—that is, "tangent" to the curve—at the point $(a, f(a))$, as in figure 4.8.

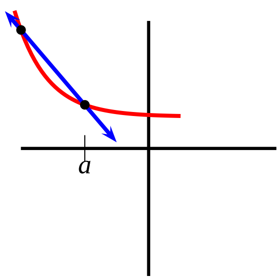

Figure 4.6: Left secant through $(a, f(a))$.

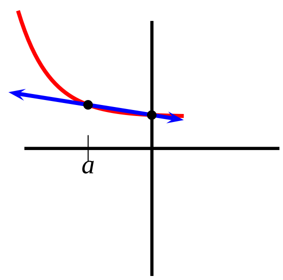

Figure 4.7: Right secant through $(a, f(a))$.

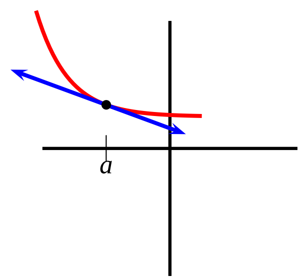

Figure 4.8: Line grazing $(a, f(a))$.

In the simplest of cases, if our curve is the line $L : y = mx + k$, then in a neighbourhood of the point $x = a$ the tangent line to L should be itself! It is not true that every curve we consider would have a unique "tangent line" at every point. For example, a curve with a sharp edge as as $y = |x|$ at $x = 0$ in figure 4.9 or the curve in figure 4.10 have infinitely many tangents at $x = 0$. The curve $y = \dfrac{1}{x}$ is not even defined at $x = 0$ and hence it does not have a tangent there. On the other hand, the parabola $y = x^2$ is "smooth" at $x = 0$ and appears to have a unique tangent there.

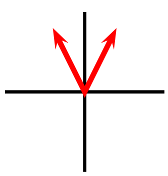

Figure 4.9: $y = |x|$ is non-smooth function at $x = 0$.

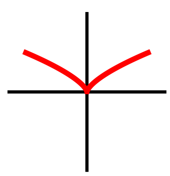

Figure 4.10: A non-smooth function at $x = 0$.

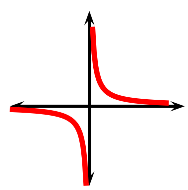

Figure 4.11: $y = \dfrac{1}{x}$.

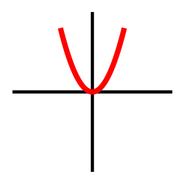

Figure 4.12: A smooth function at $x = 0$.

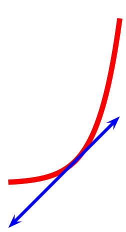

Figure 4.13: An increasing curve.

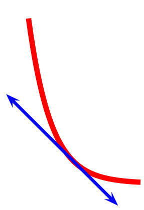

Figure 4.14: A decreasing curve.

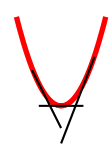

Figure 4.15: Convex curve.

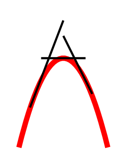

Figure 4.16: Concave curve.

STOP *Notice that gathering tangent lines at diverse points of a curve also gives us information about the monotonicity and convexity of the curve. If the tangent line at a point to a curve has positive slope, then the curve appears* *to increase. If inside an open interval the curve appears to have a maximum or a minimum point, then the tangent line there appears to be horizontal, that is, its slope is* 0*. It decreases otherwise (see figures 4.13 and 4.14). It also appears that if the slope of the tangent to a curve increases, that is, if the tangent lies always below the curve, then the curve is convex. Otherwise it is concave (see figures 4.15 and 4.16).*

Given now the formula for a function f and a point $(a, f(a))$ on the graph of f, how do we determine the tangent line to f at $(a, f(a))$? Recall that if $\varepsilon \to 0$, then $a+\varepsilon$ is in a neighbourhood of a. The slope of the secant line joining $(a+\varepsilon, f(a+\varepsilon))$ and $(a, f(a))$ is

$$\frac{f(a+\varepsilon)-f(a)}{\varepsilon} \tag{4.1}$$

We denote the value—if there is one—of 4.1— as $x \to \varepsilon$ by $f'(a)$. Hence for fixed but small ε we have

$$\frac{f(a+\varepsilon)-f(a)}{\varepsilon} \approx f'(a) \implies f(a+\varepsilon) \approx f(a)+f'(a)\varepsilon.$$

There is, generally, an error in taking the dextral side as an approximation for the sinistral side on the above formula. We will settle for having an error of the order of $O(\varepsilon^2)$, which will normally will be a good compromise for most of the formulæ we will encounter. This prepares the ground for our main definition of this section.

78 Definition Let f be a function and let $a \in \mathbf{Dom}(f)$. When there is a number $f'(a)$ such that

$$f(a+\varepsilon) = f(a)+f'(a)\varepsilon+O(\varepsilon^2) \qquad \text{as} \qquad \varepsilon \to 0$$

then we say that the function f has a *strong derivative* $f'(a)$ or that f is *strongly differentiable* at $x=a$. If we consider the set $\{x \in \mathbf{Dom}(f) : f'(x) \text{ exists}\}$ then we may form the function $x \mapsto f'(x)$ with domain $\mathbf{Dom}(f') = \{x \in \mathbf{Dom}(f) : f'(x) \text{ exists}\}$. We call the function f' the *strong derivative* of f. We will also often use the symbol $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$ to denote the function $x \mapsto f'(x)$.

79 Definition If f' is itself differentiable, then the function $(f')' = f''$ is the *second derivative* of f. It is also denoted by $\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x)$. We similarly define the third, fourth, etc., derivatives. It is customary to denote the first three derivatives of a function with primes, as in f', f'', f''', and any higher derivative with either roman numbers or with the order of the derivative enclosed in parenthesis, as in f^{iv}, f^{v} or $f^{(4)}, f^{(5)}$, etc.

4.2 Graphical Differentiation

Before we attack the problem of deducing the formula for the derivative of a function through the formula of the function, let us address the problem of obtaining an approximate value for the derivative of a function through the graph of the function. It is possible to estimate graphically the strong derivative of the function by appealing to the interpretation that the strong derivative of a function at given point is the value of the slope of the tangent at that given point.

80 Example Find an approximate graph for the derivative of f given in figure 4.17.

Solution: Observe that from the remarks following figure 4.16, we expect f' to be positive in $[-1.4; -0.6]$, since f increases there. We expect f' to be 0 at $x=-0.6$, since f appears to have a (local) maximum there. We expect f' to be negative in $[-0.6; 0; 6]$ since f decreases there. We expect f' to be 0 at $x=0.6$, since f appears to have a (local) minimum there. Finally we expect f' to be positive for $[0.6; 1.4]$ since f is increasing there.

We now perform the following steps.

1. We first divide up the domain of f into intervals of the same length, in this case we will take intervals of length 0.2.
2. For each endpoint x of an interval above, we look at the point $(x, f(x))$ on the graph of f.
3. We place a ruler so that it is tangent to the curve at $(x, f(x))$.
4. We find the slope of the ruler. Recall that any two points on the tangent line (the ruler) can serve to find the slope.
5. We tabulate the slopes obtained and we plot these values, obtaining thereby an approximate graph of f'.

In our case we obtain the following (approximate) values for $f'(x)$.

x	-1.4	-1.2	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1	1.2	1.4
$f'(x)$	4.88	3.32	2	0.92	0.08	-0.52	-0.88	-1	-0.88	-0.52	0.08	0.92	2	3.32	4.88

An approximate graph of the strong derivative appears in figure 4.18.

Figure 4.17: Example 80. $y = f(x)$

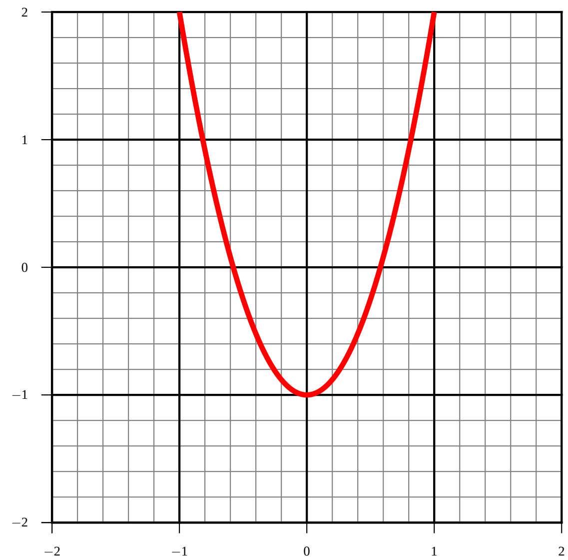

Figure 4.18: Example 80. $y = f'(x)$

81 Example Figure 4.20 gives an approximate graph of the strong derivative of the graph appearing in figure 4.19.

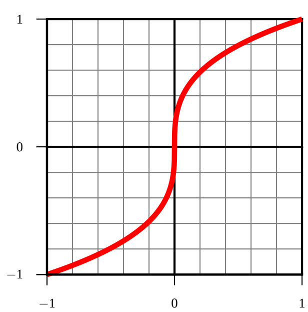

Figure 4.19: Example 81. $y = f(x)$

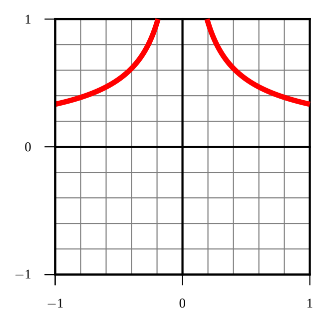

Figure 4.20: Example 81. $y = f'(x)$

To obtain it, we have served ourselves of the table below.

x	-1.0	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8	1.0
$f'(x)$	0.3	0.4	0.5	0.6	1.0	$+\infty$	1.0	0.6	0.5	0.4	0.3

82 Example Consider the function $x \mapsto \sqrt{1-x^2}$ for $x \in [-1;1]$. From example 35, its graph is the upper unit semicircle. A line from the origin to a point (a,b) on the circle has equation $y = \dfrac{b}{a}x$ (assume $ab \neq 0$). Since a line tangent to the semicircle at (a,b) is perpendicular to the line $y = \dfrac{b}{a}x$, the slope of the perpendicular line is $-\dfrac{a}{b}$ in view of Theorem 22. Hence the strong derivative at the point $x = a$ is $-\dfrac{a}{b}$. Since $b = \sqrt{1-a^2}$, we find that for $a \in]-1;1[$, the strong derivative when $x = a$ is $-\dfrac{a}{\sqrt{1-a^2}}$. We will shew how to graph the function $x \mapsto -\dfrac{x}{\sqrt{1-x^2}}$ in example **??**, but for now, an approximate tabulation gives

x	-1.0	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8	1.0
$f'(x)$	$+\infty$	1.33	0.75	0.44	0.20	0	-0.20	-0.44	-0.75	-1.33	$-\infty$

The approximate graph appears in figure 4.22.

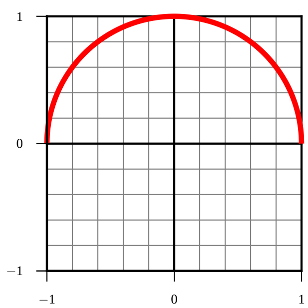

Figure 4.21: Example 82. $y = f(x)$

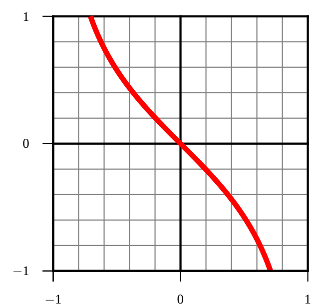

Figure 4.22: Example 82. $y = f'(x)$

4.3 Derivatives and Graphs

In this section we prove the remarks following figure 4.16 which will be the main tools for graphing in subsequent chapters.

83 Theorem If f is strongly differentiable at x then f is continuous at x.

Proof: *We have* $f(x+\varepsilon) = f(x) + f'(x)\varepsilon + O(\varepsilon^2)$. *If* $\varepsilon > 0$ *and* $\varepsilon \to 0$ *then* $f(x+\varepsilon) = f(x+)$ *and similarly if* $\varepsilon < 0$ *and* $\varepsilon \to 0$ $f(x+\varepsilon) = f(x-)$. *Hence we have* $f(x+) = f(x) = f(x-)$, *and* f *is continuous at* x. ❑

84 Theorem Let f be strongly differentiable at x. If $f'(x) > 0$ then f is increasing in a neighbourhood of x, if $f'(x) < 0$ then f is decreasing in a neighbourhood of x.

Proof: *We have* $f(x+\varepsilon) - f(x) = f'(x)\varepsilon + O(\varepsilon^2)$. *For* ε *very small, this means that*

$$f(x+\varepsilon) - f(x) \approx f'(x)\varepsilon,$$

that is, the sign of $f(x+\varepsilon) - f(x)$ *is the same as the sign of* $f'(x)\varepsilon$. *Thus if* $\varepsilon > 0$ *and* $f'(x) > 0$, *then* $f(x+\varepsilon) > f(x)$, *that is,* f *is increasing. If* $\varepsilon > 0$ *and* $f'(x) < 0$, *then* $f(x+\varepsilon) < f(x)$, *that is,* f *is decreasing. Similar conclusions are reached when considering* $\varepsilon < 0$ *and the theorem is proved.* ❑

85 Definition If f is strongly differentiable at x and $f'(x) = 0$, then we say that x is a *stationary point of* f.

86 Definition If there is a point a for which $f(x) \le f(a)$ for all x in a neighbourhood centred at $x = a$ then we say that f has a *local maximum* at $x = a$. Similarly, if there is a point b for which $f(x) \ge f(b)$ for all x in a neighbourhood centred at $x = b$ then we say that f has a *local minimum* at $x = b$.

87 Theorem If f is strongly differentiable at $x = a$, $f'(a) = 0$, and f' changes from $+$ to $-$ in a neighbourhood of a then $x = a$ is a local maximum. If f is strongly differentiable at $x = b$, $f'(b) = 0$, and f' changes from $-$ to $+$ in a neighbourhood of b then $x = b$ is a local minimum.

Proof: *By Theorem 84, when f' changes from $+$ to $-$, f is increasing and then decreasing in a neighbourhood of $x = a$. By Weierstrass's Theorem (Theorem 69), f assumes a maximum on a closed neighbourhood containing a. It cannot be to the left of a since the function is increasing there, and it cannot be to the right of a since the function is decreasing there. Hence the maximum must be at $x = a$ and so $f(x) \leq f(a)$ for x in a neighbourhood of a. The result just obtained applied to $-f$ yields the second half of the theorem.* ❑

88 Example The graph of the strong derivative f' of a function f is given in figure 4.23. Then according to Theorem 87 f has a local minimum at $x = -2$ and $x = 2$, and a local maximum at $x = 0$ and $x = 4$.

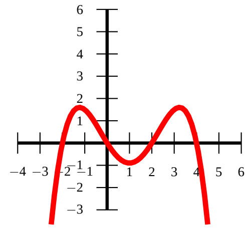

Figure 4.23: Example 88.

89 Lemma If f' increases in a neighbourhood of x, then f is convex in a neighbourhood of x. Similarly, if f' decreases in a neighbourhood of x, then f is concave in a neighbourhood of x.

Proof: *Let $\varepsilon > 0$. Then*

$$f(x) = f(x-\varepsilon+\varepsilon) = f(x-\varepsilon) + f'(x-\varepsilon)\varepsilon + O(\varepsilon^2),$$
$$f(x) = f(x+\varepsilon-\varepsilon) = f(x+\varepsilon) - f'(x+\varepsilon)\varepsilon + O(\varepsilon^2).$$

Adding,

$$2f(x) = f(x+\varepsilon) + f(x-\varepsilon) + \left(f'(x-\varepsilon) - f'(x+\varepsilon)\right)\varepsilon + O(\varepsilon^2).$$

For ε very small we then have

$$2f(x) \approx f(x+\varepsilon) + f(x-\varepsilon) + \left(f'(x-\varepsilon) - f'(x+\varepsilon)\right)\varepsilon.$$

If f' is increasing then $f'(x-\varepsilon) - f'(x+\varepsilon) < 0$. Since $\varepsilon > 0$ this implies that

$$2f(x) < f(x+\varepsilon) + f(x-\varepsilon),$$

which means that f is convex in a neighbourhood of x. This result now applied to $-f$ gives the second half of the theorem. ❑

90 Theorem A twice strongly differentiable -function f is convex in a neighbourhood of $x = a$ if $f''(a) > 0$. It is concave in a neighbourhood of $x = b$ if $f''(b) < 0$.

Proof: *This follows from Lemma 89 and Theorem 84.* ❑

Chapter 5

Polynomial Functions

In this chapter we study polynomials and their graphs. In order to do the latter, we demonstrate the Power Rule, the Sum Rule, the Product Rule, and the Chain Rule for derivatives. We also study some algebraic topics related to the roots of a polynomial.

5.1 Power Functions and the Power Rule

By a *power function* we mean a function of the form $x \mapsto x^{\alpha}$, where $\alpha \in \mathbb{R}$. In this chapter we will only study the case when α is a positive integer.

If n is a positive integer, we are interested in how to graph $x \mapsto x^n$. We have already encountered a few instances of power functions. For $n = 0$, the function $x \mapsto 1$ is a constant function, whose graph is the straight line $y = 1$ parallel to the x-axis. For $n = 1$, the function $x \mapsto x$ is the identity function, whose graph is the straight line $y = x$, which bisects the first and third quadrant. For $n = 2$, we have the square function $x \mapsto x^2$ whose graph is the parabola $y = x^2$ encountered in example 27. We reproduce their graphs below in figures 5.1 through 5.3 for easy reference.

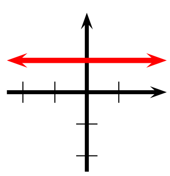

Figure 5.1: $x \mapsto 1$.

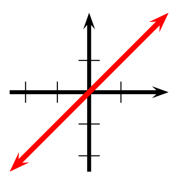

Figure 5.2: $x \mapsto x$.

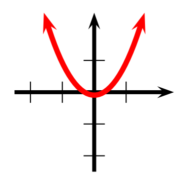

Figure 5.3: $x \mapsto x^2$.

By the groundwork from the preceding chapter, we know we can gather information about the monotonicity and convexity of the function $x \mapsto x^n$ by studying its first and second derivatives. For that we first establish a series of lemmata.

91 Lemma The strong derivative of a constant function is the 0 function. In symbols, if f is a function with assignment rule $f(x) = k$, constant, then for all x, $f'(x) = 0$.

Proof: *We have*

$$f(x+\varepsilon) = k = k + 0 \cdot \varepsilon + 0\varepsilon^2,$$

which proves the assertion. ❑

For ellipsis we will write $(k)' = 0$ *or* $\frac{\mathrm{d}}{\mathrm{d}x} k = 0$.

92 Lemma The strong derivative of the identity function $x \mapsto x$ is the constant function $x \mapsto 1$.

Proof: *If $f(x) = x$ we have*

$$f(x+\varepsilon) = x+\varepsilon = x+1\cdot\varepsilon+0\varepsilon^2,$$

and so $f'(x) = 1$. ❑

For ellipsis we will write $(x)' = 1$ or $\frac{\mathrm{d}}{\mathrm{d}x}\, x = 1$.

93 Lemma The strong derivative of the square function $x \mapsto x^2$ is the function $x \mapsto 2x$.

Proof: *If $f(x) = x^2$ we have*

$$f(x+\varepsilon) = (x+\varepsilon)^2 = x^2+2x\varepsilon+\varepsilon^2 = f(x)+2x\varepsilon+O(\varepsilon^2),$$

proving the assertion. For ellipsis we will write $(x^2)' = 2x$ or $\frac{\mathrm{d}}{\mathrm{d}x}\, x^2 = 2x$. ❑

94 Lemma The strong derivative of the cubic function $x \mapsto x^3$ is the function $x \mapsto 3x^2$.

Proof: *If $f(x) = x^3$ we have, using Lemma 93,*

$$\begin{aligned}(x+\varepsilon)^3 &= (x+\varepsilon)(x^2+2x\varepsilon+O(\varepsilon^2))\\ &= x^3+2x^2\varepsilon+O(x\varepsilon^2)+x^2\varepsilon+2x\varepsilon^2+O(\varepsilon^3)\\ &= x^3+3x^2\varepsilon+O(\varepsilon^2),\end{aligned}$$

as $\varepsilon \to 0$, and so $(x^3)' = 3x^2$ (or $\frac{\mathrm{d}}{\mathrm{d}x}\, x^3 = 3x^2$). ❑

We will now see that the pattern

$$\frac{\mathrm{d}}{\mathrm{d}x}\, x^0 = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\, x^1 = 1, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\, x^2 = 2x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\, x^3 = 3x^2,$$

is preserved for higher powers of the exponent. Arguing as in Lemma 94, we obtain the following theorem.

95 Theorem (Power Rule) If n is a positive integer, $\frac{\mathrm{d}}{\mathrm{d}x}\, x^n = nx^{n-1}$.

Proof: *If the strong derivative of $\frac{\mathrm{d}}{\mathrm{d}x}\, x^n = d_n(x)$, we have*

$$\begin{aligned}(x+\varepsilon)^{n+1} &= (x+\varepsilon)\left(x^n+d_n(x)\varepsilon+O(\varepsilon^2)\right)\\ &= x^{n+1}+(xd_n(x)+x^n)\,\varepsilon+O(\varepsilon^2)\end{aligned}$$

from where the strong derivative of x^{n+1} is $xd_n(x)+x^n$. Since $d_1(x) = 1$, we have by recurrence,

$$\begin{aligned}d_2(x) &= xd_1(x)+x^1 & &= x\cdot 1+x & &= 2x\\ d_3(x) &= xd_2(x)+x^2 & &= x\cdot(2x)+x^2 & &= 3x^2\\ d_4(x) &= xd_3(x)+x^3 & &= x\cdot(3x^2)+x^3 & &= 4x^3\\ d_5(x) &= xd_4(x)+x^4 & &= x\cdot(4x^3)+x^4 & &= 5x^4\\ d_6(x) &= xd_5(x)+x^5 & &= x\cdot(5x^4)+x^5 & &= 6x^5\\ d_7(x) &= xd_6(x)+x^6 & &= x\cdot(6x^5)+x^6 & &= 7x^6,\end{aligned}$$

and so, by recursion, $d_n(x) = nx^{n-1}$. ❑

96 Example We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\,x^3 = 3x^2, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\,x^7 = 7x^6, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\,x^{1000} = 1000x^{999},$$

etc.

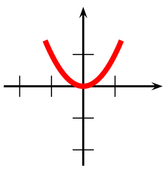

Figure 5.4: $y = x^2$.

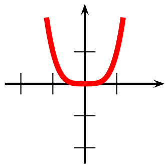

Figure 5.5: $y = x^4$.

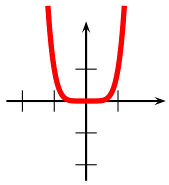

Figure 5.6: $y = x^6$.

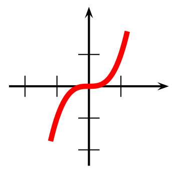

Figure 5.7: $y = x^3$.

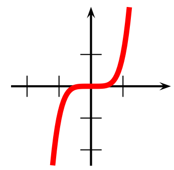

Figure 5.8: $y = x^5$.

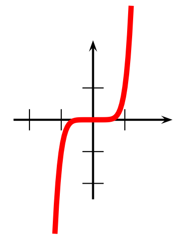

Figure 5.9: $y = x^7$.

We now address the problem of how to graph $x \mapsto x^n$.

97 Theorem Let $n \geq 2$ be an integer and $f(x) = x^n$. Then

- if n is even, f is convex, f is decreasing for $x < 0$, and f is increasing for $x > 0$. Also, $f(-\infty) = f(+\infty) = +\infty$.
- if n is odd, f is increasing, f is concave for $x < 0$, and f is convex for $x > 0$. Also, $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$.

Proof: *If $n \geq 2$ is even, $n-1 \geq 1$ is odd, and $n-2$ is even. Now $f'(x) = nx^{n-1}$ and $f''(x) = n(n-1)x^{n-2}$. Since $x^{n-2} > 0$ for all $x \neq 0$, $f''(x) > 0$ for $x \neq 0$ and so it is convex. Since $x < 0 \implies x^{n-1} < 0 \implies f'(x) < 0$, f is decreasing for $x < 0$. A similar argument shews that f is increasing for $x > 0$. It is clear that $f(-\infty) = f(+\infty) = +\infty$. If $n \geq 3$ is odd, $n-1 \geq 1$ is even, and $n-2$ is odd. Now $f'(x) = nx^{n-1}$ and $f''(x) = n(n-1)x^{n-2}$. Since*

$x^{n-1} > 0$ for all $x \neq 0$, $f'(x) > 0$ for $x \neq 0$ and so it is increasing. Since $x < 0 \implies x^{n-2} < 0 \implies f''(x) < 0$, f is concave for $x < 0$. A similar argument shews that f is convex for $x > 0$. It is clear that $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$. ❑

The graphs of $y = x^2$, $y = x^4$, $y = x^6$, etc., resemble one other. For $-1 \leq x \leq 1$, the higher the exponent, the flatter the graph (closer to the x-axis) will be, since

$$|x| < 1 \implies \cdots < x^6 < x^4 < x^2 < 1.$$

For $|x| \geq 1$, the higher the exponent, the steeper the graph will be since

$$|x| > 1 \implies \cdots > x^6 > x^4 > x^2 > 1.$$

Similarly for the graphs of $y = x^3$, $y = x^5$, $y = x^7$ etc. This information is summarised in the tables below.

x	$-\infty$	0	$+\infty$
$f(x) = x^n$	$\nearrow$	0	$\nearrow$

Table 5.1: $x \mapsto x^n$, with $n > 0$ integer and odd.

x	$-\infty$	0	$+\infty$
$f(x) = x^n$	$\searrow$	0	$\nearrow$

Table 5.2: $x \mapsto x^n$, with $n > 0$ integer and even.

98 Example Figures 5.10 through 5.12 shew a few transformations of the function $x \mapsto x^3$.

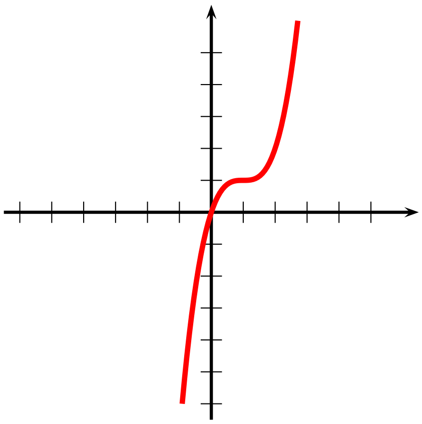

Figure 5.10: $y = (x-1)^3 + 1$

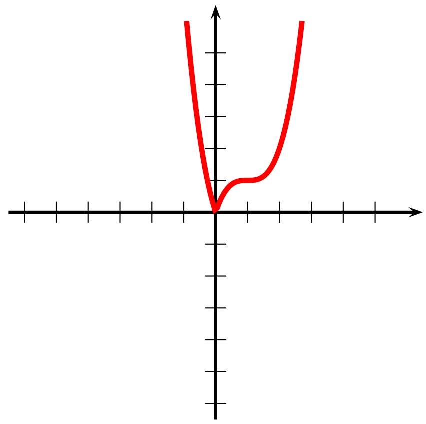

Figure 5.11: $y = |(x-1)^3 + 1|$

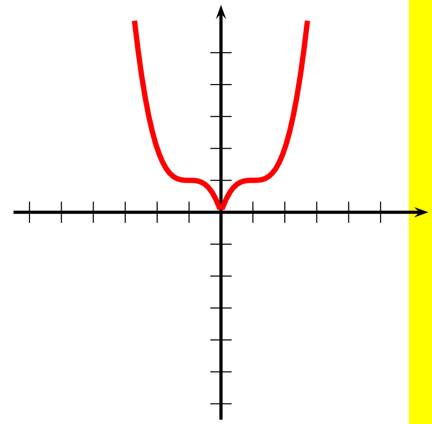

Figure 5.12: $y = (|x|-1)^3 + 1$

5.2 Sum Rule

In this section we prove two more differentiation rules.

99 Theorem (Constant-Times-Function Rule) If $k \in \mathbb{R}$ is a real number constant, and f is a strongly differentiable function at x, then kf is strongly differentiable at x and $(kf)'(x) = kf'(x)$.

Proof: *We have*

$$\begin{aligned} (kf)(x+\varepsilon) &= k(f(x+\varepsilon)) \\ &= k\left(f(x) + f'(x)\varepsilon + O(\varepsilon^2)\right) \\ &= kf(x) + kf'(x)\varepsilon + O(\varepsilon^2), \end{aligned}$$

from where the theorem follows. ❑

100 Example We have proved that $(x^2)' = 2x$. Hence $(-3x^2)' = -3(x^2)' = -3(2x) = -6x$.

101 Theorem (Sum Rule) If f, g are strongly differentiable functions at x, then $f+g$ is strongly differentiable at x and

$$(f+g)'(x) = f'(x) + g'(x).$$

Proof: *We have*

$$\begin{aligned} f(x+\varepsilon) + g(x+\varepsilon) &= (f(x) + f'(x)\varepsilon + O(\varepsilon^2)) + (g(x) + g'(x)\varepsilon + O(\varepsilon^2)) \\ &= (f(x) + g(x)) + (f'(x) + g'(x))\varepsilon + O(\varepsilon^2), \end{aligned}$$

from where the theorem follows. ❑

102 Example We have proved that $(x^2)' = 2x$ and that $(x)' = 1$. Hence $(x^2 + x)' = (x^2)' + (x)' = 2x + 1$.

103 Example Let $f(x) = 2x^3 - x^2 + 5x - 1$. Find $f'(1)$ and $f''(-1)$.

Solution: We have

$$f'(x) = 2(3x^2) - 2x + 5 = 6x^2 - 2x + 5, \qquad f''(x) = 2(6x) - 2 = 12x - 2.$$

Hence $f'(1) = 9$ and $f''(-1) = -14$.

5.3 Affine Functions

104 Definition Let m, k be real number constants. A function of the form $x \mapsto mx + k$ is called an *affine function*. In the particular case that $m = 0$, we call $x \mapsto k$ a *constant function*. If, however, $k = 0$, then we call the function $x \mapsto mx$ a *linear function*.

By virtue of Theorem 17, the graph of the function $x \mapsto mx + k$ is a straight line. Since the derivative of $x \mapsto mx + k$ is $(mx+k)' = m$, we see that $x \mapsto mx + k$ is strictly increasing if $m > 0$ and strictly decreasing if $m < 0$ in view of Theorem 84. If $m \neq 0$ then $mx + k = 0 \implies x = -\frac{k}{m}$, meaning that $x \mapsto mx + k$ has a unique zero (crosses the x-axis) at $x = -\frac{k}{m}$. This information is summarised in the following tables.

x	$-\infty$	$-\frac{k}{m}$	$+\infty$
$f(x) = mx + k$	$\nearrow$	0	$\nearrow$

Table 5.3: $x \mapsto mx + k$, with $m > 0$.

x	$-\infty$	$-\frac{k}{m}$	$+\infty$
$f(x) = mx + k$	$\searrow$	0	$\searrow$

Table 5.4: $x \mapsto mx + k$, with $m < 0$.

5.4 Quadratic Functions

105 Definition Let a, b, c be real numbers, with $a \neq 0$. A function of the form

$$f: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & ax^2 + bx + c \end{array}$$

is called a *quadratic function*.

106 Theorem Let $a \neq 0, b, c$ be real numbers and let $x \mapsto ax^2 + bx + c$ be a quadratic function. Then its graph is a parabola. If $a > 0$ the parabola has a local minimum at $x = -\frac{b}{2a}$ and it is convex. If $a < 0$ the parabola has a local maximum at $x = -\frac{b}{2a}$ and it is concave.

Proof: *Put $f(x) = ax^2 + bx + c$. Completing squares,*

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}, \end{aligned}$$

and hence this is a horizontal translation $-\frac{b}{2a}$ units and a vertical translation $\frac{4ac-b^2}{4a}$ units of the square function $x \mapsto x^2$ and so it follows from example 34 and Theorems 40 and 48, that the graph of f is a parabola.

We have $f'(x) = 2ax + b$. Assume first that $a > 0$. Then

$$f'(x) > 0 \iff x > -\frac{b}{2a}, \qquad f'(x) < 0 \iff x < -\frac{b}{2a}.$$

Thus the function decreases for values $< -\frac{b}{2a}$ and increases otherwise. Hence in view of Theorem 87, it must have a minimum at $x = -\frac{b}{2a}$. Since $f''(x) = 2a > 0$, f is convex by virtue of Theorem 90. The case when $a < 0$ can be similarly treated. ❑

The information of Theorem 106 is summarised in the following tables.

x	$-\infty$	$-\dfrac{b}{2a}$	$+\infty$
$f(x) = ax^2 + bx + c$	$\searrow$	0	$\nearrow$

Table 5.5: $x \mapsto ax^2 + bx + c$, with $a > 0$.

x	$-\infty$	$-\dfrac{b}{2a}$	$+\infty$
$f(x) = ax^2 + bx + c$	$\nearrow$	0	$\searrow$

Table 5.6: $x \mapsto ax^2 + bx + c$, with $a < 0$.

107 Definition The point $\left(-\dfrac{b}{2a}, \dfrac{4ac - b^2}{4a}\right)$ lies on the parabola and it is called the *vertex* of the parabola $y = ax^2 + bx + c$. The quantity $b^2 - 4ac$ is called the *discriminant* of $ax^2 + bx + c$. The equation

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

is called the *canonical equation of the parabola* $y = ax^2 + bx + c$.

The parabola $x \mapsto ax^2 + bx + c$ is symmetric about the vertical line $x = -\dfrac{b}{2a}$ passing through its vertex.

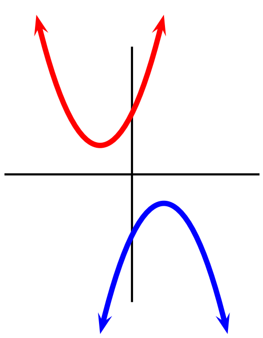

Figure 5.13: No real zeroes.

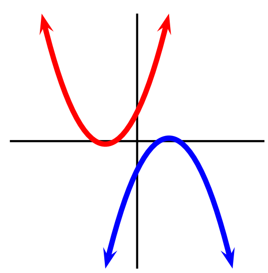

Figure 5.14: One real zero.

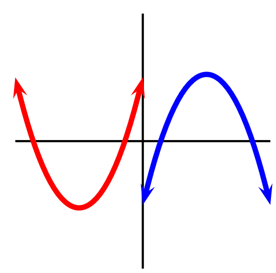

Figure 5.15: Two real zeros.

108 Corollary (Quadratic Formula) The roots of the equation $ax^2 + bx + c = 0$ are given by the formula

$$ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{5.1}$$

If $a \neq 0, b, c$ are real numbers and $b^2 - 4ac = 0$, the parabola $x \mapsto ax^2 + bx + c$ is tangent to the x-axis and has one (repeated) real root. If $b^2 - 4ac > 0$ then the parabola has two distinct real roots. Finally, if $b^2 - 4ac < 0$ the parabola has two complex roots.

Proof: *By Theorem 106 we have*

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a},$$

and so

$$\begin{aligned} ax^2+bx+c=0 &\iff \left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2} \\ &\iff x+\frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2|a|} \\ &\iff x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}, \end{aligned}$$

where we have dropped the absolute values on the last line because the only effect of having $a<0$ is to change from $\pm$ to $\mp$.

If $b^2-4ac=0$ then the vertex of the parabola is at $\left(-\frac{b}{2a},0\right)$ on the x-axis, and so the parabola is tangent there. Also, $x=-\frac{b}{2a}$ would be the only root of this equation. This is illustrated in figure 5.14.

If $b^2-4ac>0$, then $\sqrt{b^2-4ac}$ is a real number $\neq 0$ and so $\frac{-b-\sqrt{b^2-4ac}}{2a}$ and $\frac{-b+\sqrt{b^2-4ac}}{2a}$ are distinct numbers. This is illustrated in figure 5.15.

If $b^2-4ac<0$, then $\sqrt{b^2-4ac}$ is a complex number $\neq 0$ and so $\frac{-b-\sqrt{b^2-4ac}}{2a}$ and $\frac{-b+\sqrt{b^2-4ac}}{2a}$ are distinct complex numbers. This is illustrated in figure 5.13. ❑

STOP *If a quadratic has real roots, then the vertex lies on a line crossing the midpoint between the roots.*

109 Example Consider the quadratic function $f(x)=x^2-5x+3$.

❶ Find $f'(x)$. Solve $f'(x)=0$ and hence find the vertex of f. Determine the intervals of monotonicity of f.

❷ Write this parabola in canonical form.

❸ Determine $f''(x)$ and find the convexity intervals of f.

❹ Find the x-intercepts and y-intercepts of f.

❺ Graph $y=f(x)$, $y=|f(x)|$, and $y=f(|x|)$.

❻ Determine the set of real numbers x for which $f(x)>0$.

Solution:

❶ We have $f'(x)=2x-5$. Now, $2x-5=0 \implies x=\frac{5}{2}$. At $x=\frac{5}{2}$ we have $f\left(\frac{5}{2}\right)=-\frac{13}{4}$, whence the vertex is at $\left(\frac{5}{2},-\frac{13}{4}\right)$. Also,

$$f'(x)>0 \implies 2x-5>0 \implies x>\frac{5}{2},$$

and f will be increasing for $x>\frac{5}{2}$. It will be decreasing for $x<\frac{5}{2}$.

❷ Completing squares

$$y=x^2-5x+3=\left(x-\frac{5}{2}\right)^2-\frac{13}{4}.$$

❸ We have $f''(x)=(2x)'=2$. Since $f''(x)=2>0$ for all real values x, f is concave for all real values of x.

❹ For $x=0$, $f(0)=0^2-5\cdot 0+3=3$, and hence $y=f(0)=3$ is the y-intercept. By the quadratic formula,

$$f(x)=0 \iff x^2-5x+3=0 \iff x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(3)}}{2(1)}=\frac{5\pm\sqrt{13}}{2}.$$

Observe that $\frac{5-\sqrt{13}}{2}\approx 0.697224362$ and $\frac{5+\sqrt{13}}{2}\approx 4.302775638$.

❺ The graphs appear in figures 5.16 through 5.18.

❻ From the graph in figure 5.16, $x^2-5x+3>0$ for values $x\in \left]-\infty\,;\frac{5-\sqrt{13}}{2}\right[$ or $x\in \left]\frac{5+\sqrt{13}}{2}\,;+\infty\right[$.

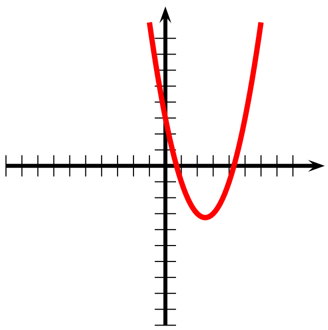

Figure 5.16: $y=x^2-5x+3$

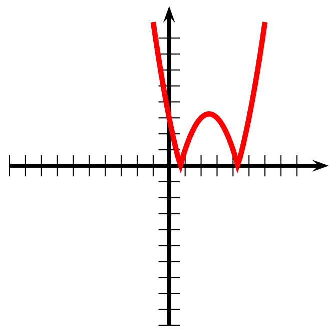

Figure 5.17: $y=|x^2-5x+3|$

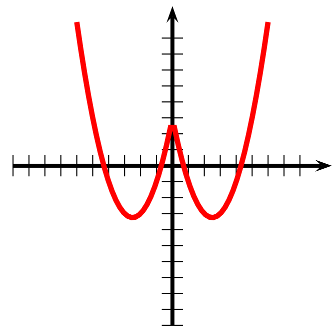

Figure 5.18: $y=|x|^2-5|x|+3$

110 Corollary If $a\neq 0, b, c$ are real numbers and if $b^2-4ac<0$, then ax^2+bx+c has the same sign as a.

Proof: *Since*

$$ax^2+bx+c=a\left(\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right),$$

and $4ac-b^2>0$, $\left(\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right)>0$ *and so* ax^2+bx+c *has the same sign as* a. ❑

111 Example Prove that the quantity $q(x)=2x^2+x+1$ is positive regardless of the value of x.

Solution: The discriminant is $1^2-4(2)(1)=-7<0$, hence the roots are complex. By Corollary 110, since its leading coefficient is $2>0$, $q(x)>0$ regardless of the value of x. Another way of seeing this is to complete squares and notice the inequality

$$2x^2+x+1=2\left(x+\frac{1}{4}\right)^2+\frac{7}{8}\geq\frac{7}{8},$$

since $\left(x+\frac{1}{4}\right)^2$ being the square of a real number, is ≥ 0.

By Corollary 108, if $a\neq 0, b, c$ are real numbers and if $b^2-4ac\neq 0$ then the numbers

$$r_1=\frac{-b-\sqrt{b^2-4ac}}{2a}\qquad\text{and}\qquad r_2=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

are distinct solutions of the equation $ax^2+bx+c=0$. Since

$$r_1+r_2=-\frac{b}{a},\qquad\text{and}\qquad r_1r_2=\frac{c}{a},$$

any quadratic can be written in the form

$$ax^2+bx+c=a\left(x^2+\frac{bx}{a}+\frac{c}{a}\right)=a\left(x^2-(r_1+r_2)x+r_1r_2\right)=a(x-r_1)(x-r_2).$$

We call $a(x-r_1)(x-r_2)$ a *factorisation* of the quadratic ax^2+bx+c.

112 Example A quadratic polynomial p has $1 \pm \sqrt{5}$ as roots and it satisfies $p(1) = 2$. Find its equation.

Solution: Observe that the sum of the roots is

$$r_1 + r_2 = 1 - \sqrt{5} + 1 + \sqrt{5} = 2$$

and the product of the roots is

$$r_1 r_2 = (1 - \sqrt{5})(1 + \sqrt{5}) = 1 - (\sqrt{5})^2 = 1 - 5 = -4.$$ [1]

Hence p has the form

$$p(x) = a\left(x^2 - (r_1 + r_2)x + r_1 r_2\right) = a(x^2 - 2x - 4).$$

Since

$$2 = p(1) \implies 2 = a(1^2 - 2(1) - 4) \implies a = -\frac{2}{5},$$

the polynomial sought is

$$p(x) = -\frac{2}{5}\left(x^2 - 2x - 4\right).$$

5.5 Product Rule and Chain Rule

We now develop tools for differentiating more complex formulæ.

113 Theorem (Product Rule) If f, g are are strongly differentiable functions at x, then fg is strongly differentiable at x and then $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

Proof: *We have*

$$\begin{aligned} f(x+\varepsilon)g(x+\varepsilon) &= \left(f(x) + f'(x)\varepsilon + O(\varepsilon^2)\right)\left(g(x) + g'(x)\varepsilon + O(\varepsilon^2)\right) \\ &= f(x)g(x) + \left(f'(x)g(x) + f(x)g'(x)\right)\varepsilon + O(\varepsilon^2), \end{aligned}$$

from where the theorem follows. ❑

It is **not true** *in general that* $(fg)' = f'g'$.

114 Example Let $f(x) = x^3$ and $g(x) = x^4$. Then from the Product Rule

$$(x^7)' = (x^3 \cdot x^4)' = x^4(x^3)' + x^3(x^4)' = x^4(3x^2) + x^4(4x^2) = 3x^6 + 4x^6 = 7x^6,$$

which is what we expect from the Power Rule.

By recurrence we can apply the product rule to more than two functions.

115 Example

$$\begin{aligned} (x(x+x^2)(1+x+x^2))' &= (x)'(x+x^2)(1+x+x^2) + x(x+x^2)'(1+x+x^2) + x(x+x^2)(1+x+x^2)' \\ &= 1(x+x^2)(1+x+x^2) + x(1+2x)(1+x+x^2) + x(x+x^2)(1+2x) \\ &= (x+2x^2+2x^3+x^4) + (x+3x^2+3x^3+2x^4) + (x^2+3x^3+2x^4) \\ &= 2x + 6x^2 + 8x^3 + 5x^4. \end{aligned}$$

116 Theorem (Chain Rule) If g is strongly differentiable at x and f is strongly differentiable at $g(x)$, then $f \circ g$ is strongly differentiable at x and $(f \circ g)'(x) = f'(g(x))\, g'(x)$

[1] As a shortcut for this multiplication you may wish to recall the *difference of squares identity*: $(a-b)(a+b) = a^2 - b^2$.

Proof: *We have, putting* $\varepsilon_1 = g'(x)\varepsilon + O(\varepsilon^2)$,

$$\begin{aligned} f(g(x+\varepsilon)) &= f(g(x)+g'(x)\varepsilon+O(\varepsilon^2)) \\ &= f(g(x)+\varepsilon_1) \\ &= f(g(x))+f'(g(x))\varepsilon_1+O(\varepsilon_1^2) \\ &= f(g(x))+f'(g(x))(g'(x)\varepsilon+O(\varepsilon^2))+O((g'(x)\varepsilon+O(\varepsilon^2))^2) \\ &= f(g(x))+f'(g(x))g'(x)\varepsilon+O(\varepsilon^2) \end{aligned}$$

and the theorem follows. ❑

117 Example Consider $h(x) = (x+1)^2$. Then $h(x) = (f\circ g)(x)$ with $f(x) = x^2$ and $g(x) = x+1$. Hence

$$((x+1)^2)' = (f\circ g)'(x) = f'(g(x))g'(x) = 2(x+1)^1(1) = 2x+2.$$

118 Example

$$((x^2+x)^3)' = 3(x^2+x)^2(2x+1).$$

119 Example Using the Product Rule and the Chain Rule,

$$(x(x+a)^2)' = (x+a)^2(x)'+x((x+a)^2)' = (x+a)^2+2x(x+a) = (x+a)(3x+a).$$

120 Example Using the Product Rule and the Chain Rule,

$$\begin{aligned} (x(x+a)^2(x+b)^3)' &= (x+a)^2(x+b)^3(x)'+x(x+b)^3((x+a)^2)'+x(x+a)^2((x+b)^3)' \\ &= (x+a)^2(x+b)^3+2x(x+a)(x+b)^3+3x(x+a)^2(x+b)^2 \\ &= (x+a)(x+b)^2((x+a)(x+b)+2x(x+b)+3x(x+a)) \\ &= (x+a)(x+b)^2(6x^2+x(4a+3b)+ab). \end{aligned}$$

121 Example Let f be strongly differentiable with $f(4) = a$ and $f'(4) = b$. If $g(x) = x^2f(x^2)$, find $g'(2)$.

Solution: Using both the Product Rule and the Chain Rule

$$g'(x) = 2xf(x^2)+x^2f'(x^2)(2x) = 2xf(x^2)+2x^3f'(x^2).$$

Hence $g'(2) = 2(2)f(4)+2(8)f'(4) = 4a+16b$.

5.6 Polynomials

5.6.1 Roots

In sections 5.3 and 5.4 we learned how to find the roots of equations (in the unknown x) of the type $ax+b = 0$ and $ax^2+bx+c = 0$, respectively. We would like to see what can be done for equations where the power of x is higher than 2. We recall that

122 Definition A *polynomial* $p(x)$ of degree $n\in\mathbb{N}$ is an expression of the form

$$p(x) = a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0, \qquad a_n\neq 0, \qquad a_k\in\mathbb{R},$$

where the a_k are constants. If the a_k are all integers then we say that p has integer coefficients, and we write $p(x)\in\mathbb{Z}[x]$; if the a_k are real numbers then we say that p has real coefficients and we write $p(x)\in\mathbb{R}[x]$; etc. The degree of the polynomial p is denoted by $\deg p$. The coefficient a_n is called the *leading coefficient* of $p(x)$. A *root* of p is a solution to the equation $p(x) = 0$.

123 Example Here are a few examples of polynomials.

- $a(x) = 2x+1 \in \mathbb{Z}[x]$, is a polynomial of degree 1, and leading coefficient 2. It has $x = -\dfrac{1}{2}$ as its only root. A polynomial of degree 1 is also known as an *affine function*.
- $b(x) = \pi x^2 + x - \sqrt{3} \in \mathbb{R}[x]$, is a polynomial of degree 2 and leading coefficient π. By the quadratic formula b has the two roots
$$x = \frac{-1+\sqrt{1+4\pi\sqrt{3}}}{2\pi} \qquad \text{and} \qquad x = \frac{-1-\sqrt{1+4\pi\sqrt{3}}}{2\pi}.$$
A polynomial of degree 2 is also called a *quadratic polynomial* or *quadratic function*.
- $C(x) = 1 \equiv 1 \cdot x^0$[2], is a constant polynomial, of degree 0. It has no roots, since it is never zero.

124 Theorem The degree of the product of two polynomials is the sum of their degrees. In symbols, if p, q are polynomials, $\deg pq = \deg p + \deg q$.

Proof: *If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, and $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$, with $a_n \neq 0$ and $b_m \neq 0$ then upon multiplication,*
$$p(x)q(x) = (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)(b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0) = a_n b_m x^{m+n} + \cdots +,$$
with non-vanishing leading coefficient $a_n b_m$. ❑

125 Example The polynomial $p(x) = (1+2x+3x^3)^4(1-2x^2)^5$ has leading coefficient $3^4(-2)^5 = -2592$ and degree $3\cdot 4 + 2\cdot 5 = 22$.

126 Example What is the degree of the polynomial identically equal to 0? Put $p(x) \equiv 0$ and, say, $q(x) = x+1$. Then by Theorem 124 we must have $\deg pq = \deg p + \deg q = \deg p + 1$. But pq is identically 0, and hence $\deg pq = \deg p$. But if $\deg p$ were finite then
$$\deg p = \deg pq = \deg p + 1 \implies 0 = 1^{[3]},$$
nonsense. Thus the 0-polynomial does not have any finite degree. We attach to it, by convention, degree $-\infty$.

127 Definition If all the roots of a polynomial are in $\mathbb{Z}$ (integer roots), then we say that the *polynomial splits or factors over* $\mathbb{Z}$. If all the roots of a polynomial are in $\mathbb{Q}$ (rational roots), then we say that the *polynomial splits or factors over* $\mathbb{Q}$. If all the roots of a polynomial are in $\mathbb{C}$ (complex roots), then we say that the *polynomial splits (factors) over* $\mathbb{C}$.

STOP *Since $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, any polynomial splitting on a smaller set immediately splits over a larger set.*

128 Example The polynomial $l(x) = x^2 - 1 = (x-1)(x+1)$ splits over $\mathbb{Z}$. The polynomial $p(x) = 4x^2 - 1 = (2x-1)(2x+1)$ splits over $\mathbb{Q}$ but not over $\mathbb{Z}$. The polynomial $q(x) = x^2 - 2 = (x-\sqrt{2})(x+\sqrt{2})$ splits over $\mathbb{R}$ but not over $\mathbb{Q}$. The polynomial $r(x) = x^2 + 1 = (x-i)(x+i)$ splits over $\mathbb{C}$ but not over $\mathbb{R}$. Here $i = \sqrt{-1}$ is the imaginary unit.

5.6.2 Taylor Polynomials

In order to motivate the following theorem, let us consider the next example.

129 Example Write x^2 as a sum of powers of $x-1$.

Solution: Observe that $x = x-1+1$ and use the identity $(a+b)^2 = a^2 + 2ab + b^2$ to obtain
$$x^2 = (x-1+1)^2 = (x-1)^2 + 2(x-1) + 1.$$

If such an identity is not known, one can proceed as follows, giving an idea of a general procedure. Put
$$x^2 = a + b(x-1) + c(x-1)^2,$$

[2] The symbol $\equiv$ is read "identically equal to" and it means that both expressions are always the same, regardless of the value of the input parameter.
[3] Much to the chagrin of our Vice-President for Academic Affairs—who claims that $1 = 2$—it is not true that $0 = 1$.

where we stop at the second power since x^2 has degree 2. Let $x = 1$. Then $1 = a$. Differentiate to obtain

$$2x = b + 2c(x-1).$$

Let again $x = 1$. This gives $2 = b$. Differentiate a second time to obtain

$$2 = 2c,$$

whence $c = 1$. Hence we have $a = 1, b = 2, c = 1$ and so

$$x^2 = a + b(x-1) + c(x-1)^2 = 1 + 2(x-1) + (x-1)^2,$$

as before.

130 Theorem (Taylor Polynomials) Let $a \in \mathbb{R}$. Then any polynomial $p(x)$ of degree n can be written as

$$p(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_n(x-a)^n,$$

for some constants b_k.

Proof: *First observe that we stop at $(x-a)^n$ since p has degree n. Differentiating k times we obtain*

$$p^{(k)}(x) = b_k k! + (k+1)(k)\cdots(2)b_{k+1}(x-a) + \cdots + (n)(n-1)\cdots(n-k+1)b_n(x-a)^{n-k}.$$ [4]

Letting $x = a$ we obtain

$$b_k = \frac{p^{(k)}(a)}{k!},$$

proving the theorem. ❑

131 Definition The expansion

$$p(x) = p(a) + p'(a)(x-a) + \frac{p''(a)}{2!}(x-a)^2 + \cdots + \frac{p^{(n)}(a)}{n!}(x-a)^n \tag{5.2}$$

is known as the *Taylor polynomial expansion* about $x = a$ of p.

132 Example Find the Taylor polynomial expansion about $x = -2$ of $p(x) = x^3 + 2x + 1$.

Solution: We have

$$p'(x) = 3x^2 + 2, \qquad p''(x) = 6x, \qquad p'''(x) = 6.$$

Hence

$$p(-2) = -11, \qquad p'(-2) = 14, \qquad p''(-2) = -12, \qquad p'''(-2) = 6,$$

and

$$x^3 + 2x + 1 = -11 + 14(x+2) + \frac{-12}{2}(x+2)^2 + \frac{6}{6}(x+2)^3 = -11 + 14(x+2) - 6(x+2)^2 + (x+2)^3.$$

5.6.3 Ruffini's Factor Theorem

133 Theorem (Ruffini's Factor Theorem) The polynomial $p(x)$ is divisible by $x - a$ if and only if $p(a) = 0$. Thus if p is a polynomial of degree n, then $p(a) = 0$ if and only if $p(x) = (x-a)q(x)$ for some polynomial q of degree $n-1$.

Proof: *The Taylor expansion of p about $x = a$ is*

$$p(x) = p(a) + (x-a)\left(p'(a) + \frac{p''(a)}{2!}(x-a) + \cdots + \frac{p^{(n)}(a)}{n!}(x-a)^{n-1}\right),$$

from where the result quickly follows. ❑

[4] The symbol $k!$—read "k factorial"—is the product $1 \cdot 2 \cdots k$. Thus for example $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$. We define $0! = 1$.

134 Example Find the value of a so that the polynomial

$$t(x) = x^3 - 3ax^2 + 2$$

be divisible by $x+1$.

Solution: By Ruffini's Theorem 133, we must have

$$0 = t(-1) = (-1)^3 - 3a(-1)^2 + 2 \implies a = \frac{1}{3}.$$

135 Definition Let a be a root of a polynomial p. We say that a is a root of *multiplicity* m if $p(x)$ is divisible by $(x-a)^m$ but not by $(x-a)^{m+1}$. This means that p can be written in the form $p(x) = (x-a)^m q(x)$ for some polynomial q with $q(a) \neq 0$.

136 Corollary The number a is a root of multiplicity m if an only if

$$p(a) = p'(a) = p''(a) = \cdots = p^{(m-1)}(a) = 0, \qquad p^{(m)}(a) \neq 0.$$

> **Proof:** *This follows immediately by considering the Taylor expansion of p about $x = a$.* ❑

137 Example Factor the polynomial $p(x) = x^5 - 5x^4 + 11x^3 - 13x^2 + 8x - 2$ over $\mathbb{Z}[x]$.

Solution: We see that $p(1) = 0, p'(1) = 0, p''(1) = 0, p'''(1) \neq 0$. Hence $(x-1)^3 = x^3 - 3x^2 + 3x - 1$ divides p. By long division

$$\begin{array}{r|rrrrrr}
 & & & x^2 & -2x & +2 & \\ \cline{2-7}
x^3-3x^2+3x-1 & x^5 & -5x^4 & +11x^3 & -13x^2 & +8x & -2 \\
 & -x^5 & +3x^4 & -3x^3 & +x^2 & & \\ \cline{2-5}
 & & -2x^4 & +8x^3 & -12x^2 & +8x & \\
 & & 2x^4 & -6x^3 & +6x^2 & -2x & \\ \cline{3-6}
 & & & 2x^3 & -6x^2 & +6x & -2 \\
 & & & -2x^3 & +6x^2 & -6x & +2 \\ \cline{4-7}
 & & & & & & 0
\end{array}$$

and so

$$x^5 - 5x^4 + 11x^3 - 13x^2 + 8x - 2 = (x-1)^3(x^2 - 2x + 2).$$

Observe that $x^2 - 2x + 2$ does not factor over $\mathbb{Z}[x]$ and hence we are finished.

138 Corollary If a polynomial of degree n had any roots at all, then it has at most n roots.

> **Proof:** *If it had at least $n+1$ roots then it would have at least $n+1$ factors of degree 1 and hence degree $n+1$ at least, a contradiction.* ❑

Notice that the above theorem only says that if a polynomial has any roots, then it must have at most its degree number of roots. It does not say that a polynomial must possess a root. That all polynomials have at least one root is much more difficult to prove. We will quote the theorem, without a proof.

139 Theorem (Fundamental Theorem of Algebra) A polynomial of degree at least one with complex number coefficients has at least one complex root.

> STOP *The Fundamental Theorem of Algebra implies then that a polynomial of degree n has* exactly *n roots (counting multiplicity).*

A more useful form of Ruffini's Theorem is given in the following corollary.

140 Corollary If the polynomial p with integer coefficients,

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

has a rational root $\frac{s}{t} \in \mathbb{Q}$ (here $\frac{s}{t}$ is assumed to be in lowest terms), then s divides a_0 and t divides a_n.

Proof: *We are given that*

$$0 = p\left(\frac{s}{t}\right) = a_n\left(\frac{s^n}{t^n}\right) + a_{n-1}\left(\frac{s^{n-1}}{t^{n-1}}\right) + \cdots + a_1\left(\frac{s}{t}\right) + a_0.$$

Clearing denominators,

$$0 = a_n s^n + a_{n-1}s^{n-1}t + \cdots + a_1 s t^{n-1} + a_0 t^n.$$

This last equality implies that

$$-a_0 t^n = s(a_n s^{n-1} + a_{n-1}s^{n-2}t + \cdots + a_1 t^{n-1}).$$

Since both sides are integers, and since s and t have no factors in common, then s must divide a_0. We also gather that

$$-a_n s^n = t(a_{n-1}s^{n-1} + \cdots + a_1 s t^{n-2} + a_0 t^{n-1}),$$

from where we deduce that t divides a_n, concluding the proof. ❑

141 Example Factorise $a(x) = x^3 - 3x - 5x^2 + 15$ over $\mathbb{Z}[x]$ and over $\mathbb{R}[x]$.

Solution: By Corollary 140, if $a(x)$ has integer roots then they must be in the set $\{-1, 1, -3, 3, -5, 5\}$. We test $a(\pm 1), a(\pm 3), a(\pm 5)$ to see which ones vanish. We find that $a(5) = 0$. By the Factor Theorem, $x - 5$ divides $a(x)$. Using long division,

$$\begin{array}{r|l} & x^2 \qquad - 3 \\ \hline x-5 & x^3 - 5x^2 - 3x + 15 \\ & \underline{-x^3 + 5x^2} \\ & \qquad\quad - 3x + 15 \\ & \qquad\quad \underline{3x - 15} \\ & \qquad\qquad\quad 0 \end{array}$$

we find

$$a(x) = x^3 - 3x - 5x^2 + 15 = (x-5)(x^2 - 3),$$

which is the required factorisation over $\mathbb{Z}[x]$. The factorisation over $\mathbb{R}[x]$ is then

$$a(x) = x^3 - 3x - 5x^2 + 15 = (x-5)(x-\sqrt{3})(x+\sqrt{3}).$$

142 Example Factorise $b(x) = x^5 - x^4 - 4x + 4$ over $\mathbb{Z}[x]$ and over $\mathbb{R}[x]$.

Solution: By Corollary 140, if $b(x)$ has integer roots then they must be in the set $\{-1, 1, -2, 2, -4, 4\}$. We quickly see that $b(1) = 0$, and so, by the Factor Theorem, $x - 1$ divides $b(x)$. By long division

$$\begin{array}{r|l} & x^4 \qquad - 4 \\ \hline x-1 & x^5 - x^4 - 4x + 4 \\ & \underline{-x^5 + x^4} \\ & \qquad\quad - 4x + 4 \\ & \qquad\quad \underline{4x - 4} \\ & \qquad\qquad\quad 0 \end{array}$$

we see that

$$b(x) = (x-1)(x^4-4) = (x-1)(x^2-2)(x^2+2),$$

which is the desired factorisation over $\mathbb{Z}[x]$. The factorisation over $\mathbb{R}$ is seen to be

$$b(x) = (x-1)(x-\sqrt{2})(x+\sqrt{2})(x^2+2).$$

Since the discriminant of x^2+2 is $-8<0$, x^2+2 does not split over $\mathbb{R}$.

143 Lemma Complex roots of a polynomial with real coefficients occur in conjugate pairs, that is, if p is a polynomial with real coefficients and if $u+vi$ is a root of p, then its conjugate $u-vi$ is also a root for p. Here $i=\sqrt{-1}$ is the imaginary unit.

Proof: *Assume*

$$p(x) = a_0 + a_1x + \cdots + a_nx^n$$

and that $p(u+vi)=0$. *Since the conjugate of a real number is itself, and conjugation is multiplicative (Theorem 190), we have*

$$\begin{aligned} 0 &= \overline{0} \\ &= \overline{p(u+vi)} \\ &= \overline{a_0 + a_1(u+vi) + \cdots + a_n(u+vi)^n} \\ &= \overline{a_0} + \overline{a_1(u+vi)} + \cdots + \overline{a_n(u+vi)^n} \\ &= a_0 + a_1(u-vi) + \cdots + a_n(u-vi)^n \\ &= p(u-vi), \end{aligned}$$

whence $u-vi$ is also a root. ❑

Since the complex pair root $u \pm vi$ would give the polynomial with real coefficients

$$(x-u-vi)(x-u+vi) = x^2 - 2ux + (u^2+v^2),$$

we deduce the following theorem.

144 Theorem Any polynomial with real coefficients can be factored in the form

$$A(x-r_1)^{m_1}(x-r_2)^{m_2}\cdots(x-r_k)^{m_k}(x^2+a_1x+b_1)^{n_1}(x^2+a_2x+b_2)^{n_2}\cdots(x^2+a_lx+b_l)^{n_l},$$

where each factor is distinct, the m_i, l_k are positive integers and A, r_i, a_i, b_i are real numbers.

5.7 Graphs of Polynomials

We start with the following theorem, which we will state without proof.

145 Theorem A polynomial function $x \mapsto p(x)$ is an everywhere continuous function.

146 Theorem Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \quad a_n \neq 0$, be a polynomial with real number coefficients. Then

$$p(-\infty) = (\text{signum}\,(a_n))(-1)^n\infty, \qquad p(+\infty) = (\text{signum}\,(a_n))\infty.$$

Thus a polynomial of odd degree will have opposite signs for values of large magnitude and different sign, and a polynomial of even degree will have the same sign for values of large magnitude and different sign.

Proof: *If $x \neq 0$ then*

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = a_nx^n\left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \sim a_nx^n,$$

since as $x \to \pm\infty$, the quantity in parenthesis tends to 1 and so the eventual sign of $p(x)$ is determined by a_nx^n, which gives the result. ❑

147 Corollary A polynomial of odd degree with real number coefficients always has a real root.

> **Proof:** *Since a polynomial of odd degree eventually changes sign, since it is continuous, the corollary follows from Bolzano's Intermediate Value Theorem 67.* ❑

148 Example Consider the polynomial $p(x) = x^3 + 4x^2 + x - 6$.

1. Prove that p splits over $\mathbb{Z}$ and find its factorisation. Also, determine its y-intercept.
2. Determine $p(-\infty)$ and $p(+\infty)$
3. Find p' and determine the intervals of monotonicity of p.
4. Determine any local extrema of p.
5. Find p'' and determine the inflexion points of p and its convexity intervals.
6. Obtain an approximate graph of p.

Solution:

1. By Corollary 140, if there are integral roots of p they must divide -6. A quick inspection shews that $p(1) = 0$ and so $x-1$ divides $p(x)$. By long division
$$\begin{array}{rrrrr} & & x^2 & +5x & +6 \\ \hline x-1) & x^3 & +4x^2 & +x & -6 \\ & -x^3 & +x^2 & & \\ \hline & & 5x^2 & +x & \\ & & -5x^2 & +5x & \\ \hline & & & 6x & -6 \\ & & & -6x & +6 \\ \hline & & & & 0 \end{array}$$
whence
$$p(x) = (x-1)(x^2+5x+6) = (x-1)(x+2)(x+3).$$
This means that p crosses the x-axis at $x = -3$, $x = -2$, and $x = 1$. Its y-intercept is $(0, p(0)) = (0, -6)$.

2. Since the leading coefficient of p is $1 > 0$ and since p has odd degree, by Theorem 146, $p(x) \sim (x)(x)(x) = x^3$, as $x \to +\infty$ and so $p(-\infty) = -\infty$ and $p(+\infty) = +\infty$.

3. $p'(x) = 3x^2 + 8x + 1$, whose graph is a convex parabola. Using the Quadratic Formula
$$3x^2 + 8x + 1 = 0 \iff x = \frac{-4-\sqrt{13}}{3} \quad \text{or} \quad x = \frac{-4+\sqrt{13}}{3}$$
and so $x \approx -2.54$ or $x \approx -0.13$. Since p' is a convex parabola this means that
$$p'(x) > 0 \iff x \in \left]-\infty\,; \frac{-4-\sqrt{13}}{3}\right[\cup \left]\frac{-4+\sqrt{13}}{3}\,; +\infty\right[,$$
and so p is increasing (approximately) in the intervals $]-\infty\,; -2.54[$ and $]-0.13\,; +\infty[$.

4. Since at $x = -2.54$ p' changes sign from $+$ to $-$, p has a local maximum there by virtue of Theorem 87, which is $p(-2.54) \approx 0.88$. Also, p' changes sign from $-$ to $+$ at $x = -0.13$ and so p has a local minimum there, which is $p(-0.13) \approx -6.06$.

5. We find $p''(x) = 6x + 8$. Now, $p''(x) = 0 \implies x = -\frac{4}{3} \approx -1.33$ and $p(-1.33) \approx -2.61$. Hence p changes convexity (approximately) at $(-1.33, -2.61)$.

6. The graph of p can be found in figure 5.19.

149 Example Consider the polynomial $p(x) = x^3 + x + 1$.

1. Prove that p is strictly increasing.
2. Prove that p has no positive roots.
3. Determine $p(-\infty)$ and $p(+\infty)$
4. Prove that p has a unique real root and find an interval $[a;b]$ of length $< \frac{1}{4}$ containing this root.
5. Find p'' and determine the inflexion points of p and its convexity intervals.
6. Obtain an approximate graph of p.

Solution:

1. We have $p'(x) = 3x^2 + 1 \geq 1 > 0$ since x^2 is always positive.[5] Since the derivative of p is always strictly positive, p is always strictly increasing.
2. Since p is strictly increasing, $p(x) > p(0) = 1$ for $x > 0$. Hence values $x > 0$ can never make p zero.
3. By Theorem 146, $p(-\infty) = -\infty$ and $p(+\infty) = +\infty$.
4. Since p changes sign, it must have a root. Since p is strictly increasing, it can cross the x-axis only once. Now, observe that
$$p(0) = 1, \qquad p(-1) = -1$$
so the root must lie in $[-1;0]$. We bisect this interval and find $p(-0.5) \approx 0.375$, so the root must lie in $[-1;-0.5]$. We again bisect this interval and find that $p(-0.75) \approx -0.171875$, so the root must lie in $[-0.75;-0.5]$. Again, we bisect this interval and find that $p(-0.625) \approx 0.13$, so the root must lie in $[-0.75;-0.625]$. We now stop since we have reached an interval of within the desired length.
5. $p''(x) = 6x$ and so p is convex for $x > 0$ and concave for $x < 0$.
6. An approximate graph is shewn in figure 5.20

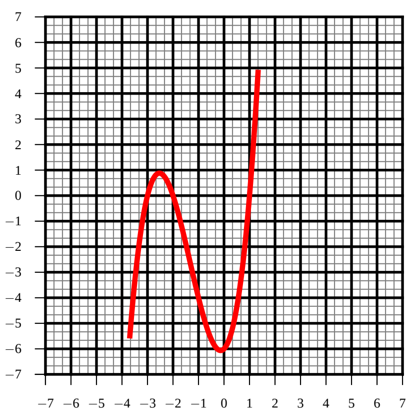

Figure 5.19: Example 148.

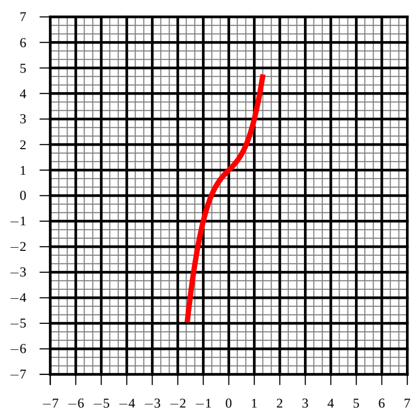

Figure 5.20: Example 149.

We now consider polynomials with real number coefficients and that split in $\mathbb{R}$. Such polynomials have the form

$$p(x) = a(x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_k)^{m_k},$$

where $a \neq 0$ and the r_i are real numbers and the $m_i \geq 1$ are integers. Graphing such polynomials will be achieved by referring to the following theorem.

[5] Another way of seeing that $3x^2 + 1 > 0$ always is by checking its discriminant.

150 Theorem Let $a \neq 0$ and the r_i are real numbers and the m_i be positive integers. Then the graph of the polynomial

$$p(x) = a(x-r_1)^{m_1}(x-r_2)^{m_2}\cdots(x-r_k)^{m_k},$$

- crosses the x-axis at $x = r_i$ if m_i is odd.
- is tangent to the x-axis at $x = r_i$ if m_i is even.
- has a convexity change at $x = r_i$ if $m_i \geq 3$ and m_i is odd.

Proof: *Since the local behaviour of $p(x)$ is that of $c(x-r_i)^{m_i}$ (where c is a real number constant) near r_i, the theorem follows at once from Theorem 97.* ❑

151 Example Make a rough sketch of the graph of $y = (x+2)x(x-1)$. Determine where it achieves its local extrema and their values. Determine where it changes convexity.

Solution: We have $p(x) = (x+2)x(x-1) \sim (x)\cdot x(x) = x^3$, as $x \to +\infty$. Hence $p(-\infty) = (-\infty)^3 = -\infty$ and $p(+\infty) = (+\infty)^3 = +\infty$. This means that for large negative values of x the graph will be on the negative side of the y-axis and that for large positive values of x the graph will be on the positive side of the y-axis. By Theorem 150, the graph crosses the x-axis at $x = -2$, $x = 0$, and $x = 1$.

Now, by the Product Rule,

$$\begin{aligned} p'(x) &= x(x-1)+(x+2)(x-1)+(x+2)x \\ &= 3x^2+2x-2. \end{aligned}$$

Using the quadratic formula,

$$3x^2+2x-2=0 \implies x = -\frac{1}{3}-\frac{\sqrt{7}}{3} \approx -1.22; \quad x = -\frac{1}{3}+\frac{\sqrt{7}}{3} \approx 0.55.$$

From geometrical considerations, $x \approx -1.22$ will be the x-coordinate of a local maximum, with y-coordinate $p(-1.22) \approx 2.11$ and $x \approx 0.55$ will be the x-coordinate of a local minimum, with y-coordinate $p(0.55) \approx -0.63$.

Also

$$p''(x) = 6x+2,$$

so $p''(x) > 0$ for $x > -\frac{1}{3}$ and $p''(x) < 0$ for $x < -\frac{1}{3}$. This means that p is convex for $x > -\frac{1}{3}$ and concave for $x < -\frac{1}{3}$. The graph is shewn in figure 5.21.

152 Example Make a rough sketch of the graph of $y = (x+2)^3x^2(1-2x)$.

Solution: We have $(x+2)^3x^2(1-2x) \sim x^3\cdot x^2(-2x) = -2x^6$. Hence if $p(x) = (x+2)^3x^2(1-2x)$ then $p(-\infty) = -2(-\infty)^6 = -\infty$ and $p(+\infty) = -2(+\infty)^6 = -\infty$, which means that for both large positive and negative values of x the graph will be on the negative side of the y-axis. By Theorem 150, in a neighbourhood of $x = -2$, $p(x) \sim 20(x+2)^3$, so the graph crosses the x-axis changing convexity at $x = -2$. In a neighbourhood of 0, $p(x) \sim 8x^2$ and the graph is tangent to the x-axis at $x = 0$. In a neighbourhood of $x = \frac{1}{2}$, $p(x) \sim \frac{25}{16}(1-2x)$, and so the graph crosses the x-axis at $x = \frac{1}{2}$.

Now,

$$\begin{aligned} p'(x) &= 3(x+2)^2x^2(1-2x)+2(x+2)^3x(1-2x)-2(x+2)^3x^2 \\ &= x(x+2)^2(3x(1-2x)+2(x+2)(1-2x)-2(x+2)x) \\ &= -x(x+2)^2(12x^2+7x-4), \end{aligned}$$

and $p'(x) = 0$ when $x = 0, -2, -\frac{7}{24}+\frac{\sqrt{241}}{24} \approx 0.36$, $-\frac{7}{24}-\frac{\sqrt{241}}{24} \approx -0.94$. From geometrical considerations, $x = 0$ and $x = -2$ are local minima, both with y-coordinate $y = 0$, and both $x \approx 0.36$ (with corresponding $y = p(0.36) \approx 0.48$) and $x \approx -0.94$ (with corresponding y-coordinate $y = p(-0.94) \approx 3.03$) are local maxima. The graph is shewn in figure 5.22.

153 Example Make a rough sketch of the graph of $y=(x+2)^2x(1-x)^2$.

Solution: The dominant term of $(x+2)^2x(1-x)^2$ is $x^2\cdot x(-x)^2=x^5$. Hence if $p(x)=(x+2)^2x(1-x)^2$ then $p(-\infty)=(-\infty)^5=-\infty$ and $p(+\infty)=(+\infty)^5=+\infty$, which means that for large negative values of x the graph will be on the negative side of the y-axis and for large positive values of x the graph will be on the positive side of the y-axis. By Theorem 150, the graph crosses the x-axis changing convexity at $x=-2$, it is tangent to the x-axis at $x=0$ and it crosses the x-axis at $x=\frac{1}{2}$. The graph is shewn in figure 5.23.

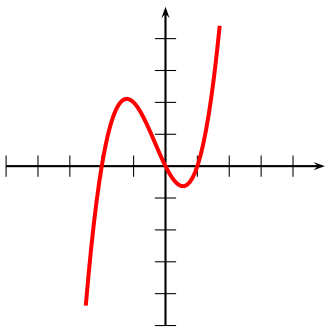
Figure 5.21: $y=(x+2)x(x-1)$.

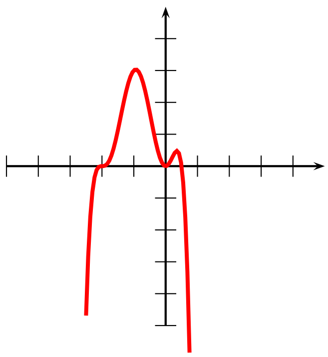
Figure 5.22: $y=(x+2)^3x^2(1-2x)$.

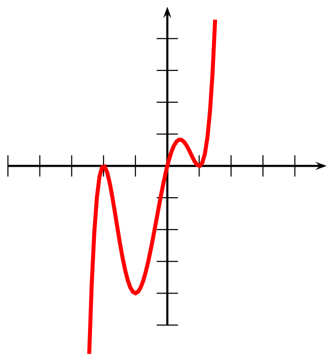
Figure 5.23: $y=(x+2)^2x(1-x)^2$.

Chapter 6

Rational Functions and Algebraic Functions

6.1 Inverse Power Functions

We now proceed to investigate the behaviour of functions of the type $x \mapsto \frac{1}{x^n}$, where $n > 0$ is an integer.

154 Theorem The strong derivative of the reciprocal function $x \mapsto \frac{1}{x}$ is the function $x \mapsto -\frac{1}{x^2}$.

Proof: *Put* $f(x) = \frac{1}{x}$ *and* $g(x) = x$. *Observe that* $g'(x) = 1$ *and* $f(x)g(x) = 1$. *Hence by the product rule*

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\, 1 = f'(x)g(x) + f(x)g'(x) = xf'(x) + \frac{1}{x},$$

and solving for $f'(x)$ *we obtain* $f'(x) = -\frac{1}{x^2}$.

An alternate proof from the definition proceeds as follows. Let $f(x) = \frac{1}{x}$ *and* $x \neq 0$. *First observe the algebraic identity*

$$\frac{1}{1+t} = 1 - t + \frac{t^2}{1+t}.$$

Hence, if $x \neq 0$ *is fixed,*

$$\frac{1}{x+\varepsilon} = \frac{1}{x} \cdot \frac{1}{1+\varepsilon/x} = \frac{1}{x}\left(1 - \frac{\varepsilon}{x} + \frac{\varepsilon^2}{x^2} \cdot \frac{1}{(1+\varepsilon/x)}\right).$$

Now, since $\varepsilon \to 0$ *we will have, eventually,* $|\varepsilon| < \frac{|x|}{2}$. *Hence* $\frac{2}{3} < \frac{1}{(1+\varepsilon/x)} < 2$. *This means that*

$$\frac{\varepsilon^2}{x^2} \cdot \frac{1}{(1+\varepsilon/x)} = O(\varepsilon^2),$$

where the implied constant depends on (the fixed value of) x, *and so*

$$\frac{1}{x+\varepsilon} = \frac{1}{x}\left(1 - \frac{\varepsilon}{x} + O(\varepsilon^2)\right) = \frac{1}{x} - \frac{\varepsilon}{x^2} + O(\varepsilon^2),$$

from where the assertion follows. ❑

155 Theorem If $n > 0$ is an integer and $x \neq 0$,

$$\left(\frac{1}{x^n}\right)' = \left(x^{-n}\right)' = -nx^{-n-1}.$$

Proof: *Let $f(x) = x^n$, $g(x) = \frac{1}{x}$, $h(x) = \frac{1}{x^n}$. Then $h = f \circ g$. By the Chain Rule (Theorem 116) and Theorem 154,*

$$h'(x) = f'(g(x))g'(x) = n\left(\frac{1}{x}\right)^{n-1}\left(-\frac{1}{x^2}\right) = -\frac{n}{x^{n+1}} = -nx^{-n-1},$$

as it was to be demonstrated. ❑

STOP *Theorems 95 and 155 say that if α is an integer, then $(x^\alpha)' = \alpha x^{\alpha-1}$.*

With the derivatives of reciprocal powers determined, we can now address how to graph them.

156 Theorem Let $n > 0$ be an integer. Then

- if n is even, $x \mapsto \frac{1}{x^n}$ is increasing for $x < 0$, decreasing for $x > 0$ and convex for all $x \neq 0$.
- if n is odd, $x \mapsto \frac{1}{x^n}$ is decreasing for all $x \neq 0$, concave for $x < 0$, and convex for $x > 0$.

Thus $x \mapsto \frac{1}{x^n}$ has a pole of order n at $x = 0$ and a horizontal asymptote at $y = 0$.

Proof: *Let $h(x) = \frac{1}{x^n}$. By Theorem 155, $h'(x) = -\frac{n}{x^{n+1}}$ and $h''(x) = \frac{n(n+1)}{x^{n+2}}$. If n is odd, then $n+1$ is even and $n+2$ is odd. Hence $h'(x) > 0$ for $x \neq 0$, proving that h is increasing and $h''(x)$ has the same sign as x, proving that h is concave for $x < 0$ and convex for $x > 0$. A similar argument is used for when n is even, completing the proof.* ❑

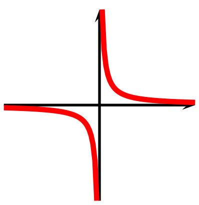

Figure 6.1: $x \mapsto \frac{1}{x}$

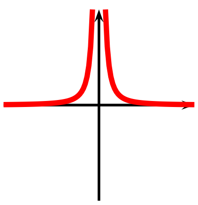

Figure 6.2: $x \mapsto \frac{1}{x^2}$

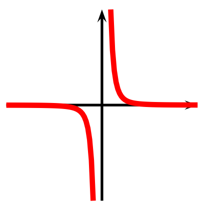

Figure 6.3: $x \mapsto \frac{1}{x^3}$

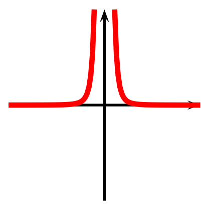

Figure 6.4: $x \mapsto \frac{1}{x^4}$

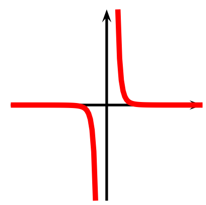

Figure 6.5: $x \mapsto \frac{1}{x^5}$

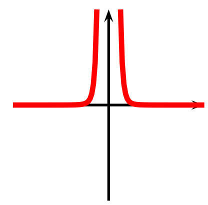

Figure 6.6: $x \mapsto \frac{1}{x^6}$

157 Example A few functions $x \mapsto \frac{1}{x^n}$ are shewn in figures 6.1 through 6.6.

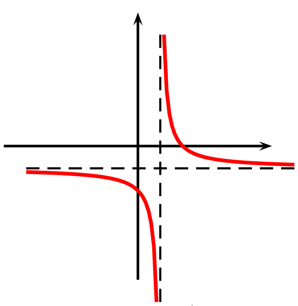

Figure 6.7: $x \mapsto \frac{1}{x-1} - 1$

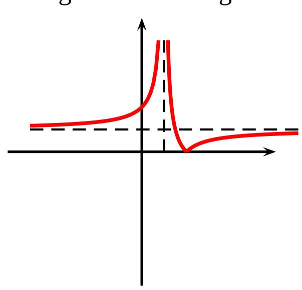

Figure 6.8: $x \mapsto \left|\frac{1}{x-1} - 1\right|$

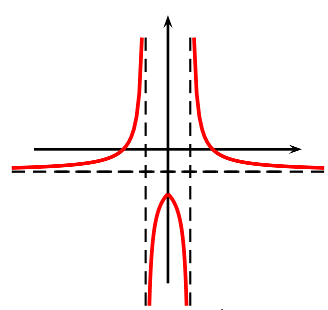

Figure 6.9: $x \mapsto \frac{1}{|x|-1} - 1$

158 Example Figures 6.7 through 6.9 shew a few transformations of $x \mapsto \frac{1}{x}$.

6.2 The Quotient Rule

159 Theorem If g is strongly differentiable at x and $g(x) \neq 0$ then

$$\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{(g(x))^2}.$$

Proof: *Let* $a(x) = \frac{1}{x}$. *Then* $\frac{1}{g} = a \circ g$. *By the Chain Rule (Theorem 116) and Theorem* **??**,

$$\left(\frac{1}{g}\right)'(x) = a'(g(x))g'(x) = -\frac{g'(x)}{(g(x))^2},$$

as we needed to shew. ❏

160 Corollary (Quotient Rule) If f, g are strongly differentiable at x and if $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Proof: *Using the Product Rule (Theorem 113) and Theorem 159,*

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \left(f \cdot \frac{1}{g}\right)'(x) \\ &= f'(x)\frac{1}{g(x)} + f(x)\left(\frac{1}{g}\right)'(x) \\ &= f'(x)\frac{1}{g(x)} + f(x)\left(-\frac{g'(x)}{(g(x))^2}\right) \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2},\end{aligned}$$

as desired. ❏

161 Example Find $b'(x)$ if $b(x) = \frac{1+x+x^2}{(x-x^2)^2}$

Solution: Use the Quotient Rule and the Chain Rule:

$$\begin{aligned}b'(x) &= \frac{(1+2x)(x-x^2)^2 - (1+x+x^2)(2(1-2x)(x-x^2))}{(x-x^2)^4} \\ &= \frac{(1+2x)(x-x^2) - (1+x+x^2)(2(1-2x))}{(x-x^2)^3} \\ &= -\frac{-2+3x+3x^2+2x^3}{(x-x^2)^3}.\end{aligned}$$

6.3 Rational Functions

162 Definition By a *rational function* $x \mapsto r(x)$ we mean a function r whose assignment rule is of the $r(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x) \neq 0$ are polynomials.

We now provide a few examples of graphing rational functions.

163 Example Draw the curve $x \mapsto \frac{x^2}{x^2+1}$.

Solution: Put $a(x) = \dfrac{x}{x^2+1}$. Observe that $a(-x) = -a(x)$, which means that a is an odd function and hence symmetric about the origin. Also

$$a'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2}.$$

Since $(x^2+1)^2$ is always positive, a' changes sign when $1-x^2 = (1-x)(1+x)$ changes sign. Hence $a'(x) \geq 0$ if $x \in [-1;1]$ and $a'(x) < 0$ otherwise. This means that a is increasing for $x \in [-1;1]$ and decreasing otherwise. Moreover

$$a''(x) = \frac{d}{dx}\,\frac{1-x^2}{(x^2+1)^2} = \frac{-2x(x^2+1)^2 - 2(2x)(1-x^2)(x^2+1)}{(x^2+1)^4} = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

Again a'' will change sign when $2x(x^2-3) = x(x-\sqrt{3})(x+\sqrt{3})$ changes sign. By means of a sign diagram we see that $a''(x) \geq 0$ for $x \in [-\sqrt{3};0] \cup [\sqrt{3};+\infty[$, and so a is convex for $x \in [-\sqrt{3};0] \cup [\sqrt{3};+\infty[$ and concave otherwise. The graph is shewn in figure 6.10.

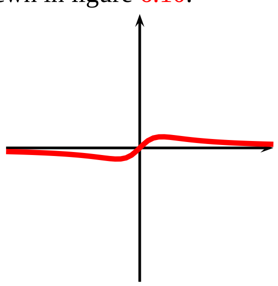

Figure 6.10: $x \mapsto \dfrac{x}{x^2+1}$

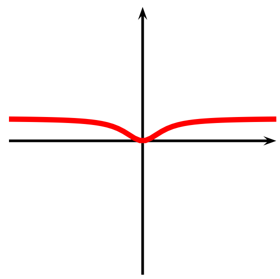

Figure 6.11: $x \mapsto \dfrac{x^2}{x^2+1}$

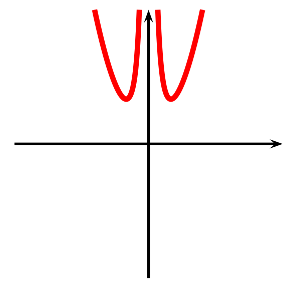

Figure 6.12: $x \mapsto x^2 + \dfrac{1}{x^2}$

164 Example Draw the curve $x \mapsto \dfrac{x^2}{x^2+1}$.

Solution: Put $b(x) = \dfrac{x^2}{x^2+1}$. Observe that $b(-x) = b(x)$, which means that b is an even function and hence symmetric about the y-axis. Also

$$b'(x) = \frac{(2x)(x^2+1) - 2x(x^2)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}.$$

Since $(x^2+1)^2$ is always positive, b' changes sign when x changes sign. Hence $b'(x) \geq 0$ if $x \geq 0$ and $b'(x) < 0$ otherwise. This means that b is increasing for $x \geq 0$ and decreasing otherwise. Moreover

$$b''(x) = \frac{d}{dx}\,\frac{2x}{(x^2+1)^2} = \frac{2(x^2+1)^2 - 2(2x)(2x)(x^2+1)}{(x^2+1)^4} = \frac{2-6x^2}{(x^2+1)^3}.$$

Again b'' will change sign when $2-6x^2 = 2(1-\sqrt{3}x)(1+\sqrt{3}x)$ changes sign. By means of a sign diagram we see that $b''(x) \geq 0$ for $x \in \left[-\dfrac{1}{\sqrt{3}};\dfrac{1}{\sqrt{3}}\right]$, and so b is convex for $x \in \left[-\dfrac{1}{\sqrt{3}};\dfrac{1}{\sqrt{3}}\right]$ and concave otherwise. The graph is shewn in figure 6.11.

165 Example Draw the curve $x \mapsto x^2 + \dfrac{1}{x^2}$.

Solution: Put $c(x) = x^2 + \dfrac{1}{x^2}$. Observe that $c(-x) = c(x)$, which means that c is an even function and hence symmetric about the y-axis. Also

$$c'(x) = 2x - \frac{2}{x^3} = \frac{2(x^4-1)}{x^3} = \frac{2(x-1)(x+1)(x^2+1)}{x^3}.$$

We make a sign diagram investigating the sign changes of c' near $x=-1$, $x=0$, and $x=1$. From this we gather that c is increasing for $x \in [-1;0[\cup [1;+\infty[$. Moreover

$$c''(x) = \frac{d}{dx}\left(2x - \frac{2}{x^3}\right) = 2 + \frac{6}{x^4}.$$

We see that c'' is always positive and hence it is always convex. The graph is shewn in figure 6.12.

Analogous to theorem 150, we now consider rational functions $x \mapsto r(x) = \dfrac{p(x)}{q(x)}$ where p and q are polynomials with no factors in common and splitting in $\mathbb{R}$.

166 Theorem Let $a \neq 0$ and the r_i are real numbers and the m_i be positive integers. Then the rational function with assignment rule

$$r(x) = K\frac{(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_k)^{m_k}}{(x-b_1)^{n_1}(x-b_2)^{n_2}\cdots(x-b_l)^{n_l}},$$

- has zeroes at $x = a_i$ and poles at $x = b_j$.
- crosses the x-axis at $x = a_i$ if m_i is odd.
- is tangent to the x-axis at $x = a_i$ if m_i is even.
- has a convexity change at $x = a_i$ if $m_i \geq 3$ and m_i is odd.
- both $r(b_j-)$ and $r(b_j+)$ blow to infinity. If n_i is even, then they have the same sign infinity: $r(b_i+) = r(b_i-) = +\infty$ or $r(b_i+) = r(b_i-) = -\infty$. If n_i is odd, then they have different sign infinity: $r(b_i+) = -r(b_i-) = +\infty$ or $r(b_i+) = -r(b_i-) = -\infty$.

Proof: *Since the local behaviour of $r(x)$ is that of $c(x-r_i)^{t_i}$ (where c is a real number constant) near r_i, the theorem follows at once from Theorem 97 and 156.* ❑

167 Example Draw a rough sketch of $x \mapsto \dfrac{(x-1)^2(x+2)}{(x+1)(x-2)^2}$.

Solution: Put $r(x) = \dfrac{(x-1)^2(x+2)}{(x+1)(x-2)^2}$. By Theorem 166, r has zeroes at $x = 1$, and $x = -2$, and poles at $x = -1$ and $x = 2$. As $x \to 1$, $r(x) \sim \dfrac{3}{2}(x-1)^2$, hence the graph of r is tangent to the axes, and positive, around $x = 2$. As $x \to -2$, $r(x) \sim -\dfrac{9}{16}(x+2)$, hence the graph of r crosses the x-axis at $x = -2$, coming from positive y-values on the left of $x = -2$ and going to negative y=values on the right of $x = -2$. As $x \to -1$, $r(x) \sim \dfrac{4}{9(x+1)}$, hence the graph of r blows to $-\infty$ to the left of $x = -1$ and to $+\infty$ to the right of $x = -1$. As $x \to 2$, $r(x) \sim \dfrac{4}{3(x-2)^2}$, hence the graph of r blows to $+\infty$ both from the left and the right of $x = 2$. Also we observe that

$$r(x) \sim \frac{(x)^2(x)}{(x)(x)^2} = \frac{x^3}{x^3} = 1,$$

and hence r has the horizontal asymptote $y = 1$. The graph of r can be found in figure 6.13.

168 Example Draw a rough sketch of $x \mapsto \dfrac{(x-3/4)^2(x+3/4)^2}{(x+1)(x-1)}$.

Solution: Put $r(x) = \dfrac{(x-3/4)^2(x+3/4)^2}{(x+1)(x-1)}$. First observe that $r(x) = r(-x)$, and so r is even. By Theorem 166, r has zeroes at $x = \pm\dfrac{3}{4}$, and poles at $x = \pm 1$. As $x \to \dfrac{3}{4}$, $r(x) \sim -\dfrac{36}{7}(x-3/4)^2$, hence the graph of r is tangent to the axes, and negative, around $x = 3/4$, and similar behaviour occurs around $x = -\dfrac{3}{4}$. As $x \to 1$, $r(x) \sim \dfrac{49}{512(x-1)}$, hence the graph of r blows to $-\infty$ to the left of $x = 1$ and to $+\infty$ to the right of $x = 1$. As $x \to -1$, $r(x) \sim -\dfrac{49}{512(x-1)}$, hence the graph of r blows to $+\infty$ to the left of $x = -1$ and to $-\infty$ to the right of $x = -1$. Also, as $x \to +\infty$,

$$r(x) \sim \frac{(x)^2(x)^2}{(x)(x)} = x^2,$$

so $r(+\infty) = +\infty$ and $r(-\infty) = +\infty$. The graph of r can be found in figure 6.14.

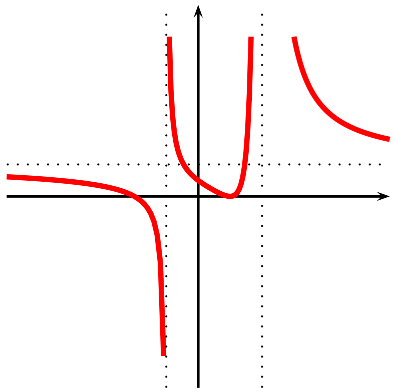

Figure 6.13: $x \mapsto \dfrac{(x-1)^2(x+2)}{(x+1)(x-2)^2}$

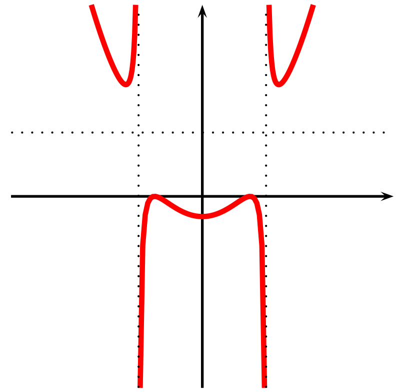

Figure 6.14: $x \mapsto \dfrac{(x-3/4)^2(x+3/4)^2}{(x+1)(x-1)}$

6.4 Algebraic Functions

169 Definition We will call *algebraic function* a function whose assignment rule can be obtained from a rational function by a finite combination of additions, subtractions, multiplications, divisions, exponentiations to a rational power.

170 Theorem Let $q \neq 0$ be an integer. The strong derivative of the function $x \mapsto x^{1/q}$ is the function $x \mapsto \dfrac{1}{q} \cdot x^{1/q-1}$, whenever this last makes sense.

Proof: *Put $f(x) = x^{1/q}$, assuming this quantity is real. Then $(f(x))^q = x$. Differentiating both sides using the Chain Rule we have*

$$q(f(x))^{q-1} f'(x) = \Longrightarrow qx^{(q-1)/q} f'(x) = 1.$$

Solving for f' gives

$$f'(x) = \frac{1}{q} \cdot x^{1/q-1},$$

if this quantity is a real number, proving the result. ❑

STOP *Theorems 95, 155, and 170, when combined with the Chain Rule, say that if α is a rational number, then $(x^\alpha)' = \alpha x^{\alpha-1}$.*

With the derivatives of rational powers determined, we can now address how to graph them.

171 Theorem Let $|q| \geq 2$ be an integer. If

- if q is even then $x \mapsto x^{1/q}$ is increasing and concave for $q \geq 2$ and decreasing and convex for $q \leq -2$ for all $x > 0$ and it is undefined for $x < 0$.
- if q is odd then $x \mapsto x^{1/q}$ is everywhere increasing and convex for $x < 0$ but concave for $x > 0$ if $q \geq 3$. If $q \leq -3$ then $x \mapsto x^{1/q}$ is decreasing and concave for $x < 0$ and increasing and convex for $x > 0$.

Proof: *Let $h(x) = x^{1/q}$. By Theorem 170, $h'(x) = \dfrac{x^{(1-q)/q}}{q}$ and $h''(x) = \dfrac{(1-q)x^{(2q-1)/q}}{q^2}$.*

Assume first that q is even. Then $x^{1/q}$ is not real for $x<0$ so we assume that $x>0$. The quantity $h'(x)=\dfrac{x^{(1-q)/q}}{q}$ is >0 for $q\geq 2$ and negative for $q\leq -2$. If $q\geq 2$ then $h''(x)=\dfrac{(1-q)x^{(2q-1)/q}}{q^2}<0$ and if $q\leq -2$ then $h''(x)>0$. Hence h is increasing and concave for $q\geq 2$ and decreasing and convex for $q\leq -2$

Assume now that q is odd. Then $1-q$ is even and the sign of the quantity $h'(x)=\dfrac{1}{q}\cdot\left(x^{1/q}\right)^{1-q}$ is depends on the sign of $\dfrac{1}{q}$. Since $2q-1$ is odd, the sign of $h''(x)=\dfrac{(1-q)x^{(2q-1)/q}}{q^2}$ is $\mathrm{signum}\,((1-q)(x))$. We have: if $q\geq 3$, $h'(x)>0$, $h''(x)<0$ for $x>0$ and $h''(x)>0$ for $x<0$. Hence for $q\geq 3$, h is increasing and it is convex for $x<0$ but concave for $x>0$. If $q\leq -3$ then h is decreasing and it is concave for $x<0$ and decreasing convex for $x>0$.
❏

A few of the functions $x\mapsto x^{1/q}$ are shewn in figures 6.15 through 6.26.

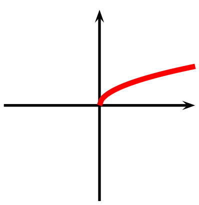
Figure 6.15: $x\mapsto x^{1/2}$

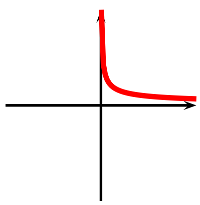
Figure 6.16: $x\mapsto x^{-1/2}$

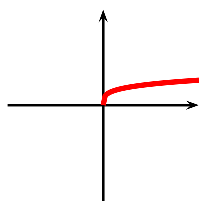
Figure 6.17: $x\mapsto x^{1/4}$

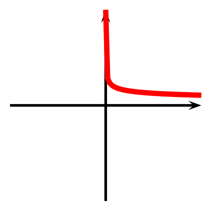
Figure 6.18: $x\mapsto x^{-1/4}$

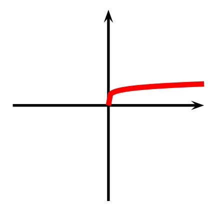
Figure 6.19: $x\mapsto x^{1/6}$

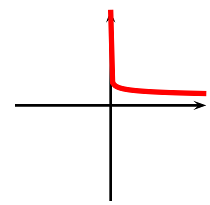
Figure 6.20: $x\mapsto x^{-1/6}$

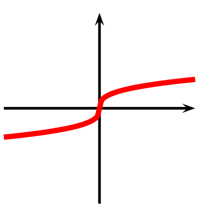
Figure 6.21: $x\mapsto x^{1/3}$

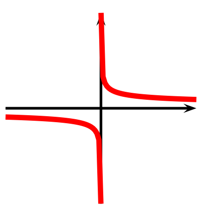
Figure 6.22: $x\mapsto x^{-1/3}$

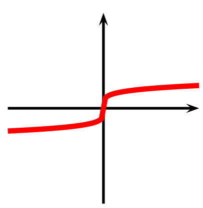
Figure 6.23: $x\mapsto x^{1/5}$

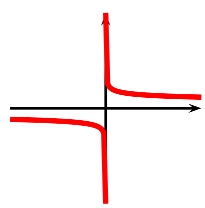
Figure 6.24: $x\mapsto x^{-1/5}$

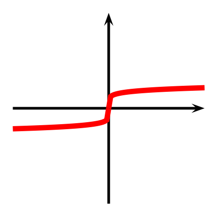
Figure 6.25: $x\mapsto x^{1/7}$

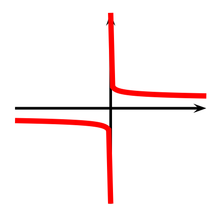
Figure 6.26: $x\mapsto x^{-1/7}$

We finish this section with an example.

172 Example Consider the function $y=f(x)=\sqrt{x-1}+\sqrt{2-x}$.

1. For which x will the output of f be a real number?
2. Find $f'(x)$.
3. By examining f', prove that f increases for $x<\dfrac{3}{2}$ and decreasing for $x>\dfrac{3}{2}$.
4. Prove that for all x in the domain of f one has $f(x)\leq\sqrt{2}$.
5. Find $f''(x)$.
6. Determine in which intervals f is convex or concave.
7. Graph f.

Solution:

1. We need, simultaneously, $x \geq 1$ and $2 \geq x$. This means that $x \in [1;2]$.
2. Via the Chain Rule:
$$\frac{\mathrm{d}}{\mathrm{d}x}(x-1)^{1/2} + (2-x)^{1/2} = \frac{1}{2}(x-1)^{-1/2} - \frac{1}{2}(2-x)^{-1/2} = \frac{1}{2(x-1)^{1/2}} - \frac{1}{2(2-x)^{1/2}}$$
3. f has a stationary point when $f'(x) = 0$, that is, if
$$\frac{1}{2(x-1)^{1/2}} = \frac{1}{2(2-x)^{1/2}} \implies x-1 = 2-x \implies x = \frac{3}{2},$$
so f' has only one zero in $[1;2]$. Since f' is continuous in $]1;2[$ and has only one zero there, it must be negative in a portion of the interval and positive in the other. Examining values in $]1;\frac{3}{2}[$ we see that f' is positive for there and negative in $]\frac{3}{2};2[$.
4. By the above, $x = \frac{3}{2}$ is a global maximum in $[1;2]$, and hence
$$f(x) \leq f\left(\frac{3}{2}\right) = \sqrt{\frac{3}{2}-1}\sqrt{2-\frac{3}{2}} = 2\sqrt{\frac{1}{2}} = \sqrt{2}.$$
5. Via the Chain Rule:
$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}(x-1)^{-1/2} - \frac{1}{2}(2-x)^{-1/2} = -\frac{1}{4}(x-1)^{-3/2} - \frac{1}{4}(2-x)^{-3/2} = -\frac{1}{4(x-1)^{3/2}} - \frac{1}{4(2-x)^{3/2}}.$$
6. Observe that
$$f''(x) = -\frac{1}{4(x-1)^{3/2}} - \frac{1}{4(2-x)^{3/2}} = -\frac{1}{4}\left(\frac{1}{(x-1)^{3/2}} + \frac{1}{(2-x)^{3/2}}\right).$$
Since the quantity in parenthesis is always positive, f'' is always negative, and hence it is everywhere concave.
7. The graph appears in figure 6.27.

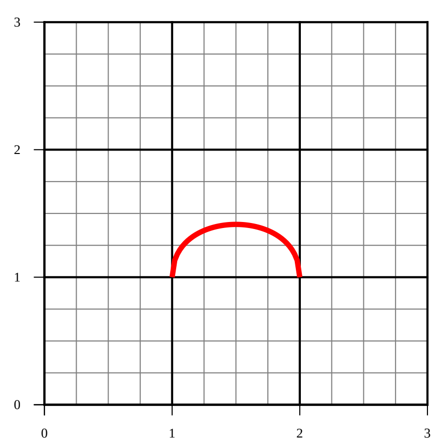

Figure 6.27: $y = \sqrt{x-1} + \sqrt{2-x}$

Appendix **A**

Big O Notation

We will now study the order of magnitude of functions, in other words, how big functions are in a neighbourhood of a point.

173 Definition Let f, g be two functions. We write

$$f(x) = O(g(x)) \qquad \text{as} \qquad x \to a,$$

read "$f(x)$ is big oh of $g(x)$ as x tends to a" if there is a positive constant C such that for all x sufficiently close to a we have $|f(x)| \leq C|g(x)|$. An equation of the type $f(x) = h(x) + O(g(x))$ means that $f(x) - h(x) = O(g(x))$. Here $h(x)$ is called the *principal term* and $O(g(x))$ is called the *error term*.

We are mainly interested in the above definition when $a = 0$ or $a = +\infty$.

174 Example As $x \to +\infty$, we have $x + x^2 = O(x^2)$. In fact, if $1 < x$ then $x < x^2$ which means that

$$x + x^2 < x^2 + x^2 = 2x^2.$$

Thus for any x larger than 1, $x + x^2$ is bounded by a constant times x^2.

175 Example As $x \to 0$, we have $x + x^2 = O(x)$. In fact, if $|x| < 1$, then $x^2 < |x| < 1$, and hence

$$|x + x^2| < |x| + |x^2| < |x| + |x| = 2|x|$$

when $|x| < 1$.

STOP *The equality $f(x) = O(g(x))$ is not symmetric. For example, as $x \to +\infty$ we have $x = O(x^2)$ but $x^2 \neq O(x)$.*

We have the following theorem.

176 Theorem Let r, s be real numbers with $r < s$. If $x \to +\infty$ then $x^r = O(x^s)$. If $x \to 0$ then $x^s = O(x^r)$.

> **Proof:** *First observe that $s - r > 0$. Now, if $|x| > 1$, then $|x|^{s-r} > 1$ since we are raising a number larger than 1 to a positive power. Thus*
>
> $$|x| > 1 \implies |x|^{s-r} > 1 \implies |x|^s > |x|^r \implies x^r = O(x^s),$$
>
> *for $|x| > 1$ and certainly as $x \to +\infty$.*
>
> *Also, if $|x| < 1$, then $|x|^{s-r} < 1$ since we are raising a number smaller than 1 to a positive power. Thus*
>
> $$|x| < 1 \implies |x|^{s-r} < 1 \implies |x|^s < |x|^r \implies x^s = O(x^r).$$
>
> ❏

The following properties of the O symbol are easy to prove and will be left as an exercise.

177 Theorem The O symbol has the following properties.

$$
\begin{aligned}
f(x) &= O(f(x)) && \text{(A.1)}\\
kO(f(x)) &= O(f(x)) && \text{(A.2)}\\
O(f(x))+O(f(x)) &= O(f(x)) && \text{(A.3)}\\
O(O(f(x))) &= O(f(x)) && \text{(A.4)}\\
O(f(x))O(g(x)) &= O(f(x)g(x)) && \text{(A.5)}\\
O(f(x)g(x)) &= f(x)O(g(x)) && \text{(A.6)}\\
& && \text{(A.7)}
\end{aligned}
$$

178 Example As $x \to +\infty$ we have $O(x^3)+O(x^4) = O(x^4)$, since the x^4 term dominates over the x^3 for large x.

179 Example As $x \to 0$ we have $O(x^3)+O(x^4) = O(x^3)$, since the x^3 term dominates over the x^4 for small x.

180 Example $x+2x^2+3x^3 = O(x^3)$ as $x \to +\infty$. This means that for x sufficiently large, $x+2x^2+3x^3$ is dominated by x^3.

181 Example $x+2x^2+3x^3 = O(x)$ as $x \to 0$. This means that for x sufficiently small, $x+2x^2+3x^3$ is dominated by x.

182 Example We have, as $x \to +\infty$,

$$
\begin{aligned}
(2x^3+O(x))(-3x^2+O(x)) &= -6x^5+O(-3x^3)+O(2x^4)+O(x^2)\\
&= -6x^5+O(x^3)+O(x^4)+O(x^2)\\
&= -6x^5+O(x^4).
\end{aligned}
$$

Sometimes it is more important to know which term in a given sum dominates when the variable tends to a determinate quantity. In the next definition we will concentrate in the cases when the variable tends to 0 or $+\infty$.

183 Definition If

$$0 \le n_1 < n_2 < \cdots < n_q$$

is a sequence of integers then the polynomial

$$a_{n_1}x^{n_1}+a_{n_2}x^{n_2}+\cdots+a_{n_q}x^{n_q}$$

has dominant term $a_{n_1}x^{n_1}$ as $x \to 0$ and we write

$$a_{n_1}x^{n_1}+a_{n_2}x^{n_2}+\cdots+a_{n_q}x^{n_q} \sim a_{n_1}x^{n_1}, \quad x \to 0,$$

read "$a_{n_1}x^{n_1}+a_{n_2}x^{n_2}+\cdots+a_{n_q}x^{n_q}$ *is asymptotic to* $a_{n_1}x^{n_1}$ as $x \to 0$."

Similarly, if $x \to \pm\infty$ then the polynomial

$$a_{n_1}x^{n_1}+a_{n_2}x^{n_2}+\cdots+a_{n_q}x^{n_q}$$

has dominant term $a_{n_q}x^{n_q}$ as $x \to \pm\infty$ and we write

$$a_{n_1}x^{n_1}+a_{n_2}x^{n_2}+\cdots+a_{n_q}x^{n_q} \sim a_{n_q}x^{n_q}, \quad x \to \pm\infty,$$

read "$a_{n_1}x^{n_1}+a_{n_2}x^{n_2}+\cdots+a_{n_q}x^{n_q}$ *is asymptotic to* $a_{n_q}x^{n_q}$ as $x \to \pm\infty$."

184 Example We have

$$
\begin{aligned}
4x+3x^2+2x^3+x^4 &\sim 4x, \quad x \to 0,\\
1+4x+3x^2+2x^3+x^4 &\sim 1, \quad x \to 0,\\
4x+3x^2+2x^3+x^4 &\sim x^4, \quad x \to \pm\infty,
\end{aligned}
$$

etc.

Appendix B

Complex Numbers

B.1 Arithmetic of Complex Numbers

We use the symbol i to denote the *imaginary unit* $i = \sqrt{-1}$. Then $i^2 = -1$.

185 Example Find $\sqrt{-25}$.

Solution: $\sqrt{-25} = 5i$.

Since $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i$, etc., the powers of i repeat themselves cyclically in a cycle of period 4.

186 Definition If a, b are real numbers then the object $a + bi$ is called a *complex number*.

We use the symbol $\mathbb{C}$ to denote the set of all complex numbers. If $a, b, c, d \in \mathbb{R}$, then the sum of the complex numbers $a+bi$ and $c+di$ is naturally defined as

$$(a+bi)+(c+di) = (a+c)+(b+d)i \tag{B.1}$$

The product of $a+bi$ and $c+di$ is obtained by multiplying the binomials:

$$(a+bi)(c+di) = ac + adi + bci + bdi^2 = (ac-bd)+(ad+bc)i \tag{B.2}$$

187 Example Find the sum $(4+3i)+(5-2i)$ and the product $(4+3i)(5-2i)$.

Solution: One has

$$(4+3i)+(5-2i) = 9+i$$

and

$$(4+3i)(5-2i) = 20-8i+15i-6i^2 = 20+7i+6 = 26+7i.$$

188 Definition Let $z \in \mathbb{C}, (a,b) \in \mathbb{R}^2$ with $z = a+bi$. The *conjugate* $\overline{z}$ of z is defined by

$$\overline{z} = \overline{a+bi} = a-bi \tag{B.3}$$

189 Example The conjugate of $5+3i$ is $\overline{5+3i} = 5-3i$. The conjugate of $2-4i$ is $\overline{2-4i} = 2+4i$.

STOP *The conjugate of a real number is itself, that is, if $a \in \mathbb{R}$, then $\overline{a} = a$. Also, the conjugate of the conjugate of a number is the number, that is, $\overline{\overline{z}} = z$.*

64

190 Theorem The function $z : \mathbb{C} \to \mathbb{C},\ z \mapsto \overline{z}$ is multiplicative, that is, if z_1, z_2 are complex numbers, then

$$\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2} \tag{B.4}$$

Proof: *Let $z_1 = a + bi, z_2 = c + di$ where a, b, c, d are real numbers. Then*

$$\begin{aligned} \overline{z_1 z_2} &= \overline{(a+bi)(c+di)} \\ &= \overline{(ac-bd)+(ad+bc)i} \\ &= (ac-bd)-(ad+bc)i \end{aligned}$$

Also,

$$\begin{aligned} \overline{z_1} \cdot \overline{z_2} &= (\overline{a+bi})(\overline{c+di}) \\ &= (a-bi)(c-di) \\ &= ac - adi - bci + bdi^2 \\ &= (ac-bd)-(ad+bc)i, \end{aligned}$$

which establishes the equality between the two quantities. ❏

191 Example Express the quotient $\dfrac{2+3i}{3-5i}$ in the form $a+bi$.

Solution: One has

$$\frac{2+3i}{3-5i} = \frac{2+3i}{3-5i} \cdot \frac{3+5i}{3+5i} = \frac{-9+19i}{34} = \frac{-9}{34} + \frac{19i}{34}$$

192 Definition The *modulus* $|a+bi|$ of $a+bi$ is defined by

$$|a+bi| = \sqrt{(a+bi)(\overline{a+bi})} = \sqrt{a^2+b^2} \tag{B.5}$$

Observe that $z \mapsto |z|$ is a function mapping $\mathbb{C}$ to $\mathbb{R}_+$.

193 Example Find $|7+3i|$.

Solution: $|7+3i| = \sqrt{(7+3i)(7-3i)} = \sqrt{7^2+3^2} = \sqrt{58}$.

194 Example Find $|\sqrt{7}+3i|$.

Solution: $|\sqrt{7}+3i| = \sqrt{(\sqrt{7}+3i)(\sqrt{7}-3i)} = \sqrt{7+3^2} = 4$.

195 Theorem The function $z \mapsto |z|,\ \mathbb{C} \to \mathbb{R}_+$ is multiplicative. That is, if z_1, z_2 are complex numbers then

$$|z_1 z_2| = |z_1||z_2| \tag{B.6}$$

Proof: *By Theorem 190, conjugation is multiplicative, hence*

$$\begin{aligned} |z_1 z_2| &= \sqrt{z_1 z_2 \overline{z_1 z_2}} \\ &= \sqrt{z_1 z_2 \overline{z_1} \cdot \overline{z_2}} \\ &= \sqrt{z_1 \overline{z_1} z_2 \overline{z_2}} \\ &= \sqrt{z_1 \overline{z_1}} \sqrt{z_2 \overline{z_2}} \\ &= |z_1||z_2| \end{aligned}$$

whence the assertion follows. ❏

196 Example Write $(2^2+3^2)(5^2+7^2)$ as the sum of two squares.

Solution: The idea is to write $2^2+3^2=|2+3i|^2$, $5^2+7^2=|5+7i|^2$ and use the multiplicativity of the modulus. Now

$$\begin{aligned}(2^2+3^2)(5^2+7^2) &= |2+3i|^2|5+7i|^2\\ &= |(2+3i)(5+7i)|^2\\ &= |-11+29i|^2\\ &= 11^2+29^2\end{aligned}$$

B.2 Equations involving Complex Numbers

Recall that if $ux^2+vx+w=0$ with $u\neq 0$, then the roots of this equation are given by the *Quadratic Formula*

$$x=-\frac{v}{2u}\pm\frac{\sqrt{v^2-4uw}}{2u} \tag{B.7}$$

The quantity v^2-4uw under the square root is called the *discriminant* of the quadratic equation $ux^2+vx+w=0$. If u,v,w are real numbers and this discriminant is negative, one obtains complex roots.

Complex numbers thus occur naturally in the solution of quadratic equations. Since $i^2=-1$, one sees that $x=i$ is a root of the equation $x^2+1=0$. Similary, $x=-i$ is also a root of x^2+1.

197 Example Solve $2x^2+6x+5=0$

Solution: Using the quadratic formula

$$x=-\frac{6}{4}\pm\frac{\sqrt{-4}}{4}=-\frac{3}{2}\pm i\frac{1}{2}$$

In solving the problems that follow, the student might profit from the following identities.

$$s^2-t^2=(s-t)(s+t) \tag{B.8}$$

$$s^{2k}-t^{2k}=(s^k-t^k)(s^k+t^k),\ k\in\mathbb{N} \tag{B.9}$$

$$s^3-t^3=(s-t)(s^2+st+t^2) \tag{B.10}$$

$$s^3+t^3=(s+t)(s^2-st+t^2) \tag{B.11}$$

198 Example Solve the equation $x^4-16=0$.

Solution: One has $x^4-16=(x^2-4)(x^2+4)=(x-2)(x+2)(x^2+4)$. Thus either $x=-2, x=2$ or $x^2+4=0$. This last equation has roots $\pm 2i$. The four roots of $x^4-16=0$ are thus $x=-2,x=2,x=-2i,x=2i$.

199 Example Find the roots of $x^3-1=0$.

Solution: $x^3-1=(x-1)(x^2+x+1)$. If $x\neq 1$, the two solutions to $x^2+x+1=0$ can be obtained using the quadratic formula, getting $x=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$.

200 Example Find the roots of $x^3+8=0$.

Solution: $x^3+8=(x+2)(x^2-2x+4)$. Thus either $x=-2$ or $x^2-2x+4=0$. Using the quadratic formula, one sees that the solutions of this last equation are $x=1\pm i\sqrt{3}$.

201 Example Solve the equation $x^4+9x^2+20=0$.

Solution: One sees that

$$x^4+9x^2+20=(x^2+4)(x^2+5)=0$$

Thus either $x^2+4=0$, in which case $x=\pm 2i$ or $x^2+5=0$ in which case $x=\pm i\sqrt{5}$. The four roots are $x=\pm 2i, \pm i\sqrt{5}$