More Discrete Mathematics

David A. SANTOS dsantos@ccp.edu

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Preface

These notes were created because I saw a need to them. There are four main topics that we will cover in the class:

1. Closed forms for sums and recurrences.
2. Estimates for sums and recurrences.
3. Basic programming algorithms and their complexity.
4. Graph theoretic methods.

Topic 1 is classic, as I learned from [Boo], [HaKn] and [Chr]. The more recent [KGP] and [Wil] have also become instant classics on the subject.

Topic 2 is very well explained in analytic number theory books (see reference [MoVa]) and computer science books (once again, refer to [KGP]). The approaches to asymptotics are somewhat different in the two fields, but nevertheless, both have produced extraordinary methods for dealing with asymptotic estimates. The classic [DeB] is also worth noting.

There is an abundance of *advanced* books for topic 3, with [CLRS] and [Knu] being standard references. There are very few books, however, that explain basic algorithmic constructs at a level understandable to a novice, with the notable exceptions [Ser] and [She]. Their examples are in Pascal or pseudocode, which, for our purposes, will not do. Hence, I have translated many of their examples into Maple code, and also added many problems of my own.

I haven't included any material on topic 4 here. There is no shortage of good books, both at the elementary and advanced level in graph theory. My favourites are [BoMu] and [HaRi]. As the semester progresses, I will write some Maple™ labs that will include graph theory, and then I will add them here.

Some of the material here uses Calculus, although Calculus is not part of the course prerequisites. Most of the students taking this course, however, have seen one or two semesters of Calculus. Those of you not having seen Calculus can skip over those parts. In some cases there are alternative derivations for some of the results here that do not involve Calculus, but I didn't want to write an encyclopaedic work, and hence I used the most expedient methods available, in many cases using Calculus. Perhaps some day I will include alternative proofs without Calculus, but I do not foresee having the time.

David A. SANTOS

dsantos@ccp.edu

1 Sums and Recursions

1.1 Some Finite Sums and Products

Recall that by $\sum_{k=1}^{n} a_k$ we mean $a_1 + a_2 + \cdots + a_n$ and that by $\prod_{k=1}^{n} a_k$ we mean $a_1 a_2 \cdots a_n$. We will often write this as

$$\sum_{1 \le k \le n} a_k = a_1 + a_2 + \cdots + a_n, \qquad \prod_{1 \le k \le n} a_k = a_1 a_2 \cdots a_n.$$

Our interest is to obtain *closed forms* for some classic choices of the a_k, that is, a formula for the sum or the product that is a hopefully simpler expression involving n and not involving sums or products of individual terms. The are many approaches for obtaining such sums in the simple cases that we will investigate here. We will only provide a sample of them. The interested reader may consult the works of [Chr], [HaKn], [KGP], or [Wil] for a more comprehensive treatment.

Perhaps the simplest cases are when we have

$$(a_2 - a_1) + (a_3 - a_2) + \cdots + (a_n - a_{n-1}) = a_n - a_1,$$

and

$$\frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdots \frac{a_n}{a_{n-1}} = \frac{a_n}{a_1},$$

in which case we say that the sum or the product *telescopes*.

We start by adding up a finite geometric series.

1 Theorem (Finite Geometric Series) Let $x \neq 1$. Then $\sum_{0 \le k \le n} x^k = 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$.

Proof: *Put*

$$S = 1 + x + x^2 + \cdots + x^n.$$

Then

$$xS = x + x^2 + x^3 + \cdots + x^{n+1}$$

shifts every exponent one unit. Subtracting,

$$S - xS = (1 + x + x^2 + \cdots + x^n) - (x + x^2 + x^3 + \cdots + x^{n+1}) = 1 - x^{n+1} \implies (1-x)S = 1 - x^{n+1} \implies S = \frac{1 - x^{n+1}}{1 - x},$$

since $x \neq 1$, obtaining the result. ❑

☞ *More important than remembering the* formula *above is remembering the* method *of how this formula was obtained. After many examples it will become clear that the same method applies to a wide variety of problems: in Mathematics thus there are more problems than methods.*

The above closed form is obtained readily using Maple™ . You must press ENTER after entering the semicolon.

```
> sum(x^k,k=0..n);
```

Putting $N = n + 1$ in the above formula, we are provided with the following factorisation, which might be useful in certain situations.

$$x^N - 1 = (x - 1)(x^{N-1} + x^{N-2} + \cdots + x + 1). \tag{1.1}$$

For example,

$$x^2-1=(x-1)(x+1), \quad x^3-1=(x-1)(x^2+x+1), \quad x^4-1=(x-1)(x^3+x^2+x+1),$$

etc. The above simple formula gives rise, upon differentiation, to other few well known formulæ.

2 Corollary Let $x \neq 1$. Then $\sum_{1\le k\le n} kx^{k-1} = \frac{1-x^{n+1}}{(1-x)^2} - \frac{(n+1)x^n}{1-x}$.

Proof: *By Theorem 1 we may set for* $x \neq 1$,

$$f(x) = 1+x+x^2+\cdots+x^n = \frac{1-x^{n+1}}{1-x}.$$

Differentiating both sides,

$$f'(x) = x+2x+3x^2+\cdots+nx^{n-1} = \frac{1-x^{n+1}}{(1-x)^2} - \frac{(n+1)x^n}{1-x},$$

obtaining the result.

Aliter: *This is an example of a so-called* arithmetic-geometric sequence. *We use the same trick that we used for adding a geometric sum,*

$$S = 1+2x+3x^2+\cdots+nx^{n-1} \implies xS = x+2x^2+3x^3+\cdots+nx^n.$$

Subtracting,

$$S-xS = 1+(2x-x)+(3x^2-2x^2)+\cdots+(nx^{n-1}-(n-1)x^{n-1})-nx^n = (1+x+x^2+\cdots+x^{n-1})-nx^n = \frac{1-x^n}{1-x}-nx^n,$$

upon adding the geometric sum. This reduces to

$$\begin{aligned}
(1-x)S &= \frac{1-x^n}{1-x} - nx^n \\
&= \frac{1-x^n-nx^n+nx^{n+1}}{1-x} \\
&= \frac{1-x^n-nx^n+(n+1)x^{n+1}-x^{n+1}}{1-x} \\
&= \frac{1-x^{n+1}}{1-x} + \frac{-(n+1)x^n+(n+1)x^{n+1}}{1-x} \\
&= \frac{1-x^{n+1}}{1-x} - x^n\left(\frac{(n+1)(1-x)}{1-x}\right) \\
&= \frac{1-x^{n+1}}{1-x} - (n+1)x^n,
\end{aligned}$$

from where we get the result. ❑

The Maple™ commands to obtain this sum are

```
> sum(k*x^(k-1),k=0..n);
```

3 Corollary $\sum_{1\le k\le n} k = \frac{n(n+1)}{2}$.

Proof: *We will provide three essentially different proofs for this classic result. The first proof can be simply obtained by letting* $x=1$ *in Corollary 2, whence*

$$\sum_{1\le k\le n} k = \lim_{x\to 1}\left(\frac{-x^n n + x^{n+1}n - x^n + 1}{(1-x)^2}\right) = \frac{n(n+1)}{2},$$

upon using L'Hôpital's Rule twice.

Our second proof is known as Gauß 's trick. It depends on the fact that any sum can be added the same forwards as backwards, and since we are adding an arithmetic progression, the terms at the beginning compensate the terms at the end to obtain equal quantities. If

$$S = 1 + 2 + 3 + \cdots + n$$

then

$$S = n + (n-1) + \cdots + 1.$$

Adding these two quantities,

$$\begin{array}{ccccccccc}
S & = & 1 & + & 2 & + & \cdots & + & n \\
S & = & n & + & (n-1) & + & \cdots & + & 1 \\
\hline
2S & = & (n+1) & + & (n+1) & + & \cdots & + & (n+1) \\
 & = & n(n+1), & & & & & &
\end{array}$$

since there are n summands. This gives $S = \dfrac{n(n+1)}{2}$, as was to be proved.

For our third proof we convert the given sum into a telescoping sum. Observe that

$$k^2 - (k-1)^2 = 2k - 1.$$

From this

$$\begin{array}{ccc}
1^2 - 0^2 & = & 2 \cdot 1 - 1 \\
2^2 - 1^2 & = & 2 \cdot 2 - 1 \\
3^2 - 2^2 & = & 2 \cdot 3 - 1 \\
\vdots & \vdots & \vdots \\
n^2 - (n-1)^2 & = & 2 \cdot n - 1
\end{array}$$

Adding both columns,

$$n^2 - 0^2 = 2(1 + 2 + 3 + \cdots + n) - n.$$

Solving for the sum,

$$1 + 2 + 3 + \cdots + n = n^2/2 + n/2 = \frac{n(n+1)}{2}.$$

❑

4 Corollary $\displaystyle\sum_{1 \le k \le n} k^2 = \frac{n(n+1)(2n+1)}{6}.$

Proof: *We will provide two essentially different proofs for this classic result, which essentially resemble the first and third proofs of Corollary 3. If in Corollary 2 we put*

$$g(x) = \sum_{1 \le k \le n} kx^{k-1} = \frac{1 - x^{n+1}}{(1-x)^2} - \frac{(n+1)x^n}{1-x}$$

then put

$$h(x) = xg(x) = \sum_{1\le k\le n} kx^k = \frac{x-x^{n+2}}{(1-x)^2} - \frac{(n+1)x^{n+1}}{1-x}$$

and differentiating,

$$h'(x) = \sum_{1\le k\le n} k^2x^{k-1} = -\frac{-2x^n n - x^n + 1 + x - x^{n+1} - x^n n^2 + 2x^{n+1}n^2 - x^{n+2}n^2 + 2x^{n+1}n}{(-1+x)^3},$$

and letting $x = 1$ we obtain

$$\sum_{1\le k\le n} k^2 = \lim_{x\to 1}\left(-\frac{-2x^n n - x^n + 1 + x - x^{n+1} - x^n n^2 + 2x^{n+1}n^2 - x^{n+2}n^2 + 2x^{n+1}n}{(-1+x)^3}\right) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n(n+1)(2n+1)}{6},$$

using L'Hôpital's Rule three times.

For the second proof, observe that

$$k^3 - (k-1)^3 = 3k^2 - 3k + 1.$$

Hence

$$\begin{array}{lcl}
1^3 - 0^3 & = & 3\cdot 1^2 - 3\cdot 1 + 1 \\
2^3 - 1^3 & = & 3\cdot 2^2 - 3\cdot 2 + 1 \\
3^3 - 2^3 & = & 3\cdot 3^2 - 3\cdot 3 + 1 \\
\vdots & \vdots & \vdots \\
n^3 - (n-1)^3 & = & 3\cdot n^2 - 3\cdot n + 1
\end{array}$$

Adding both columns,

$$n^3 - 0^3 = 3(1^2 + 2^2 + 3^2 + \cdots + n^2) - 3(1 + 2 + 3 + \cdots + n) + n.$$

From the preceding example $1 + 2 + 3 + \cdots + n = \cdot n^2/2 + n/2 = \dfrac{n(n+1)}{2}$ *so*

$$n^3 - 0^3 = 3(1^2 + 2^2 + 3^2 + \cdots + n^2) - \frac{3}{2}\cdot n(n+1) + n.$$

Solving for the sum,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n^3}{3} + \frac{1}{2}\cdot n(n+1) - \frac{n}{3}.$$

After simplifying we obtain

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

as desired. ❑

The alert reader will note that the leading term in $\sum_{1\le k\le n} k$ is $\dfrac{n^2}{2}$ and the leading term in $\sum_{1\le k\le n} k^2$ is $\dfrac{n^3}{3}$. This is analogous to $\int_0^n x\,\mathrm{d}x = \dfrac{n^2}{2}$ and $\int_0^n x^2\,\mathrm{d}x = \dfrac{n^3}{3}$. This is no coincidence, since an integral is essentially a sum. The Calculus of Finite Differences develops a "discrete derivative" and a "discrete integral" whereby our sums can be obtained by a process akin to integration.

The method above of writing a sum as a telescopic sum is the basis for the Calculus of Finite Differences. A good reference for this is [Boo]. We present a few more examples using this method.

5 Theorem $\sum_{2\leq k\leq n}\frac{1}{(k-1)k}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)\cdot n}=\frac{n-1}{n}.$

Proof: *Observe that*

$$\frac{1}{(k-1)k}=\frac{1}{k-1}-\frac{1}{k}.$$

Thus

$$\begin{array}{ccc}
\frac{1}{1\cdot 2} & = & \frac{1}{1}-\frac{1}{2}\\
\frac{1}{2\cdot 3} & = & \frac{1}{2}-\frac{1}{3}\\
\frac{1}{3\cdot 4} & = & \frac{1}{3}-\frac{1}{4}\\
\vdots & \vdots & \vdots\\
\frac{1}{(n-1)\cdot n} & = & \frac{1}{n-1}-\frac{1}{n}
\end{array}$$

Adding both columns,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)\cdot n}=1-\frac{1}{n}=\frac{n-1}{n}.$$

❑

The alert reader will see how to generalise the method above. For example, to sum $\sum_{1\leq k\leq n}\frac{1}{k(k+1)(k+2)(k+3)}$, write the general term as the difference

$$\frac{1}{k(k+1)(k+2)(k+3)}=\frac{1}{3k(k+1)(k+2)}-\frac{1}{3(k+1)(k+2)(k+3)}.$$

This gives

$$\begin{aligned}
\sum_{1\leq k\leq n}\frac{1}{k(k+1)(k+2)(k+3)} &= \sum_{1\leq k\leq n}\left(\frac{1}{3k(k+1)(k+2)}-\frac{1}{3(k+1)(k+2)(k+3)}\right)\\
&= \frac{1}{3\cdot 1\cdot 2\cdot 3}-\frac{1}{2\cdot(n+1)(n+2)(n+3)}\\
&= \frac{1}{18}-\frac{1}{3\cdot(n+1)(n+2)(n+3)}.
\end{aligned}$$

Again, observing the difference

$$k(k+1)=\frac{k(k+1)(k+2)}{3}-\frac{(k-1)k(k+1)}{3},$$

we find

$$\begin{aligned}
1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1) &= \left(\frac{1\cdot 2\cdot 3}{3}-\frac{0\cdot 1\cdot 2}{3}\right)+\left(\frac{2\cdot 3\cdot 4}{3}-\frac{1\cdot 2\cdot 3}{3}\right)+\cdots+\left(\frac{n(n+1)(n+2)}{3}-\frac{(n-1)n(n+1)}{3}\right)\\
&= \frac{n(n+1)(n+2)}{3}-\frac{0\cdot 1\cdot 2}{3}\\
&= \frac{n(n+1)(n+2)}{3}.
\end{aligned}$$

The preceding identities were obtained by telescoping cancellation. The idea can be extended to some products. Here is a classic result.

6 Theorem Let $n \geq 1$ be an integer. Then

$$\prod_{k=1}^{n} \cos \frac{x}{2^k} = \frac{\sin x}{2^n \sin \frac{x}{2^n}}.$$

Proof: *Using* $\sin 2\theta = 2\sin\theta\cos\theta$, *and letting* $P = \prod_{k=1}^{n} \cos \frac{x}{2^k}$, *we have*

$$\begin{aligned}
\left(\sin \frac{x}{2^n}\right) P &= \left(\cos \frac{x}{2}\right)\left(\cos \frac{x}{2^2}\right)\cdots\left(\cos \frac{x}{2^n}\right)\left(\sin \frac{x}{2^n}\right) \\
&= \left(\cos \frac{x}{2}\right)\left(\cos \frac{x}{2^2}\right)\cdots\left(\cos \frac{x}{2^{n-1}}\right)\left(\frac{1}{2}\sin \frac{x}{2^{n-1}}\right) \\
&= \left(\cos \frac{x}{2}\right)\left(\cos \frac{x}{2^2}\right)\cdots\left(\cos \frac{x}{2^{n-2}}\right)\left(\frac{1}{2^2}\sin \frac{x}{2^{n-2}}\right) \\
&= \left(\cos \frac{x}{2}\right)\left(\cos \frac{x}{2^2}\right)\cdots\left(\cos \frac{x}{2^{n-3}}\right)\left(\frac{1}{2^3}\sin \frac{x}{2^{n-3}}\right) \\
&\vdots \quad \vdots \\
&= \frac{1}{2^n}\sin x
\end{aligned}$$

From where

$$\prod_{k=1}^{n} \cos \frac{x}{2^k} = \frac{\sin x}{2^n \sin \frac{x}{2^n}}.$$

❑

For the next discussion we will need the following notation. For integers $0 \leq k \leq n$, we define the symbol $\binom{n}{k}$ (read n *choose* k) as follows:

$$\binom{n}{0} = 1, \qquad \binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}.$$

For example,

$$\binom{10}{4} = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} = 210, \qquad \binom{10}{5} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 252.$$

7 Theorem (Binomial Theorem) $(1+x)^n = \sum_{0\leq k\leq n} \binom{n}{k} x^k.$

Proof: *We will give the following Calculus based proof, which essentially computes the MacLaurin expansion of* $x \mapsto (1+x)^n$. *It is clear that* $(1+x)^n$ *is a polynomial of degree* n, *hence put*

$$(1+x)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots a_n x^n.$$

We will prove that $a_k = \binom{n}{k}$. *Differentiating* k *times both sides of the above equality,*

$$n(n-1)(n-2)\cdots(n-k+1)(1+x)^{n-k} = k!a_k + (k+1)k(k-1)\cdots 2a_{k+1}x + \cdots + n(n-1)(n-2)\cdots(n-k+1)a_n x^{n-k}.$$

Setting $x = 0$ and noticing that every term after the first vanishes on the dextral side of the last equality,

$$n(n-1)(n-2)\cdots(n-k+1) = k!a_k \implies a_k = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \binom{n}{k},$$

as was required.❑

Setting $x = 1$ in the identity above we obtain the following corollary.

8 Corollary $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$

Since there are $\binom{n}{k}$ subsets of $\{1,2,\ldots,n\}$ with exactly k elements, both sides count the number of subsets of the set $\{1,2,\ldots,n\}$.

9 Example How many subsets of $\{1,2,3,\ldots,100\}$ have an even number (zero included) elements? How many have an odd number of elements.

Solution: ► *Recall that $\{1,2,3,\ldots,100\}$ has $2^{100} = 1267650600228229401496703205376$ subsets. Hence, doing a search of them one by one would be silly! The quantity*

$$\binom{100}{0} + \binom{100}{2} + \binom{100}{4} + \cdots + \binom{100}{98} + \binom{100}{100}$$

counts the number of subsets of $\{1,2,3,\ldots,100\}$ with an even number of elements, and similarly

$$\binom{100}{1} + \binom{100}{3} + \binom{100}{5} + \cdots + \binom{100}{97} + \binom{100}{99}$$

counts the number of subsets of $\{1,2,3,\ldots,100\}$ with an odd number of elements. Set

$$f(x) = (1+x)^{100} = \binom{100}{0} + \binom{100}{1}x + \binom{100}{2}x^2 + \binom{100}{3}x^3 + \cdots + \binom{100}{99}x^{99} + \binom{100}{100}x^{100}.$$

Then

$$2^{100} = f(1) = \binom{100}{0} + \binom{100}{1} + \binom{100}{2} + \binom{100}{3} + \cdots + \binom{100}{99} + \binom{100}{100},$$

and

$$0 = f(-1) = \binom{100}{0} - \binom{100}{1} + \binom{100}{2} - \binom{100}{3} + \cdots - \binom{100}{99} + \binom{100}{100}.$$

Whence,

$$\binom{100}{0} + \binom{100}{2} + \binom{100}{4} + \cdots + \binom{100}{98} + \binom{100}{100} = \frac{f(1)+f(-1)}{2} = 2^{99} = 633825300114114700748351602688,$$

and

$$\binom{100}{1} + \binom{100}{3} + \binom{100}{5} + \cdots + \binom{100}{97} + \binom{100}{99} = \frac{f(1)-f(-1)}{2} = 2^{99} = 633825300114114700748351602688.$$

Incidentally, we have proved that $\{1,2,3,\ldots,100\}$ has as many subsets with an even number of elements as with an odd number of elements. Try the Maple sequences

```
> sum(binomial(100,2*k),k=0..50);
> sum(binomial(100, 2*k-1), k = 1 ..  50)
```

◄

10 Example Find the exact value of the sum

$$\frac{\binom{10}{1}}{2}+\frac{\binom{10}{2}}{3}+\cdots+\frac{\binom{10}{10}}{11}.$$

Solution: ► *Put*

$$f(x)=(1+x)^{10}=\binom{10}{0}+\binom{10}{1}x+\binom{10}{2}x^2+\cdots+\binom{10}{10}x^{10}.$$

Integrating both sides on the interval $[0;1]$ *we obtain,*

$$\frac{2047}{11}=\int_0^1(1+x)^{10}\,\mathrm{d}x=\frac{\binom{10}{0}}{1}+\frac{\binom{10}{1}}{2}+\frac{\binom{10}{2}}{3}+\cdots+\frac{\binom{10}{10}}{11},$$

whence

$$\frac{\binom{10}{1}}{2}+\frac{\binom{10}{2}}{3}+\cdots+\frac{\binom{10}{10}}{11}=\frac{2047}{11}-\frac{\binom{10}{0}}{1}=\frac{2047}{11}-1=\frac{2036}{11}.$$

◄

Homework

11 Exercise Here is a standard interview question for prospective computer programmers: You are given a list of $1{,}000{,}001$ positive integers from the set $\{1,2,\ldots,1{,}000{,}000\}$. In your list, every member of $\{1,2,\ldots,1{,}000{,}000\}$ is listed once, except for x, which is listed twice. How do you find what x is without doing a $1{,}000{,}000$ step search?

12 Exercise Find the sum of all the integers from 1 to 1000 inclusive, which are not multiples of 3 or 5.

13 Exercise Find the sum of all integers between 1 and 100 that leave remainder 2 upon division by 6.

14 Exercise The odd natural numbers are arranged as follows:

$$(1)$$
$$(3,5)$$
$$(7,9,11)$$
$$(13,15,17,19)$$
$$(21,23,25,27,29)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Find the sum of the nth row.

15 Exercise Shew that

$$1+3+5+\cdots+2n-1=n^2.$$

16 Exercise Prove using the binomial theorem that $(k+1)^4=k^4+4k^3+6k^2+4k+1$. Then use the difference

$$(k+1)^4-k^4=4k^3+6k^2+4k+1$$

and the results of Corollaries 3 and 4 to prove that

$$1^3+2^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2.$$

17 Exercise A *palindrome* is a positive integer whose decimal expansion is symmetric and does not end in 0. For example, 1, 99, 100123321001, are all palindromes. Find the sum of all palindromes of five digits, that is, find

$$10001+10101+\cdots+99999.$$

18 Exercise Find a closed formula for

$$D_n=1-2+3-4+\cdots+(-1)^{n-1}n.$$

19 Exercise Find a closed formula for

$$T_n=1^2-2^2+3^2-4^2+\cdots+(-1)^{n-1}n^2.$$

20 Exercise Find a closed form for $\sum_{1\le k\le n}3^k$.

21 Exercise Let $n\ge 1$. Find a closed form for $\sum_{0\le k\le n}\binom{n}{k}(-1)^k$.

22 Exercise Find a closed form for $\sum_{1\le k\le n}\binom{n}{k}3^k$.

23 Exercise Evaluate the double sum $\sum_{1\le i\le n}\sum_{1\le k\le n}1$.

24 Exercise Evaluate the double sum $\sum_{1\leq i\leq n}\sum_{1\leq k\leq i} 1$.

25 Exercise Evaluate the double sum $\sum_{1\leq i\leq n}\sum_{1\leq k\leq i} k$.

26 Exercise Evaluate the double sum $\sum_{1\leq i\leq n}\sum_{1\leq k\leq n} ik$.

27 Exercise Legend says that the inventor of the game of chess, Sissa ben Dahir, asked the King Shirham of India to place a grain of wheat on the first square of the chessboard, 2 on the second square, 4 on the third square, 8 on the fourth square, etc..

1. How many grains of wheat are to be put on the last (64-th) square?,
2. How many grains, total, are needed in order to satisfy the greedy inventor?,
3. Given that 15 grains of wheat weigh approximately one gramme, what is the approximate weight, in kg, of wheat needed?,
4. Given that the annual production of wheat is 350 million tonnes, how many years, approximately, are needed in order to satisfy the inventor (assume that production of wheat stays constant)

28 Exercise Factor

$$1+x+x^2+\cdots+x^{80}$$

as a polynomial with integer coefficients.

29 Exercise Prove that $\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)=\frac{n+1}{2n}$.

30 Exercise Find integers a, b so that

$$(2+1)\cdot(2^2+1)\cdot\left(2^{2^2}+1\right)\cdot\left(2^{2^3}+1\right)\cdots\left(2^{2^{99}}+1\right)=2^a+b.$$

31 Exercise Prove that

$$(\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{1023} 1024)=10.$$

32 Exercise Evaluate $\sum_{k=1}^{1000}\lfloor \log_2 k\rfloor$.

33 Exercise Obtain a closed formula for $\sum_{1\leq k\leq n} k\cdot k!$. Hint: $(k+1)!=(k+1)k!$.

34 Exercise Prove, by differentiating $x\mapsto(1+x)^n$, that $\sum_{1\leq k\leq n} k\binom{n}{k}=n2^{n-1}$.

35 Exercise Prove that

$$\sum_{1\leq k\leq n} k^2\binom{n}{k}=2^{n-2}n^2+2^{n-2}n.$$

36 Exercise Prove that

$$\sum_{0\leq k\leq\lfloor n/2\rfloor}\binom{n}{2k}=\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=2^{n-1},$$

and that

$$\sum_{0\leq k\leq\lfloor n/2\rfloor}\binom{n}{2k+1}=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots=2^{n-1}.$$

(The first sum goes over all binomial coefficients with even index, the second, over the odd indices.)

37 Exercise Find the sum of all the coefficients once the following product is expanded and like terms are collected:

$$(1-x^2+x^4)^{109}(2-6x+5x^9)^{1996}.$$

38 Exercise Consider the polynomial

$$(1-x^2+x^4)^{2003}=a_0+a_1x+a_2x^2+\cdots+a_{8012}x^{8012}.$$

Find

❶ a_0

❷ $a_0+a_1+a_2+\cdots+a_{8012}$

❸ $a_0-a_1+a_2-a_3+\cdots-a_{8011}+a_{8012}$

❹ $a_0+a_2+a_4+\cdots+a_{8010}+a_{8012}$

❺ $a_1+a_3+\cdots+a_{8009}+a_{8011}$

39 Exercise Let f satisfy

$$f(n+1)=(-1)^{n+1}n-2f(n),\quad n\geq 1.$$

If $f(1)=f(1001)$ find

$$f(1)+f(2)+f(3)+\cdots+f(1000).$$

40 Exercise Prove the following identity of Catalan:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

41 Exercise Find

$$(123456789)^2-(123456787)\cdot(123456791),$$

mentally.

42 Exercise Given that 1002004008016032 has a prime factor $p > 250000$, find it.

43 Exercise Shew that

$$\csc 2 + \csc 4 + \csc 8 + \cdots + \csc 2^n = \cot 1 - \cot 2^n.$$

44 Exercise Find the exact value of the product

$$P = \cos\frac{\pi}{7} \cdot \cos\frac{2\pi}{7} \cdot \cos\frac{4\pi}{7}.$$

45 Exercise Shew that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{9999}{10000} < \frac{1}{100}.$$

46 Exercise Let $a_1, a_2, \ldots, a_n$ be arbitrary numbers. Shew that

$$\begin{aligned} & a_1 + a_2(1+a_1) + a_3(1+a_1)(1+a_2) \\ & \quad + a_4(1+a_1)(1+a_2)(1+a_3) + \cdots \\ & \quad + a_{n-1}(1+a_1)(1+a_2)(1+a_3)\cdots(1+a_{n-2}) \\ & \qquad = (1+a_1)(1+a_2)(1+a_3)\cdots(1+a_n) - 1. \end{aligned}$$

47 Exercise Shew that

$$\tan\frac{\pi}{2^{100}} + 2\tan\frac{\pi}{2^{99}} + 2^2\tan\frac{\pi}{2^{98}} + \cdots + 2^{98}\tan\frac{\pi}{2^2} = \cot\frac{\pi}{2^{100}}.$$

48 Exercise Shew that

$$\sum_{k=1}^{n} \frac{k}{k^4+k^2+1} = \frac{1}{2}\cdot\frac{n^2+n}{n^2+n+1}.$$

49 Exercise (Lagrange's Identity) Let a_k, b_k be real numbers. Prove that

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1\le k<j\le n} (a_k b_j - a_j b_k)^2.$$

50 Exercise The sum of a certain number of consecutive positive integers is 1000. Find these integers. (There is more than one solution. You must find them all.)

1.2 Some Infinite Sums and Products

The material of this section will be treated *formally*, that is, without much rigor. We present here without proof, the following MacLaurin expansions, which we hope the reader has encountered in his Calculus courses.

51 Theorem The following expansions hold:

1. $\displaystyle \frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \quad |x| < 1$

2. $\displaystyle \sin x = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots, \quad x \in \mathbb{R}.$

3. $\displaystyle \cos x = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots, \quad x \in \mathbb{R}.$

4. $\displaystyle e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots, \quad x \in \mathbb{R}$

5. $\displaystyle \log(1+x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots, \quad |x| < 1.$

6. $\displaystyle (1+x)^\tau = \sum_{n=0}^{+\infty} \binom{\tau}{n} x^n = 1 + \tau x + \frac{\tau(\tau-1)}{2!}x^2 + \cdots + \frac{\tau(\tau-1)(\tau-2)(\tau-3)\cdots(\tau-n+1)}{n!}x^n + \cdots, \quad |x| < 1.$

The idea of the preceding section of finding a general function and then evaluating it a particular value extends to infinite sums, but care must be taken with convergence. We state the following without proof.

52 Theorem (Abel's Limit Theorem) Let $r > 0$, and suppose that $\sum_{n\geq 0} a_n r^n$ converges. Then $\sum_{n\geq 0} a_n x^n$ converges absolutely for $|x| < r$, and

$$\lim_{x\to r^-} \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0} a_n r^n.$$

53 Example Find the exact numerical value of the alternating harmonic series

$$\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

Solution: ► *This alternating series converges by Leibniz's Test. Consider more generally the MacLaurin expansion of* $x \mapsto \log(1+x)$*:*

$$f(x) = \sum_{n\geq 1} \frac{(-1)^{n-1}x^n}{n} = \log(1+x).$$

We see that $f(1) = \log 2$. *Thus*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2,$$

by Abel's Limit Theorem. The Maple™ commands to obtain this sum are

```
> sum(((-1)^(k+1))/(k),k=1..infinity);
```

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We now consider an infinite product. Letting $n \to +\infty$ in the product of theorem 6, we deduce the following result.

54 Theorem

$$\prod_{k=1}^{+\infty} \cos\frac{x}{2^k} = \lim_{n\to+\infty} \prod_{k=1}^{n} \cos\frac{x}{2^k} = \lim_{n\to+\infty} \frac{\sin x}{2^n \sin\frac{x}{2^n}} = \frac{\sin x}{x}.$$

Letting $x = \frac{\pi}{2}$ we obtain one of the earliest formulas form π.

55 Corollary (Vieta's Formula for π)

$$\frac{2}{\pi} = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\right)\cdots.$$

Some infinite sums can be recognised as being Riemann sums, and hence, allowing one to sum them. In general,

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to+\infty} \frac{b-a}{n} \sum_{k=0}^{n} f\left(a + \frac{k(b-a)}{n}\right), \tag{1.2}$$

if f is Riemann-integrable function on $[a;b]$.

56 Example Find $\lim_{n\to+\infty} \sum_{k=0}^{n} \frac{n}{n^2+k^2}$.

Solution: ▸ *We have,*

$$\sum_{k=0}^{n} \frac{n}{n^2+k^2} = \sum_{k=0}^{n} \frac{1}{n} \frac{1}{1+\frac{k^2}{n^2}} \to \int_0^1 \frac{\mathrm{d}x}{1+x^2} = \frac{\pi}{4},$$

If $f(x) = \frac{1}{1+x^2}$, $a = 0$, $b = 1$, *by (1.2).* ◂

Homework

57 Exercise A fly starts at the origin and goes 1 unit up, $1/2$ unit right, $1/4$ unit down, $1/8$ unit left, $1/16$ unit up, etc., *ad infinitum.* In what coordinates does it end up?

58 Exercise Find the exact numerical value of

$$\sum_{n\geq 0} \frac{(n+1)^2}{n!}.$$

59 Exercise Find the exact numerical value of the sum $\sum_{n=1}^{+\infty} n2^{1-n}$.

60 Exercise Find the exact numerical value of the sum $\sum_{n=1}^{+\infty} n^2 2^{1-n}$.

61 Exercise Let $\mathscr{S}$ be the set of positive integers none of whose digits in its decimal representation is a 0. Prove that the series $\sum_{n\in\mathscr{S}} \frac{1}{n}$ converges.

62 Exercise Find the exact numerical value of the sum $\sum_{n=0}^{+\infty} \arctan \frac{1}{n^2+n+1}$.

63 Exercise Using $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ deduce that

$$\frac{\sin x}{x} = \prod_{n=1}^{+\infty} \frac{4\cos^2 \frac{x}{3^n} - 1}{3}.$$

64 Exercise Find the sum of the series $\sum_{n=1}^{+\infty} \frac{1}{4n^2-1}$.

65 Exercise Prove that $\prod_{n=2}^{+\infty} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$.

66 Exercise Find the exact numerical value of the infinite sum

$$\sum_{n=1}^{+\infty} \frac{\sqrt{(n-1)!}}{(1+\sqrt{1})(1+\sqrt{2})(1+\sqrt{3})\cdots(1+\sqrt{n})}.$$

67 Exercise Find

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \cdots,$$

which is the sum of the reciprocals of all positive integers of the form $2^n 3^m$ for integers $n \geq 0, m \geq 0$.

68 Exercise (Deus Numero Impare Gaudet) Prove that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \sum_{n\geq 1} \frac{(-1)^{n+1}}{2n-1}.$$

69 Exercise Prove that

$$1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right).$$

(Hint: Expand $(1+x^3)^{-1}$) into a power series. Integrate $(1+x^3)^{-1}$ using partial fractions. Use Abel's Limit Theorem.)

70 Exercise Let $0 < x < 1$. Shew that

$$\sum_{n=1}^{\infty} \frac{x^{2^n}}{1-x^{2^{n+1}}} = \frac{x}{1-x}.$$

71 Exercise Evaluate

$$\left(\frac{1\cdot 2\cdot 4 + 2\cdot 4\cdot 8 + 3\cdot 6\cdot 12 + \cdots}{1\cdot 3\cdot 9 + 2\cdot 6\cdot 18 + 3\cdot 9\cdot 27 + \cdots}\right)^{1/3}.$$

72 Exercise Prove that

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k^2}} = \log(1+\sqrt{2}).$$

73 Exercise (Gram's Product) Prove that

$$\prod_{k=2}^{+\infty} \frac{k^3-1}{k^3+1} = \frac{2}{3}.$$

1.3 Some Identities with Complex Numbers

We use the symbol i to denote the *imaginary unit* $i = \sqrt{-1}$. Then $i^2 = -1$. Since $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, etc., the powers of i repeat themselves cyclically in a cycle of period 4.

74 Example For any positive integer α one has

$$i^{\alpha} + i^{\alpha+1} + i^{\alpha+2} + i^{\alpha+3} = i^{\alpha}(1 + i + i^2 + i^3) = i^{\alpha}(1 + i - 1 - i) = 0.$$

75 Definition If a, b are real numbers then the object $z = a + bi$ is called a *complex number*. We use the symbol $\mathbb{C}$ to denote the set of all complex numbers. $a = \Re z$ is the *real part* of z and $b = \Im z$ is the *imaginary* part of z.

If $a, b, c, d \in \mathbb{R}$, then the sum of the complex numbers $a + bi$ and $c + di$ is naturally defined as

$$(a + bi) + (c + di) = (a + c) + (b + d)i \tag{1.3}$$

The product of $a + bi$ and $c + di$ is obtained by multiplying the binomials:

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i \tag{1.4}$$

Complex numbers can be given a geometric representation in the *Argand diagram* (see figure 1.1), where the horizontal axis carries the real parts and the vertical axis the imaginary ones.

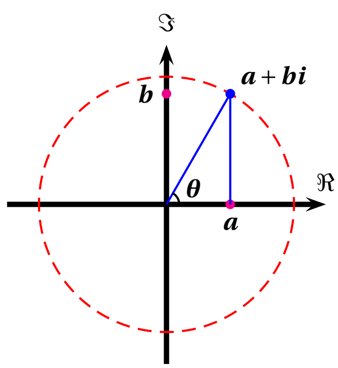

Figure 1.1: Argand's diagram.

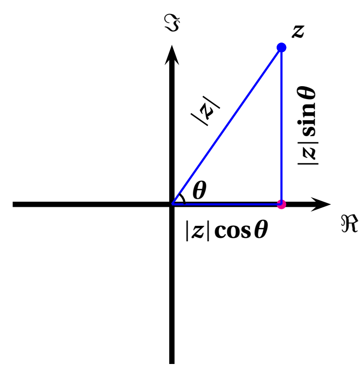

Figure 1.2: Polar Form of a Complex Number.

76 Definition Let $z \in \mathbb{C}, (a, b) \in \mathbb{R}^2$ with $z = a + bi$. The *conjugate* $\overline{z}$ of z is defined by

$$\overline{z} = \overline{a + bi} = a - bi \tag{1.5}$$

☞ *The conjugate of a real number is itself, that is, if $a \in \mathbb{R}$, then $\overline{a} = a$. Also, the conjugate of the conjugate of a number is the number, that is, $\overline{\overline{z}} = z$.*

77 Theorem The function $z : \mathbb{C} \to \mathbb{C}$, $z \mapsto \overline{z}$ is multiplicative, that is, if z_1, z_2 are complex numbers, then

$$\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2} \tag{1.6}$$

Proof: *Let* $z_1 = a+bi, z_2 = c+di$ *where* a,b,c,d *are real numbers. Then*

$$\begin{aligned}\overline{z_1 z_2} &= \overline{(a+bi)(c+di)}\\ &= \overline{(ac-bd)+(ad+bc)i}\\ &= (ac-bd)-(ad+bc)i\end{aligned}$$

Also,

$$\begin{aligned}\overline{z_1}\cdot\overline{z_2} &= (\overline{a+bi})(\overline{c+di})\\ &= (a-bi)(c-di)\\ &= ac-adi-bci+bdi^2\\ &= (ac-bd)-(ad+bc)i,\end{aligned}$$

which establishes the equality between the two quantities. ❑

78 Definition The *modulus* $|a+bi|$ of $a+bi$ is defined by

$$|a+bi| = \sqrt{(a+bi)(\overline{a+bi})} = \sqrt{a^2+b^2} \tag{1.7}$$

Observe that $z \mapsto |z|$ is a function mapping $\mathbb{C}$ to $[0;+\infty[$.

Given a complex number $z = a+bi$ on the Argand diagram, consider the angle $\theta \in]-\pi;\pi]$ that a straight line segment passing through the origin and through z makes with the positive real axis. Considering the polar coordinates of z we gather

$$z = |z|(\cos\theta + i\sin\theta), \qquad \theta \in]-\pi;\pi], \tag{1.8}$$

which we call the *polar form* of the complex number z. The angle θ is called the *argument* of the complex number z.

79 Example Find the polar form of $\sqrt{3}-i$.

Solution: ► *First observe that* $|\sqrt{3}-i| = \sqrt{\sqrt{3}^2+1^2} = 2$. *Now, if*

$$\sqrt{3}-i = 2(\cos\theta + i\sin\theta),$$

we need $\cos\theta = \frac{\sqrt{3}}{2}$, $\sin\theta = -\frac{1}{2}$. *This happens for* $\theta \in]-\pi;\pi]$ *when* $\theta = -\frac{\pi}{6}$. *Therefore,*

$$\sqrt{3}-i = 2(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)$$

is the required polar form. ◄

80 Theorem The function $z \mapsto |z|$, $\mathbb{C} \to [0;+\infty[$ is multiplicative. That is, if z_1, z_2 are complex numbers then

$$|z_1 z_2| = |z_1||z_2| \tag{1.9}$$

Proof: *By Theorem 77, conjugation is multiplicative, hence*

$$\begin{aligned} |z_1z_2| &= \sqrt{z_1z_2\overline{z_1z_2}} \\ &= \sqrt{z_1z_2\overline{z_1}\cdot\overline{z_2}} \\ &= \sqrt{z_1\overline{z_1}z_2\overline{z_2}} \\ &= \sqrt{z_1\overline{z_1}}\sqrt{z_2\overline{z_2}} \\ &= |z_1||z_2| \end{aligned}$$

whence the assertion follows. ❑

81 Example Write $(2^2+3^2)(5^2+7^2)$ as the sum of two squares.

Solution: ► *The idea is to write* $2^2+3^2=|2+3i|^2$, $5^2+7^2=|5+7i|^2$ *and use the multiplicativity of the modulus. Now*

$$\begin{aligned} (2^2+3^2)(5^2+7^2) &= |2+3i|^2|5+7i|^2 \\ &= |(2+3i)(5+7i)|^2 \\ &= |-11+29i|^2 \\ &= 11^2+29^2 \end{aligned}$$

◄

We now present some identities involving complex numbers. Let us start with the following classic result.

If we allow complex numbers in our MacLaurin expansions, we readily obtain Euler's Formula.

82 Theorem (Euler's Formula) Let $x\in\mathbb{R}$. Then

$$e^{ix}=\cos x+i\sin x.$$

Proof: *Using the MacLaurin expansion's of* $x\mapsto e^x$, $x\mapsto\cos x$, *and* $x\mapsto\sin x$, *we gather*

$$\begin{aligned} e^{ix} &= \sum_{k=0}^{+\infty}\frac{(ix)^n}{n!} \\ &= \sum_{k=0}^{+\infty}\frac{(ix)^{2n}}{(2n)!}+\sum_{k=0}^{+\infty}\frac{(ix)^{2n+1}}{(2n+1)!} \\ &= \sum_{k=0}^{+\infty}\frac{(-1)^nx^{2n}}{(2n)!}+i\sum_{k=0}^{+\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!} \\ &= \cos x+i\sin x. \end{aligned}$$

❑

Taking complex conjugates,

$$e^{-ix} = \overline{e^{ix}} = \overline{\cos x + i \sin x} = \cos x - i \sin x.$$

Solving for $\sin x$ we obtain

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{1.10}$$

Similarly,

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{1.11}$$

83 Corollary (De Moivre's Theorem) Let $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. Then

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

Proof: *We have*

$$(\cos x + i \sin x)^n = (e^{ix})^n = e^{ixn} = \cos nx + i \sin nx,$$

by theorem 82.

Aliter: *An alternative proof without appealing to Euler's identity follows. We first assume that* $n > 0$ *and give a proof by induction. For* $n = 1$ *the assertion is obvious, as*

$$(\cos x + i \sin x)^1 = \cos 1 \cdot x + i \sin 1 \cdot x.$$

Assume the assertion is true for $n - 1 > 1$*, that is, assume that*

$$(\cos x + i \sin x)^{n-1} = \cos(n-1)x + i \sin(n-1)x.$$

Using the addition identities for the sine and cosine,

$$\begin{aligned}
(\cos x + i \sin x)^n &= (\cos x + i \sin x)(\cos x + i \sin x)^{n-1} \\
&= (\cos x + i \sin x)(\cos(n-1)x + i \sin(n-1)x). \\
&= (\cos x)(\cos(n-1)x) - (\sin x)(\sin(n-1)x) + i((\cos x)(\sin(n-1)x) + (\cos(n-1)x)(\sin x)). \\
&= \cos(n-1+1)x + i \sin(n-1+1)x \\
&= \cos nx + i \sin nx,
\end{aligned}$$

proving the theorem for $n > 0$.

Assume now that $n < 0$. *Then* $-n > 0$ *and we may used what we just have proved for positive* *integers we have*

$$
\begin{aligned}
(\cos x + i\sin x)^n &= \frac{1}{(\cos x + i\sin x)^{-n}} \\
&= \frac{1}{\cos(-nx) + i\sin(-nx)} \\
&= \frac{1}{\cos nx - i\sin nx} \\
&= \frac{\cos nx + i\sin nx}{(\cos nx + i\sin nx)(\cos nx - i\sin nx)} \\
&= \frac{\cos nx + i\sin nx}{\cos^2 nx + \sin^2 nx} \\
&= \cos nx + i\sin nx,
\end{aligned}
$$

proving the theorem for $n < 0$. *If* $n = 0$, *then since* $\sin$ *and* $\cos$ *are not simultaneously zero, we get* $1 = (\cos x + i\sin x)^0 = \cos 0x + i\sin 0x = \cos 0x = 1$, *proving the theorem for* $n = 0$.

❑

84 Example Prove that

$$\cos 3x = 4\cos^3 x - 3\cos x, \qquad \sin 3x = 3\sin x - 4\sin^3 x.$$

Solution: ► *Using Euler's identity and the Binomial Theorem,*

$$
\begin{aligned}
\cos 3x + i\sin 3x &= e^{3ix} \\
&= (e^{ix})^3 = (\cos x + i\sin x)^3 \\
&= \cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x \\
&= \cos^3 x + 3i(1 - \sin^2 x)\sin x - 3\cos x(1 - \cos^2 x) - i\sin^3 x,
\end{aligned}
$$

we gather the required identities. ◄

The following corollary is immediate.

85 Corollary (Roots of Unity) If $n > 0$ is an integer, the n numbers $e^{2\pi ik/n} = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$, $0 \le k \le n-1$, are all different and satisfy $(e^{2\pi ik/n})^n = 1$.

86 Example For $n = 2$, the square roots of unity are the roots of

$$x^2 - 1 = 0 \implies x \in \{-1, 1\}.$$

For $n = 3$ we have $x^3 - 1 = (x-1)(x^2+x+1) = 0$ hence if $x \ne 1$ then $x^2 + x + 1 = 0 \implies x = \frac{-1 \pm i\sqrt{3}}{2}$. Hence the cubic roots of unity are

$$\left\{-1, \frac{-1 - i\sqrt{3}}{2}, \frac{-1 + i\sqrt{3}}{2}\right\}.$$

Or, we may find them trigonometrically,

$$\begin{array}{rcccl} e^{2\pi i\cdot 0/3} & = & \cos\frac{2\pi\cdot 0}{3}+i\sin\frac{2\pi\cdot 0}{3} & = & 1, \\ e^{2\pi i\cdot 1/3} & = & \cos\frac{2\pi\cdot 1}{3}+i\sin\frac{2\pi\cdot 1}{3} & = & -\frac{1}{2}+i\frac{\sqrt{3}}{2} \\ e^{2\pi i\cdot 2/3} & = & \cos\frac{2\pi\cdot 2}{3}+i\sin\frac{2\pi\cdot 2}{3} & = & -\frac{1}{2}-i\frac{\sqrt{3}}{2} \end{array}$$

For $n=4$ they are the roots of $x^4-1=(x-1)(x^3+x^2+x+1)=(x-1)(x+1)(x^2+1)=0$, which are clearly

$$\{-1,1,-i,i\}.$$

Or, we may find them trigonometrically,

$$\begin{array}{rcccl} e^{2\pi i\cdot 0/4} & = & \cos\frac{2\pi\cdot 0}{4}+i\sin\frac{2\pi\cdot 0}{4} & = & 1, \\ e^{2\pi i\cdot 1/4} & = & \cos\frac{2\pi\cdot 1}{4}+i\sin\frac{2\pi\cdot 1}{4} & = & i \\ e^{2\pi i\cdot 2/4} & = & \cos\frac{2\pi\cdot 2}{4}+i\sin\frac{2\pi\cdot 2}{4} & = & -1 \\ e^{2\pi i\cdot 3/4} & = & \cos\frac{2\pi\cdot 3}{4}+i\sin\frac{2\pi\cdot 3}{4} & = & -i \end{array}$$

For $n=5$ they are the roots of $x^5-1=(x-1)(x^4+x^3+x^2+x+1)=0$. To solve $x^4+x^3+x^2+x+1=0$ observe that since clearly $x\neq 0$, by dividing through by x^2, we can transform the equation into

$$x^2+\frac{1}{x^2}+x+\frac{1}{x}+1=0.$$

Put now $u=x+\frac{1}{x}$. Then $u^2-2=x^2+\frac{1}{x^2}$, and so

$$x^2+\frac{1}{x^2}+x+\frac{1}{x}+1=0 \implies u^2-2+u+1=0 \implies u=\frac{-1\pm\sqrt{5}}{2}.$$

Solving both equations

$$x+\frac{1}{x}=\frac{-1-\sqrt{5}}{2}, \qquad x+\frac{1}{x}=\frac{-1+\sqrt{5}}{2},$$

we get the four roots

$$\left\{\frac{-1-\sqrt{5}}{4}-i\frac{\sqrt{10-2\sqrt{5}}}{4},\quad \frac{-1-\sqrt{5}}{4}+i\frac{\sqrt{10-2\sqrt{5}}}{4},\quad \frac{\sqrt{5}-1}{4}-i\frac{\sqrt{2\sqrt{5}+10}}{4},\quad \frac{\sqrt{5}-1}{4}+i\frac{\sqrt{2\sqrt{5}+10}}{4}\right\},$$

or, we may find, trigonometrically,

$$\begin{array}{rcccl} e^{2\pi i\cdot 0/5} & = & \cos\frac{2\pi\cdot 0}{5}+i\sin\frac{2\pi\cdot 0}{5} & = & 1, \\ e^{2\pi i\cdot 1/5} & = & \cos\frac{2\pi\cdot 1}{5}+i\sin\frac{2\pi\cdot 1}{5} & = & \left(\frac{\sqrt{5}-1}{4}\right)+i\left(\frac{\sqrt{2}\cdot\sqrt{5+\sqrt{5}}}{4}\right), \\ e^{2\pi i\cdot 2/5} & = & \cos\frac{2\pi\cdot 2}{5}+i\sin\frac{2\pi\cdot 2}{5} & = & \left(\frac{-\sqrt{5}-1}{4}\right)+i\left(\frac{\sqrt{2}\cdot\sqrt{5-\sqrt{5}}}{4}\right), \\ e^{2\pi i\cdot 3/5} & = & \cos\frac{2\pi\cdot 3}{5}+i\sin\frac{2\pi\cdot 3}{5} & = & \left(\frac{-\sqrt{5}-1}{4}\right)-i\left(\frac{\sqrt{2}\cdot\sqrt{5-\sqrt{5}}}{4}\right), \\ e^{2\pi i\cdot 4/5} & = & \cos\frac{2\pi\cdot 4}{5}+i\sin\frac{2\pi\cdot 4}{5} & = & \left(\frac{\sqrt{5}-1}{4}\right)-i\left(\frac{\sqrt{2}\cdot\sqrt{5+\sqrt{5}}}{4}\right), \end{array}$$

See figures 1.3 through 1.5.

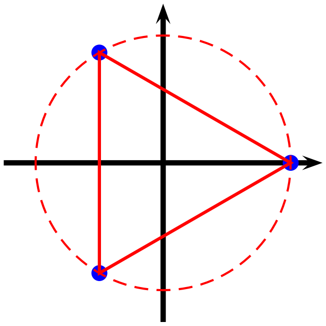

Figure 1.3: Cubic Roots of 1.

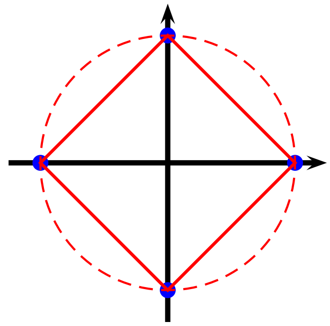

Figure 1.4: Quartic Roots of 1.

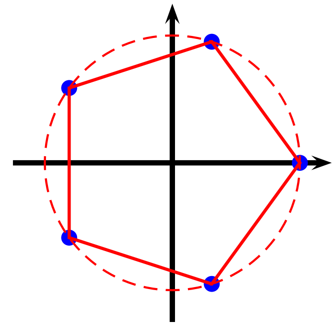

Figure 1.5: Quintic Roots of 1.

By the Fundamental Theorem of Algebra the equation $x^n - 1 = 0$ has exactly n complex roots, which gives the following result.

87 Corollary Let $n > 0$ be an integer. Then

$$x^n - 1 = \prod_{k=0}^{n-1} (x - e^{2\pi ik/n}).$$

88 Theorem We have,

$$1 + x + x^2 + \cdots + x^{n-1} = \begin{cases} 0 & x = e^{\frac{2\pi ik}{n}}, \quad 1 \leq k \leq n-1, \\ n & x = 1. \end{cases}$$

Proof: *Since* $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$, *from Corollary 87, if* $x \neq 1$,

$$x^{n-1} + x^{n-2} + \cdots + x + 1 = \prod_{k=1}^{n-1} (x - e^{2\pi ik/n}).$$

If ϵ *is a root of unity different from* 1, *then* $\epsilon = e^{2\pi ik/n}$ *for some* $k \in [1; n-1]$, *and this proves the theorem. Alternatively,*

$$1 + \epsilon + \epsilon^2 + \epsilon^3 + \cdots + \epsilon^{n-1} = \frac{\epsilon^n - 1}{\epsilon - 1} = 0.$$

This gives the result. ❑

89 Theorem Let $n \geq 1$ be an integer. Then $\dfrac{n}{2^{n-1}} = \displaystyle\prod_{k=1}^{n-1} \sin \frac{k\pi}{n}$.

Proof: *Differentiating both sides of the equality*

$$x^n - 1 = \prod_{k=0}^{n-1} (x - e^{2\pi ik/n}),$$

and letting $x = 1$,

$$\begin{aligned}
n &= (1 - e^{2\pi i/n})(1 - e^{4\pi i/n})(1 - e^{6\pi i/n})\cdots(1 - e^{2(n-1)\pi i/n}) \\
&= e^{(1+2+3+\cdots+(n-1))\pi i/n}(e^{-\pi i/n} - e^{\pi i/n})(e^{-2\pi i/n} - e^{2\pi i/n})(e^{-3\pi i/n} - e^{3\pi/n})\cdots(e^{-(n-1)\pi i/n} - e^{(n-1)\pi i/n}) \\
&= e^{(n-1)\pi i/2}\left(-2i\sin\frac{\pi}{n}\right)\left(-2i\sin\frac{2\pi}{n}\right)\cdots\left(-2i\sin\frac{(n-1)\pi}{n}\right) \\
&= e^{(n-1)\pi i/2}(-i)^{n-1}2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} \\
&= (e^{\pi i/2})^{n-1}(-i)^{n-1}2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} \\
&= i^{n-1}(-i)^{n-1}2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} \\
&= (-i^2)^{n-1}2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} \\
&= 2^{n-1}\prod_{k=1}^{n-1}\sin\frac{k\pi}{n},
\end{aligned}$$

giving the result. ❑

90 Example Prove that the improper integral $I = \int_0^{\pi} \log\sin x \, dx = -\pi\log 2$.

Solution: ► *We will deduce this in two ways. From Theorem 89,*

$$\sum_{k=1}^{n-1}\log\sin\frac{k\pi}{n} = \log n - (n-1)\log 2.$$

By (1.2), we see that

$$\int_0^{\pi}\log\sin x\,dx = \lim_{n\to+\infty}\frac{\pi}{n}\sum_{k=1}^{n-1}\log\sin\frac{k\pi}{n} = \lim_{n\to+\infty}\frac{\pi}{n}\left(\log n - (n-1)\log 2\right) = -\pi\log 2,$$

as claimed.

Aliter: *From* $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ *we get*

$$\begin{aligned}
I &= \int_0^{\pi}\log 2\,dx + \int_0^{\pi}\log\sin\frac{x}{2}\,dx + \int_0^{\pi}\log\cos\frac{x}{2}\,dx \\
&= \pi\log 2 + 2\int_0^{\pi/2}\log\sin y\,dy + 2\int_0^{\pi/2}\log\cos y\,dy.
\end{aligned}$$

Setting $y = \frac{\pi}{2} - u$ *and using* $\sin(\pi - u) = \sin u = \cos\left(\frac{\pi}{2} - u\right)$ *we see that*

$$\int_0^{\pi/2}\log\sin y\,dy = \int_0^{\pi/2}\log\cos y\,dy \implies 2\int_0^{\pi/2}\log\sin y\,dy = \int_0^{\pi/2}\left(\log\sin u + \log\sin(\pi - x)\right)du = \int_0^{\pi}\log\sin u\,du = I,$$

from where

$$I = \pi\log 2 + 2I \implies I = -\pi\log 2.$$

◄

91 Example Justify that $\sum_{n=1}^{+\infty}\frac{\sin n}{n} = \frac{\pi - 1}{2}$.

Solution: ► *We start by assuming that* $\sum_{n=1}^{+\infty}\frac{e^{izn}}{n}=-\log(1-e^{iz})$ *for* $z\in\mathbb{R}$, *in analogy to the MacLaurin expansion of* $x\mapsto\log(1+x)$ *for real* x. *Then letting* $z=1$,

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{\cos n+i\sin n}{n} &= \sum_{n=1}^{+\infty}\frac{e^{in}}{n}\\ &= -\log(1-e^{i})\\ &= -\log e^{i/2}(e^{-i/2}-e^{i/2})\\ &= -\log e^{i/2}-\log 2i\left(-\sin\frac{1}{2}\right)\\ &= -\log(-2i)-\frac{i}{2}-\log\left(\sin\frac{1}{2}\right)\end{aligned}$$

Since $-2i=2e^{-\pi i/2}$, $-\log(-2i)=-\log 2+\frac{\pi i}{2}$. *Thus we get*

$$\sum_{n=1}^{+\infty}\frac{\cos n+i\sin n}{n}=-\log 2-\log\left(\sin\frac{1}{2}\right)+i\left(\frac{\pi}{2}-\frac{1}{2}\right).$$

Equating real and imaginary parts we verify our claim.

The formal argument above can be rigorously proved by means of Fourier Analysis, but this is beyond our scope.

◄

Theorem 88 is quite useful for "multisecting" a power series.

92 Example Find the sum $S=\sum_{k=0}^{9}\binom{27}{3k}$.

Solution: ► *We use the fact that for* $\epsilon_1=-1/2+i\sqrt{3}/2$ *and* $\epsilon_2=-1/2-i\sqrt{3}/2$ *are cubic roots of unity and hence satisfy*

$$\epsilon_k^3=1,\ \text{and}\ 1+\epsilon_k+\epsilon_k^2=0,\ k=1,2.$$

Thus

$$\epsilon_k^s+\epsilon_k^{s+1}+\epsilon_k^{s+2}=0,\ k=1,2,\ s\in\mathbb{Z}. \tag{1.12}$$

From this

$$\begin{aligned}(1+1)^{27} &= \binom{27}{0}+\binom{27}{1}+\binom{27}{2}+\binom{27}{4}+\cdots+\binom{27}{26}+\binom{27}{27}\\ (1+\epsilon_1)^{27} &= \binom{27}{0}+\binom{27}{1}\epsilon_1+\binom{27}{2}\epsilon_1^2+\binom{27}{3}\epsilon_1^3+\cdots+\binom{27}{27}\epsilon_1^{27}\\ (1+\epsilon_2)^{27} &= \binom{27}{0}+\binom{27}{1}\epsilon_2+\binom{27}{2}\epsilon_2^2+\binom{27}{3}\epsilon_2^3+\cdots+\binom{27}{27}\epsilon_2^{27}\end{aligned}$$

Summing column-wise and noticing that because of (1.12) only the terms $0,3,6,\ldots,27$ *survive,*

$$2^{27}+(1+\epsilon_1)^{27}+(1+\epsilon_2)^{27}=3\binom{27}{0}+3\binom{27}{3}+3\binom{27}{6}+\cdots+3\binom{27}{27}.$$

By DeMoivre's Theorem, $(1-1/2+i\sqrt{3}/2)^{27}=\cos 9\pi+i\sin 9\pi=-1$ *and* $(1-1/2-i\sqrt{3}/2)^{27}=\cos 45\pi+i\sin 45\pi=-1$. *Thus*

$$\binom{27}{0}+\binom{27}{3}+\binom{27}{6}+\cdots+\binom{27}{27}=\frac{1}{3}(2^{27}-2).$$

◄

The procedure of example 92 can be generalised as follows. Suppose that

$$f(x) = \sum_{k=0}^{\infty} c_k x^k.$$

If $\omega = e^{2\pi i/q}$, $q \in \mathbb{N}$, $q > 1$, then $\omega^q = 1$ and $1 + \omega + \omega^2 + \omega^3 + \cdots + \omega^{q-1} = 0$. Then in view of

$$\frac{1}{q} \sum_{1 \le b \le q} \omega^{kb} = \begin{cases} 1 & \text{if } q \text{ divides } k, \\ 0 & \text{else,} \end{cases}$$

we have

$$\sum_{\substack{n=0 \\ n \equiv a \mod q}}^{\infty} c_n x^n = \frac{1}{q} \sum_{b=1}^{q} \omega^{-ab} f(\omega^b x). \tag{1.13}$$

We may use complex numbers to select certain sums of coefficients of polynomials. The following problem uses the fact that if k is an integer

$$i^k + i^{k+1} + i^{k+2} + i^{k+3} = i^k(1 + i + i^2 + i^3) = 0 \tag{1.14}$$

93 Example Let

$$(1 + x^4 + x^8)^{100} = a_0 + a_1 x + a_2 x^2 + \cdots + a_{800} x^{800}.$$

Find:

❶ $a_0 + a_1 + a_2 + a_3 + \cdots + a_{800}$.

❷ $a_0 + a_2 + a_4 + a_6 + \cdots + a_{800}$.

❸ $a_1 + a_3 + a_5 + a_7 + \cdots + a_{799}$.

❹ $a_0 + a_4 + a_8 + a_{12} + \cdots + a_{800}$.

❺ $a_1 + a_5 + a_9 + a_{13} + \cdots + a_{797}$.

Solution: ► *Put*

$$p(x) = (1 + x^4 + x^8)^{100} = a_0 + a_1 x + a_2 x^2 + \cdots + a_{800} x^{800}.$$

Then

❶
$$a_0 + a_1 + a_2 + a_3 + \cdots + a_{800} = p(1) = 3^{100}.$$

❷
$$a_0 + a_2 + a_4 + a_6 + \cdots + a_{800} = \frac{p(1) + p(-1)}{2} = 3^{100}.$$

❸
$$a_1 + a_3 + a_5 + a_7 + \cdots + a_{799} = \frac{p(1) - p(-1)}{2} = 0.$$

❹
$$a_0 + a_4 + a_8 + a_{12} + \cdots + a_{800} = \frac{p(1) + p(-1) + p(i) + p(-i)}{4} = 2 \cdot 3^{100}.$$

❺
$$a_1 + a_5 + a_9 + a_{13} + \cdots + a_{797} = \frac{p(1) - p(-1) - ip(i) + ip(-i)}{4} = 0.$$

◄

Homework

94 Exercise Compute
$$\frac{(1+i)^{2004}}{(1-i)^{2000}}.$$

95 Exercise Let $i^2 = -1$. Evaluate
$$1+2i+3i^2+4i^3+5i^4+\cdots+2007i^{2006}.$$

96 Exercise Prove that
$$\cos^6 2x = \frac{1}{32}\cos 12x + \frac{3}{16}\cos 8x + \frac{15}{32}\cos 4x + \frac{5}{16}.$$

97 Exercise Prove that
$$\sqrt{3} = \tan\frac{\pi}{9} + 4\sin\frac{\pi}{9}.$$

98 Exercise Let
$$(1+x+x^2)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n}x^{2n}.$$

Find formulæ for

1. $\sum_{k=0}^{2n} a_k$
2. $\sum_{0\leq k\leq n/2} a_{2k}$
3. $\sum_{1\leq k\leq n/2} a_{2k-1}$
4. $a_0 + a_4 + a_8 + \cdots$
5. $a_1 + a_5 + a_9 + \cdots$

99 Exercise Find the exact numerical value of
$$\sum_{k=0}^{665}\binom{1995}{3k}.$$

1.4 Iteration and Recursion

100 Definition Given a function f, its *iterate at* x is $f(f(x))$, that is, we use its value as the new input. The iterates at x
$$x,\ f(x),\ f(f(x)),\ f(f(f(x))),\ldots$$
are called 0-*th iterate*, 1*st iterate*, 2*nd iterate*, 3*rd iterate*, etc. We denote the n-th iterate by $f^{[n]}$.

In some particular cases it is easy to find the nth iterate of a function, for example
$$a(x) = x^t \implies a^{[n]}(x) = x^{t^n},$$
$$b(x) = mx \implies b^{[n]}(x) = m^n x,$$
$$c(x) = mx + k \implies c^{[n]}(x) = m^n x + k\left(\frac{m^n - 1}{m-1}\right).$$
The above examples are more the exception than the rule. Even if its possible to find a closed formula for the n-th iterate some cases prove quite truculent.

101 Example Let $f(x) = \dfrac{1}{1-x}$. Find the n-th iterate of f at x, and determine the set of values of x for which it makes sense.

Solution: ► *We have*
$$f^{[2]}(x) = (f\circ f)(x) = f(f(x)) = \frac{1}{1-\frac{1}{1-x}} = \frac{x-1}{x},$$
$$f^{[3]}(x) = (f\circ f\circ f)(x) = f(f^{[2]}(x)) = f\left(\frac{x-1}{x}\right) = \frac{1}{1-\frac{x-1}{x}} = x.$$
Notice now that $f^{[4]}(x) = (f\circ f^{[3]})(x) = f(f^{[3]}(x)) = f(x) = f^{[1]}(x)$. *We see that* f *is cyclic of period* 3, *that is,*
$$f^{[1]}(x) = f^{[4]}(x) = f^{[7]}(x) = \ldots = \frac{1}{1-x},$$
$$f^{[2]}(x) = f^{[5]}(x) = f^{[8]}(x) = \ldots = \frac{x-1}{x},$$
$$f^{[3]}(x) = f^{[6]}(x) = f^{[9]}(x) = \ldots = x.$$
The formulæ above hold for $x \not\in \{0,1\}$. ◄

If there are functions ϕ and g for which
$$f \circ \phi = \phi \circ g, \tag{1.15}$$
then $f = \phi \circ g \circ \phi^{-1}$. If the iterates of g are easy to find, then[1]
$$f^{[n]} = \phi \circ g^{[n]} \circ \phi^{-1}, \tag{1.16}$$
provides the nth iterate of f.

102 Example Let $f(x) = 2x^2 - 1$. Find $f^{[n]}(x)$.

Solution: ► *Observe that since* $2\cos^2 y - 1 = \cos 2y$, *we may take* $\phi(x) = \cos x$ *and* $g(x) = 2x$ *in (1.15). Since* $g^{[n]}(x) = 2^n x$, *by virtue of (1.16),*
$$f^{[n]}(x) = \cos(2^n \arccos x).$$
This formula is valid for $|x| \le 1$. ◄

103 Example Let $f(x) = 4x(1-x)$. Find $f^{[n]}(x)$.

Solution: ► *Observe that since*
$$4\sin^2 y - 4\sin^4 y = 4\sin^2 y(1-\sin^2 y) = (2\sin y \cos y)^2 = \sin^2 2y,$$
we may take $\phi(x) = \sin^2 x$ *and* $g(x) = 2x$ *in (1.15). Since* $g^{[n]}(x) = 2^n x$, *by virtue of (1.16),*
$$f^{[n]}(x) = \sin^2(2^n \arcsin \sqrt{x}).$$
This formula is valid for $0 \le x \le 1$. ◄

104 Definition Let $c_0, c_2, \ldots, c_k$ be real constants and $f : \mathbb{N} \to \mathbb{R}$ a function. A recurrence relation of the form
$$c_0 a_n + c_1 a_{n+1} + c_2 a_{n+2} + \cdots + + c_k a_{n+k} = f(n), \qquad n \ge 0.$$
is called a *linear difference equation*. If f is identically zero, we say that the equation is *homogeneous*.

We begin by examining some simple recursions of first order.

105 Example Let $x_0 = 7$ and $x_n = 2x_{n-1}, n \ge 1.$ Find a closed form for x_n.

Solution: ► *We have*
$$\begin{array}{rcl} x_0 & = & 7 \\ x_1 & = & 2x_0 \\ x_2 & = & 2x_1 \\ x_3 & = & 2x_2 \\ \vdots & \vdots & \vdots \\ x_n & = & 2x_{n-1} \end{array}$$
Multiplying both columns,
$$x_0 x_1 \cdots x_n = 7 \cdot 2^n x_0 x_1 x_2 \cdots x_{n-1}.$$

[1]The reader who has seen Linear Algebra will recognise that this is the same idea involving powers of similar matrices.

Cancelling the common factors on both sides of the equality,

$$x_n = 7 \cdot 2^n.$$

◄

106 Example Let $x_0 = 7$ and $x_n = 2x_{n-1} + 1, n \geq 1.$ Find a closed form for x_n.

Solution: ► *We have:*

$$\begin{array}{lcl} x_0 & = & 7 \\ x_1 & = & 2x_0 + 1 \\ x_2 & = & 2x_1 + 1 \\ x_3 & = & 2x_2 + 1 \\ \vdots & \vdots & \vdots \\ x_{n-1} & = & 2x_{n-2} + 1 \\ x_n & = & 2x_{n-1} + 1 \end{array}$$

Multiply the kth row by 2^{n-k}. We obtain

$$\begin{array}{lcl} 2^n x_0 & = & 2^n \cdot 7 \\ 2^{n-1} x_1 & = & 2^n x_0 + 2^{n-1} \\ 2^{n-2} x_2 & = & 2^{n-1} x_1 + 2^{n-2} \\ 2^{n-3} x_3 & = & 2^{n-2} x_2 + 2^{n-3} \\ \vdots & \vdots & \vdots \\ 2^2 x_{n-2} & = & 2^3 x_{n-3} + 2^2 \\ 2x_{n-1} & = & 2^2 x_{n-2} + 2 \\ x_n & = & 2x_{n-1} + 1 \end{array}$$

Adding both columns, cancelling, and adding the geometric sum,

$$x_n = 7 \cdot 2^n + (1 + 2 + 2^2 + \cdots + 2^{n-1}) = 7 \cdot 2^n + 2^n - 1 = 2^{n+3} - 1.$$

Aliter: *Let $u_n = x_n + 1 = 2x_{n-1} + 2 = 2(x_{n-1} + 1) = 2u_{n-1}$. We solve the recursion $u_n = 2u_{n-1}$ as we did example 105: $u_n = 2^n u_0 = 2^n(x_0 + 1) = 2^n \cdot 8 = 2^{n+3}$. Finally, $x_n = u_n - 1 = 2^{n+3} - 1$.* ◄

107 Example (Oval's on the Plane) Let there be drawn n ovals on the plane. If an oval intersects each of the other ovals at exactly two points and no three ovals intersect at the same point, find a recurrence relation for the number of regions into which the plane is divided.

Solution: ► *Let this number be* a_n. *Plainly* $a_1 = 2$. *After the* $n-1$*th stage, the* n*th oval intersects the previous ovals at* $2(n-1)$ *points, i.e. the* n*th oval is divided into* $2(n-1)$ *arcs. This adds* $2(n-1)$ *regions to the* a_{n-1} *previously existing. Thus*

$$a_n = a_{n-1} + 2(n-1), \; a_1 = 2.$$

This is a non-homogeneous linear recurrence. To obtain a closed form, write

$$\begin{array}{lcl} a_2 & = & a_1 + 2(1), \\ a_3 & = & a_2 + 2(2), \\ a_4 & = & a_3 + 2(3), \\ \vdots & \vdots & \vdots \\ a_{n-1} & = & a_{n-2} + 2(n-2), \\ a_n & = & a_{n-1} + 2(n-1), \end{array}$$

Add these equalities and cancel common terms on the left and right,

$$a_2+a_3+a_4+\cdots+a_{n-1}+a_n = a_1+a_2+a_3+a_4+\cdots+a_{n-1}+2(1+2+\cdots+(n-1)) \implies a_n = a_1+(n-1)n = n^2-n+2,$$

upon using Corollary 3.

A Maple sequence for solving this recurrence is

```
> rsolve({a(k)=a(k-1)+2*(k-1), a(1)=2}, a(k));
```

◄

Suppose that $a_n = ar^n$, $r \neq 0$, is a solution to the homogeneous differential equation

$$c_0 a_n + c_1 a_{n+1} + c_2 a_{n+2} + \cdots + + c_k a_{n+k} = 0.$$

Then

$$c_0 r^n + c_1 r^{n+1} + c_2 r^{n+2} + \cdots + + c_k r^{n+k} = 0 \implies r^n(c_0 + c_1 r + c_2 r^2 + \cdots + + c_k r^k) = 0 \implies c_0 + c_1 r + c_2 r^2 + \cdots + + c_k r^k = 0.$$

The equation

$$c_0 + c_1 r + c_2 r^2 + \cdots + + c_k r^k = 0$$

is called the *characteristic equation* of the difference equation.[2] Clearly if bs^n, $s \neq r$, is a solution, then $ar^n + bs^n$ is also a solution. This is the so-called *superposition principle.*

We will not discuss here a general theory of how to solve difference equations, we will only focus on some examples that will be used later on. The interested reader may read [Boo] for the more general case. Let us, however, discuss the case of the second order linear homogeneous difference equation

$$c_0 x^n + c_1 x^{n+1} + c_2 x^{n+2} = 0.$$

The characteristic equation is a quadratic equation, say

$$p(x) := c_0 + c_1 x + c_2 x^2 = 0.$$

This equation has two roots r, s, and so

$$c_0 + c_1 x + c_2 x^2 = c_2(x-r)(x-s).$$

[2] In olden days these used to be called the *secular equation*.

If $r \neq s$, then by the superposition principle we have seen that $a_n = ar^n + bs^n$ for some constants a,b. What happens if $r = s$? In this case r is a double root and

$$p(x) = c_2(x-r)^2.$$

and also, $p'(x) = c_1 + 2c_2x = 2c_2(x-r)$ Since $p'(r) = 0$, we must have $c_1 + 2c_2r = 0$. Now, let us try nr^n as another solution to the difference equation. Then

$$c_0nr^n + c_1(n+1)r^{n+1} + c_2(n+2)r^{n+2} = 0 \implies nr^n(c_0 + c_1r + c_2r^2) + r^{n+1}(c_1 + 2c_2r) = nr^n0 + r^{n+1}0 = 0,$$

whence nr^n is also a solution.

108 Example (Fibonacci Numbers) The *Fibonacci sequence* is given by $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, $f_5 = 5$, and in general,

$$f_{n+1} = f_n + f_{n-1}, \qquad n \geq 1.$$

Find a closed formula for f_n.

Solution: ► *Suppose ar^n, $r \neq 0$, is a solution, then*

$$ar^{n+1} = ar^n + ar^{n-1} \implies ar^{n-1}(r^2 - r - 1) = 0 \implies r = \frac{1 \pm \sqrt{5}}{2}.$$

This means that

$$f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

for some constants A and B that we must determine. Now

$$f_0 = 0 \implies 0 = A + B, \qquad f_1 = 1 \implies 1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right).$$

Solving for A and B we find $A = \frac{1}{\sqrt{5}} = -B$. Hence

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

This closed form is called the Cauchy-Binet Formula. *To obtain the first 100 Fibonacci numbers using Maple™ use the following commands. Notice that the double bars indicate Maple that it is dealing with a sequence.*

```
> f||0:=0; f||1:=1;
> for n from 2 to 100 do f||n:=f||(n-1)+f||(n-2); od;
```

Maple also has a command `rsolve`*, that solves recursions. Let us use it to obtain the Cauchy-Binet formula. We have to change slightly our notation because Maple reads $f||n$ differently from, say, $f(n)$.*

```
> rsolve({f(k)=f(k-1)+f(k-2), f(0)=0, f(1)=1}, f(n));
```

The answer that Maple displays appears to be different than the one we obtained. Prove, by rationalising the denominator of $\frac{1}{1 \pm \sqrt{5}}$, that they are in fact equal. ◄

109 Example Find a closed formula for the recursion $a_{n+2} = a_{n+1} + 6a_n$, $a_0 = 3$ and $a_1 = 1$.

Solution: ► *Suppose ar^n, $r \neq 0$, is a solution, then*

$$ar^{n+2} = ar^{n+1} + 6ar^n \implies ar^n(r^2 - r - 6) = 0 \implies r \in \{-3, 2\}.$$

Thus the solution must be of the form

$$a_n = A(-3)^n + B2^n.$$

Using the initial conditions,

$$3 = a_0 = A + B, \qquad 1 = a_1 = -3A + 2B \implies A = 1, B = 2.$$

Thus

$$a_n = (-3)^n + 2^{n+1}.$$

◄

110 Example Find a closed form for the recursion $a_{n+2} = 6a_{n+1} - 9a_n$, $a_0 = 2$ and $a_1 = 15$.

Solution: ► *Suppose* ar^n, $r \neq 0$, *is a solution, then*

$$ar^{n+2} = 6ar^{n+1} - 9ar^n \implies ar^n(r^2 - 6r + 9) = 0 \implies r = 3,$$

a repeated root. Thus the solution must be of the form

$$a_n = A3^n + Bn3^n.$$

Using the initial conditions,

$$2 = a_0 = A, \qquad 15 = a_1 = 3A + 3B \implies A = 2, B = 3.$$

Thus

$$a_n = 2 \cdot 3^n + n3^{n+1}.$$

To obtain the first 100 *terms of this sequence and to obtain a closed form for it use the Maple™ commands*

```
>  a||0:=2; a||1:=15;
>  for n from 2 to 100 do a||n:=6*s||(n-1)-9*a||(n-2); od;
>  rsolve({a(k)=6*a(k-1)-9*a(k-2), a(0)=2, a(1)=15}, a(n));
```

◄

111 Example Find the recurrence relation for the number of n digit binary sequences with no pair of consecutive 1's.

Solution: ► *It is quite easy to see that* $a_1 = 2, a_2 = 3$. *To form* $a_n, n \geq 3$, *we condition on the last digit. If it is 0, the number of sequences sought is* a_{n-1}. *If it is 1, the penultimate digit must be 0, and the number of sequences sought is* a_{n-2}. *Thus*

$$a_n = a_{n-1} + a_{n-2}, \qquad a_1 = 2, \; a_2 = 3.$$

This recurrence looks like the Fibonacci recurrence. It is called the Lucas sequence. *We leave to the reader to prove that its closed form is*

$$a_n = \left(\frac{1+2\sqrt{5}}{5}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-2\sqrt{5}}{5}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

◄

112 Example Let $a_{n+2} - 2a_{n+1} + 2a_n = 0$ with $a_0 = 1$ and $a_1 = 1$. Find a close form for this recursion.

Solution: ► *The characteristic equation is* $r^2 - 2r + 2 = 0 \implies r \in \{1 - i, 1 + i\}$. *Hence*

$$a_n = A(1-i)^n + B(1+i)^n.$$

Using the polar forms

$$1-i=\sqrt{2}e^{-\pi i/4}, \qquad 1+i=\sqrt{2}e^{\pi i/4},$$

we may write

$$a_n = C2^{n/2}\cos\frac{\pi n}{4} + D2^{n/2}\sin\frac{\pi n}{4}.$$

Now,

$$a_0 = 1 \implies 1 = C.$$

Also,

$$a_1 = 1 \implies 1 = C2^{1/2}\cos\frac{\pi}{4} + D2^{1/2}\sin\frac{\pi}{4} = C + D \implies D = 0.$$

The general solution is thus $a_n = 2^{n/2}\cos\frac{\pi n}{4}$. ◂

Homework

113 Exercise Let $f(x) = x^2 - 2$. Use the fact that $\left(x+\frac{1}{x}\right)^2 - 2 = x^2 + \frac{1}{x^2}$ to prove that

$$f^{[n]}(x) = \left(\frac{x+\sqrt{x^2-4}}{2}\right)^{2^n} + \left(\frac{x-\sqrt{x^2-4}}{2}\right)^{2^n}$$

for $|x| \geq 2$.

114 Exercise (Lines on the Plane) Find a recurrence relation for the number of regions into which the plane is divided by n straight lines if every pair of lines intersect, but no three lines intersect.

115 Exercise Solve the recursion $a_n = 1 + \sum_{k=1}^{n-1} a_k$ for $n \geq 2$ and $a_1 = 1$.

116 Exercise Let $x_0 = 1, x_n = 3x_{n-1} - 2n^2 + 6n - 3$. Find a closed form for this recursion.

117 Exercise Find a closed form for $x_n = 2x_{n-1} + 3^{n-1}, x_0 = 2$.

118 Exercise Solve the recursion $a_n = 2a_{n/2} + 6n - 1$ for $n \geq 2$, n a power of 2, and $a_1 = 1$.

119 Exercise Let $x_0 = 2, x_n = 9x_{n-1} - 56n + 63$. Find a closed form for this recursion.

120 Exercise Let $x_0 = 7$ and $x_n = x_{n-1} + n, n \geq 1$. Find a closed formula for x_n.

121 Exercise Solve the recursion $a_n = 2a_{n-1} + n - 1$ for $n \geq 2$ and $a_1 = 1$.

122 Exercise (Putnam 1985) Let d be a real number. For each integer $m \geq 0$, define a sequence $a_m(j), j = 0, 1, 2, \cdots$ by $a_m(0) = \frac{d}{2^m}$, and $a_m(j+1) = (a_m(j+1))^2 + 2a_m(j), j \geq 0$. Evaluate

$$\lim_{n\to\infty} a_n(n).$$

123 Exercise A recursion satisfies $u_0 = 3, u_{n+1}^2 = u_n, n \geq 1$. Find a closed form for this recursion.

124 Exercise There are two urns, one is full of water and the other is empty. On the first stage, half of the contains of urn I is passed into urn II. On the second stage 1/3 of the contains of urn II is passed into urn I. On stage three, 1/4 of the contains of urn I is passed into urn II. On stage four 1/5 of the contains of urn II is passed into urn I, and so on. What fraction of water remains in urn I after the 1978th stage?

125 Exercise (Towers of Hanoi) The French mathematician Edouard Lucas furnished, in 1883, the toy seen in figure 1.6 (with eight disks), along with the following legend. The tower of Brahma had 64 disks of gold resting on three diamond needles. At the beginning of time, God placed these disks on the first needle and ordained that a group of priests should transfer them to the third needle according to the following rules:

1. The disks are initially stacked on peg A, in decreasing order (from bottom to top).
2. The disks must be moved to another peg in such a way that only one disk is moved at a time and without stacking a larger disk onto a smaller disk.

When they finish, the Tower will crumble and the world will end. Prove that if there are n disks, then

$2^n - 1$ are necessary and sufficient to perform the task according to the rules.

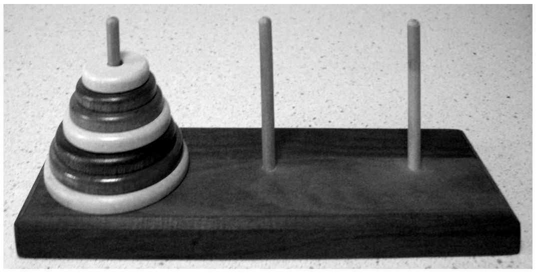

Figure 1.6: Towers of Hanoi.

126 Exercise (Josephus' Problem) In [HeKa] we find the following legend about the famous first-century Jewish historian Flavius Josephus:

> In the Jewish revolt against Rome, Josephus and 39 of his comrades were holding out against the Romans in a cave. With defeat imminent, they resolved that, like the rebels at Masada, they would rather die than be slaves to the Romans. They decided to arrange themselves in a circle. One man was designated as number one, and they proceeded clockwise killing every seventh man.... Josephus (according to the story) was among other things an accomplished mathematician; so he instantly figured out where he ought to sit in order to be the last to go. But when the time came, instead of killing himself he joined the Roman side.

In general, given a group of n men arranged in a circle under the edict that every mth man will be executed going around the circle until only one remains, the object is to find the position $L(n,m)$ in which you should stand in order to be the last survivor. The particular situation of Flavius Josephus is asking for $L(40,7)$. The general Josephus' Problem is very difficult. Prove, however, that $L(n,2) = 1 + 2n - 2^{1+\lfloor \log_2 n \rfloor}$.

127 Exercise (Monkeys and Coconuts) N men and M monkeys gather coconuts all day and then they fall asleep. The first man wakes up, separates p coconuts for each monkey, and then takes $\frac{1}{N}$ of what remains for himself and goes back to sleep. The second man wakes up, separates p coconuts for each monkey, and then takes $\frac{1}{N}$ of what remains for himself and goes back to sleep, etc. until the Nth man wakes up and does the same. In the morning everyone wakes up, and the men give p coconuts to every monkey and $\frac{1}{N}$ of what remains for themselves. Given that each division was an integer division, find the least amount of coconuts needed.

128 Exercise (Derangements) An absent-minded secretary is filling n envelopes with n letters. Find a recursion for the number D_n of ways in which she never stuffs the right letter into the right envelope.

2 Some Maple™ Programming

In this chapter we will introduce some algorithmic constructs: looping, conditional expressions, etc. An *algorithm* is a set of vividly clear instructions that must be executed in order to perform a well defined task. We will avail from the software Maple™ in order to illustrate these points. Maple™ is easy to use, and its basic syntax does not differ much from other programming languages like Pascal, C, or Java.

Our object here will be to study the logic of writing small programs. The topic of safeguarding our program against errors in inputs, and of proving our algorithms correct, although important topics in computer programming, will only distract us from our main goals, and hence we will not touched it here. Most the algorithms here will be numeric, it would be a rare occurrence if we treat non-numeric algorithms.

Programming is a difficult subject for a beginner and it requires practice and attention to detail. Most of the exercises at the end of the section are solved. I urge you to attempt them without looking at my solution. You should run each line through Maple. Since these notes were hastily put together, the writing is somewhat cryptic.

2.1 Basic Operations

Although we now have versions past Maple IX, we will use the programming constructs of Maple IX in our discussion. Our interest is to learn basic procedural programming and albeit the basic WYSIWYG constructs are easier for the the oligophrenic, we will not make use of them here.

Maple uses + for addition, − for subtraction, ∧ (circumflex accent) for exponentiation, **/** for division, **!** for the factorial. The usual algebraic precedence of operators (parentheses over exponents, over multiplication and division, over addition and subtraction) is respected. Instructions are typed after the [> prompt, and must always be ended with a semicolon, after which you must press the $\boxed{\text{ENTER}}$ key. Whitespace is ignored between characters. If a colon is used instead of a semicolon, the command is executed silently, that is, Maple does not make visible the output. To obtain a decimal approximation, either put a decimal point anywhere in the expression, or use the command `evalf()` (evaluate to floating point).

129 Example Compute $\dfrac{1^1+2^2+3^3}{(4!+5\cdot 6\cdot 7\cdot 8)^9}$. using Maple.

Solution: ► *The required command line is*

```
> (1^1+2^2+3^3)/(4!+ 5*6*7*8)^9;
```

$$\frac{1}{3785091090811379105075822592}$$

```
> evalf((1^1+2^2+3^3)/(4!+ 5*6*7*8)^9);
```

$$.2641944344\ 10^{-27}$$

```
> (1^1+2^2+3^3)/(4!+ 5.*6*7*8)^9;
```

$$.2641944344\ 10^{-27}$$

◄

The power of Maple rests on its ability to perform symbolic computations in a straightforward manner. To operate with complex numbers, use the imaginary unit I (capitalised). Maple is able to evaluate a large list of common functions, among them `sin()`, `cos()`, `tan()`, `log()`, `log[n]()` `exp()`, `max()`, `min()`, `sqrt()`, `abs()`, `floor()`, `ceil()`. Enter π as `Pi`.

130 Example Evaluate the following using Maple.

$$\sin\frac{\pi}{3}+\tan\frac{\pi}{6}, \quad (1+i)^{20}+(1-\sqrt{2})^{20}, \quad \max(\lfloor 5.6\rfloor, \log 100).$$

Solution: ► *The required command line is*

```
> (sin(Pi/3)+tan(Pi/6);
```

$$\frac{5}{6}\sqrt{3}$$

```
> (1+I)^20+(1-sqrt(2))^20;
```

$$-1024+\left(1-\sqrt{2}\right)^{20}$$

```
> (max(floor(5.6),log(100));
```

$$5$$

◄

Maple has several libraries that have tailor-made commands for Linear Algebra, Calculus, Plotting, Graph Theory, Number Theory, etc. Some combinatorial and number theoretic functions of use are the following:

1. `binomial(n,k)` computes the binomial coefficient $\binom{n}{k}$
2. `gcd(a,b)` finds the greatest common divisor of the integers a and b.
3. `lcm(a,b)` finds the least common multiple of the integers a and b.
4. `isprime(x)` determines whether the integer x is prime.
5. `ithprime(k)` gives the prime the k-th position, where $p_1=2$ is the first prime, $p_2=3$ is the second prime, etc.
6. `nextprime(x)` finds the prime just above the integer x.
7. `ifactor(x)` gives the prime factorisation of the integer x.
8. `iquo(a, b)` finds the integral quotient when the integer a is divided by the integer b.
9. `irem(a, b)` finds the integral remainder when the integer a is divided by the integer b.
10. `a mod b` finds the integer a modulo the integer b.

131 Example Use Maple to find $\gcd\left(\binom{20}{10},\binom{20}{15}\right)$.

Solution: ► *The required command line is*

```
> gcd(binomial(20,10), binomial(20,15));
```

$$1292$$

◄

132 Example Use Maple to determine whether 60637[1] is prime. Find the prime just above 60637.

Solution: ► *The required command line is*

```
> isprime(60637);
```

$$true$$

```
> nextprime(60637);
```

[1] A well-known zip code...

60647

◄

Maple is able to operate symbolically. To multiply out an algebraic expression, use `expand()`. To simplify an expression, use `simplify()`. This last command is rather limited and sometimes one needs to refine it, perhaps with the the `convert()` command. The `is` command determines whether two formulæ (involving numbers) are equal. To factor an expression use the command `factor()`.

133 Example Multiply out $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.

Solution: ► *The required command line is*

```
> expand((a+b+c)*(a^2+b^2+c^2-a*b-b*c-c*a));
```

$$a^3+b^3+c^3-3abc$$

◄

134 Example Factor $x^{10}-x^8-2x^7-x^6-x^4+x^2+2x+1$ using Maple.

Solution: ► *The required command line is*

```
> factor(x^10-x^8-2*x^7-x^6-x^4+x^2+2*x+1);
```

$$(x-1)(x+1)(x^2-x+1)(x^2-x-1)(x^2+x+1)^2$$

◄

135 Example Obtain the partial fraction expansion of $\frac{x}{x^3+1}$ using Maple.

Solution: ► *The required command line is*

```
> convert(x/(x^3-1), parfrac,x);
```

$$\frac{1}{3}\cdot\frac{1}{x-1}-\frac{1}{3}\cdot\frac{x-1}{x^2+x+1}$$

◄

136 Example Use Maple to find the exact value of $\cos\frac{\pi}{30}$.

Solution: ► *The required command line is*

```
> convert(cos(Pi/30), radical);
```

$$\frac{1}{8}\sqrt{2}\sqrt{5-\sqrt{5}}+\left(\frac{1}{8}+\frac{1}{8}\sqrt{5}\right)\sqrt{3}$$

◄

137 Example Reduce the fraction $\frac{x-1}{x^4-1}$.

Solution: ► *The required command line is*

```
> simplify((x-1)/(x^4-1));
```

$$\frac{1}{x^3+x^2+x+1}$$

◄

Maple is able to differentiate and integrate functions symbolically with the commands `diff()` and `int()`. It is also able to add or multiply numbers in sequence with the commands `sum()` and `product()`.

138 Example Find a closed formula for the sum

$$\sum_{1\le k\le n} k^2 = 1^2+2^2+\cdots+n^2.$$

.

Solution: ► *The command line appears below. So that the formula appears in a familiar shape, we factor the result.*

```
> factor(sum(k^2,k=1..n));
```

$$\frac{1}{6}n(n+1)(2n+1)$$

◄

139 Example Find the prime factorisation of the product

$$\prod_{1\le k\le 50}(2k-1) = (1)(3)(5)\cdots(99).$$

.

Solution: ► *The command line appears below.*

```
> ifactor(product(2*k-1,k=1..50));
```

$$(3)^{26}(5)^{12}(7)^8(11)^5(13)^4(17)^3(19)^3(23)^2(29)^2(31)^2(37)(41)(43)(47)(53)(59)(61)(67)(71)(73)(79)(83)(89)(97)$$

◄

140 Example Find the derivative and the integral of the function $x\mapsto \frac{x}{x^3+1}$ with respect to x. Also, find the definite integral $\int_{-1/2}^{1/2}\frac{x\mathrm{d}x}{x^3+1}$.

Solution: ► *The command lines appear below.*

```
> diff(x/(x^3+1),x);
```

$$\frac{1}{x^3+1}-\frac{3x^3}{(x^3+1)^2}$$

```
> int(x/(x^3+1),x);
```

$$-\frac{1}{3}\log(x+1)+\frac{1}{6}\log(x^2-x+1)+\frac{1}{3}\sqrt{3}\arctan\left(\frac{1}{3}(2x-1)\sqrt{3}\right)$$

```
> int(x/(x^3+1),x=-1/2..1/2);
```

$$-\frac{1}{6}\log 3-\frac{1}{6}\log 7+\frac{1}{3}\sqrt{3}\arctan\left(\frac{2}{3}\sqrt{3}\right)$$

◄

Homework

141 Exercise Find

$$(123456789)^2-(123456787)\cdot(123456791)$$

using Maple.

142 Exercise Use Maple to verify that for any integers a and b it holds that $\gcd(a,b)\cdot\operatorname{lcm}(a,b)=ab$.

143 Exercise Compute

$$\frac{(10^4+324)(22^4+324)(34^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)}$$

using Maple. Then do this by hand.

144 Exercise Find $\int \frac{\mathrm{d}x}{\sqrt{1+\sqrt{1+\sqrt{x}}}}$ both by hand and using Maple.

145 Exercise Evaluate the definite integral $\int_{-1}^{2} \max(|x-1|, x^2+2)\,dx$.

146 Exercise Find $\int \sqrt{\tan x}\; dx$ both by hand and using Maple.

147 Exercise Compute

$$\frac{(1+i)^{2004}}{(1-i)^{2000}}$$

using Maple.

148 Exercise Factor 1002004008016032 using Maple.

149 Exercise Use Maple to verify that

$$(x+y)^5 - x^5 - y^5 = 5xy(x+y)(x^2+xy+y^2)$$

and

$$(x+a)^7 - x^7 - a^7 = 7xa(x+a)(x^2+xa+a^2)^2.$$

150 Exercise Write Maple code to verify that a product of sums of squares can be written as a sum of squares, that is, verify that

$$(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2.$$

151 Exercise Let $i^2 = -1$. Evaluate

$$1 + 2i + 3i^2 + 4i^3 + 5i^4 + \cdots + 2007i^{2006}$$

using Maple.

152 Exercise Give Maple code to compute $\sum_{k=1}^{1000} \lfloor \log_2 k \rfloor$.

153 Exercise Find the exact value of $\cos \frac{\pi}{5}$ using Maple and by hand.

2.2 Sets, Lists, and Arrays

Maple has a rich variety of data structures, among them sets, lists, and arrays. Roughly speaking, a *set* corresponds to a set in combinatorics: the order of the elements is irrelevant, and repetitions are not taken into account. Sets are defined by using curly braces { }. In a *list*, the order of the elements is important and repetitions are taken into account. Lists are defined by using square brackets [ ]. Arrays are a generalisations of matrices. They can be modified and are declared with the command `array()`.

We will first consider sets and set operations. In order to facilitate our presentation, we will give names to the various objects we will define. In order to attach a name, we need the assignment operator `:=`, where there is no space between the colon and the equal sign. Maple is able to perform set operations with the commands `union`, `intersect`, and `minus`. To check whether two sets are equal we may use the command `evalb()` (evaluate boolean).

154 Example Consider the sets

$$A = \{1,2,3,a,b,c,d\}, \quad B = \{3,4,5,a,b,e,f\}.$$

Use Maple to obtain

$$A \cup B, \quad A \cap B, \quad A \setminus B,$$

and to verify that

$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

Solution: ► *We first define the sets and then perform the desired operations. The following command lines accomplish what is required.*

```
> A:={1,2,3,a,b,c,d};
```

$$A := \{1,2,3,a,b,c,d\}$$

```
> B:={3,4,5,a,b,e,f};
```

$$B := \{3,4,5,a,b,e,f\}$$

```
> A union B
```

$$\{1,2,3,4,5,f,a,b,c,d,e\}$$

```
> A intersect B;
```

$$\{3, a, b\}$$

```
> A minus B;
```

$$\{1, 2, c, d\}$$

```
> evalb((A union B) minus (A intersect B)=(A minus B) union (B minus A));
```

$$true$$

◀

We do not need to write code *in extenso* in order to define a set whose elements are in sequence, as we may use the the function `seq()`.

155 Example To define the set

$$X := \{1, 2, \ldots, 100\}$$

we type

```
> X:={seq(k,k=1..100)};
```

$$X := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10,$$

$$11, 12, 13, 14, 15, 16, 17, 18, 19, 20,$$

$$21, 22, 23, 24, 25, 26, 27, 28, 29, 30,$$

$$31, 32, 33, 34, 35, 36, 37, 38, 39, 40,$$

$$41, 42, 43, 44, 45, 46, 47, 48, 49, 50,$$

$$51, 52, 53, 54, 55, 56, 57, 58, 59, 60,$$

$$61, 62, 63, 64, 65, 66, 67, 68, 69, 70,$$

$$71, 72, 73, 74, 75, 76, 77, 78, 79, 80,$$

$$81, 82, 83, 84, 85, 86, 87, 88, 89, 90,$$

$$91, 92, 93, 94, 95, 96, 97, 98, 99, 100\}$$

A list is more or less like a set, except that repetitions are allowed and the order of the elements is respected. The function `nops()` gives the number of elements of the list. To obtain the i-th element of the list, we type `name[i]`, enclosed in square brackets. We may also access an element of the list by using the function `op(i, name)`. We may also obtain elements in a range using these operators, for example `name[low..high]` or, equivalently, `op(low..high,name)`. The command `subop(index1=newvalue1,index2=newvalue2,...,name)`.

156 Example In this problem we perform various operations with lists.

1. Create the list $L1$ consisting of the elements 4, 4, 5, 5, 2, 3, 2 in that order.
2. Create the list $L2$ consisting of the elements a, b, b, c, c, a, d in that order.
3. Concatenate $L1$ and $L2$ into a list $L3$.
4. Create a list $L4$ consisting of the first three elements of $L1$ and the last three elements of $L2$.
5. Delete the first and last elements of $L1$ and substitute, respectively, with the values x and y. Shew that $L1$ has not changed.

Solution: ► *The required commands follow.*

```
> L1:=[4,4,5,5,2,3,2];
```

$$L1 := [4,4,5,5,2,3,2]$$

```
> L2:=[a,b,b,c,c,a,d];
```

$$L2 := [a,b,b,c,c,a,d]$$

```
> L3:=[op(L1),op(L2)];
```

$$L3 := [4,4,5,5,2,3,2,a,b,b,c,c,a,d]$$

```
> L4:=[op(1..3,L1),op(-3..-1,L2)];
```

$$L4 := [4,4,5,c,a,d]$$

```
> subsop(1=x,-1=y,L1);
```

$$[x,4,5,5,2,3,y]$$

```
> L1;
```

$$[4,4,5,5,2,3,2]$$

◄

Before discussing arrays let us mention in passing a curious feature of Maple. Given a set or a list ({...} or [...]), we can retrieve the members by appending [ ] at the end. Some examples follow.

```
> {2,3,4}[ ];
```

$$2,3,4$$

```
> max({2,3,4}[ ]);
```

$$4$$

```
> [1,2,3,4,5][ ];
```

$$1,2,3,4,5$$

```
> min([1,2,3,4,5][ ]);
```

$$1$$

An *array* is a more general data structure than a list, in fact, arrays are generalisations of the matrix concept. Arrays are modifiable. Arrays are defined as `array(index_function, ranges, initial_value_lists)` All these parameters are optional. The *dimension* of an array is the number of indices used to describe it. A one-dimensional array is akin of a vector, and a two dimensional array is akin of a matrix. Thus the one-dimensional array x with n elements essentially looks like

$$x := [x[1],x[2],\ldots,x[n]]$$

and a two-dimensional array A with mn elements (with m rows and n-columns), essentially looks like

$$A := \begin{bmatrix} A[1,1] & A[1,2] & A[1,3] & \cdots & A[1,n] \\ A[2,1] & A[2,2] & A[2,3] & \cdots & A[2,n] \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A[m,1] & A[m,2] & A[m,3] & \cdots & A[m,n] \end{bmatrix}$$

In order to list the elements of an array we must use the command `eval()`.

157 Example Define the one-dimensional array $V := [5,6,7,8,9]$. Then, change its third element to be x.

Solution: ► *The required code follows.*

```
> V:=array([5,6,7,8,9]);
```

$$V := [5,6,7,8,9]$$

```
> eval(V);
```

$$[5,6,7,8,9]$$

```
> V[3]:=x;
```

$$V_3 := x$$

```
> eval(V);
```

$$[5,6,x,8,9]$$

◄

158 Example Define a 2×3 array M with $M_{11}=1$, $M_{12}=2$, $M_{13}=3$, $M_{21}=a$, $M_{22}=b$, and $M_{23}=c$. Then, redefine M_{22} to be x. Also, define an uninitialised 3×2 array N.

Solution: ►

The required code follows.

```
> M:=array([ [1,2,3],[a,b,c] ]);
```

$$M := \begin{bmatrix} 1 & 2 & 3 \\ a & b & c \end{bmatrix}$$

```
> M[2,2]:=x;
```

$$M_{2,2} := x$$

```
> eval(M);
```

$$\begin{bmatrix} 1 & 2 & 3 \\ a & x & c \end{bmatrix}$$

```
> N:=array(1..3,1..2);
```

$$N := \boldsymbol{array}(1..3, 1..2, [])$$

```
> eval(N);
```

$$\begin{bmatrix} ?_{11} & ?_{12} \\ ?_{21} & ?_{22} \\ ?_{31} & ?_{32} \end{bmatrix}$$

◄

Homework

159 Exercise Consider the set of 100 integers $X := \{1,2,\ldots,100\}$. Using Maple set operations, and Inclusion-Exclusion, find the number of elements of X which are neither multiples of 2 nor 3. Also, list all such elements.

160 Exercise Consider the set of 1000 integers $X := \{1,2,\ldots,1000\}$. Using Maple set operations, and Inclusion-Exclusion, find the number of elements of X which are neither perfect squares, nor perfect cubes, nor perfect fifth-powers. Also, list all such elements.

161 Exercise Write Maple code that will compute the sum of the elements of a list of numbers.

2.3 Functions and Procedures

Recall that in mathematics we call a *function* the collection of the following ingredients:

1. A set of inputs, called the *domain* of the function.
2. A set of *possible* outputs called the *target set* of the function.
3. A name for the function.
4. A name for a typical input (input parameter) of the function.
5. An *assignment rule* that assigns to every input a unique output.

The collection of all the above ingredients is summarised in the notation

$$f: \begin{array}{ccc} A & \to & B \\ x & \mapsto & f(x) \end{array}$$

where f is the name of the function, A is the domain, B is the target set, x is the typical input, and $f(x)$

Such definition of a function is especially useful in Computer Science. For example, if we had a function $f: \mathbb{Z}^3 \to \mathbb{R}$, we would write this in C code has `float f(int, int, int){instructions}`, where *float* refers to a floating (decimal) real number, and *int* refers to integer. This allocates sufficient space in the memory to handle the inputs and outputs. Since memory is limited, we need to know before hand how much of it to allocate. In most of your Precalculus and Calculus courses you have been misinformed when calling functions simply by their name and the assignment rule. This is unfortunate, because say talking of the "function" f with $f(x) = 3x + 1$ does not tell you anything about its domain and hence nothing about its image. It is also unfortunate because you cannot tell whether the given function is injective, surjective, etc., simply by its assignment rule. On the other hand, this simplifies matters when defining functions in Maple, we will simply have to be *very careful* that we input the right kind of inputs in our functions. Assigning the wrong kind of input to a function where no provisions have been done for type-checking can lead us to infinite loops or program crashes. Since the programs we will write here are very simple, we will not engage in this kind of safeguarding, but again, we insist that much care must be taken by the serious programmer to guard against possible confusion and wrong inputs by the user.

There are at least two ways of defining functions in Maple. The ways we will explore are not completely equivalent and one has advantages over the other, but we will not fuss with this now.

The first way we will explore is through the arrow notation `->`, which is obtained by a dash and a greater than sign, with no spaces in between. This is reminiscent with the way functions are defined in Precalculus and Calculus. To name the function f with $f(x) = 3x + 1$ you type `f:=x->3*x+1`. The sum, difference, product and quotient of functions is obtained in the expected way. To obtain $f \circ g$ we type `f@g`. To obtain the output in a specific set or list we use the command `map(x->f(x),X)`, where X is the name of the set or list.

162 Example Consider the assignment rules $f(x) = x^2 - x$ and $g(x) = 2x + 1$. Write Maple code

1. Defining both f and g.
2. Computing $(f + g - fg)(2)$.
3. Computing the set $f(A)$, where A is the set $A = \{-3, -2, -1, 0, 1, 2, 3\}$.
4. Computing the set $f(L)$, where L is the list $L = [-3, -2, -1, 0, 1, 2, 3]$.
5. Computing $(f \circ g)(2)$.
6. Computing $\underbrace{(f \circ f \circ \cdots \circ f)}_{20\ f'\text{s}}(3)$.

Solution: ► *The required commands appear below. Notice that since f is not injective on the set A, there are fewer elements in $f(A)$ than in A.*

```
> f:=x->x^2-x;
```
$$f := x \to x^2 - x$$
```
> g:=x->2*x+1;
```
$$g := x \to 2x + 1$$
```
> (f+g-f*g)(2);
```
$$-3$$
```
> A:={-3,-2,-1,0,1,2,3};
```
$$A := \{-3,-2,-1,0,1,2,3\}$$
```
> map(x->f(x), A);
```
$$\{0,2,6,12\}$$
```
> L:=[-3,-2,-1,0,1,2,3];
```
$$L := [-3,-2,-1,0,1,2,3];$$
```
> map(x->f(x), L);
```
$$[12,6,2,0,0,2,6]$$
```
> (f@g)(2);
```
$$20$$
```
> (f@20)(3);
```
$$380$$

◄

163 Example Write a function that takes a list of numbers as an input and outputs the average of the elements of the list. Test the function with the list $[-1,2,3,3,4]$.

Solution: ► *Here is a possible way.*

```
> AVERAGE:=X-> sum(X[i],i=1..nops(X))/nops(X);
```
$$AVERAGE := X \to \frac{\sum_{i=1}^{nops(X)} X_i}{nops(X)}$$
```
> AVERAGE([-1,2,3,3,4]);
```
$$\frac{11}{5}$$

◄

For our next example we need the command `coeffs(p,x)`. This returns the set of coefficients of the polynomial p in the variable x. For example, the call `coeffs(10*x^2-4*x+1,x)` returns `{10,-4,1}`.

164 Example The *height* of a polynomial $p(x)$ is the largest value of the absolute values of its coefficients. Create a function `HEIGHT` that finds the height of a given polynomial.

Solution: ► *Here is a possible answer. The idea is the following. The input is a polynomial p in the indeterminate x.* `map(abs, {coeffs(p,x)})` *creates a set with the absolute values of the coefficients of p, and appending* `[]` *at its end retrieves the numerical values which are then able to be fed to the* `max` *function.*

```
> HEIGHT:=(p,x)->max( map(abs, {coeffs(p,x)})[ ]);
```
$$HEIGHT := (p,x) \to max(map(abs,\{coeffs(p,x)\})[\,])$$

◄

Another way of defining function in Maple is by means of *procedure* declarations. We will actually prefer this method rather than the arrow method, since this method is akin to the ones used by most computer languages. To declare a procedure, we use the syntax `name:=proc(inputs) instructions  end;`.

For example, to declare the assignment rule $f(x) = 3x + 1$ we write `f:=proc(x) 3*x+1 end;`. A procedure usually returns its last evaluation, a behaviour which can be bypassed with the `RETURN` statement. The values of the input parameters cannot be changed by the procedure, and hence, if they need to be modified somehow one must first make copies of them.

165 Example Write a procedure `SWAP(x,y)` which takes two numbers x and y and exchanges them.

Solution: ► *This is a classic algorithm in introductory programming. A standard trick is to create a temporary variable, store the contents of x there, store the contents of y in x, and finally, store the contents of the temporary variable in y. Since we cannot change the values of x and y because they are input parameters, we make copies of these variables into $x1$ and $y1$.*

$$
\begin{aligned}
&> SWAP := proc(x, y)\\
&x1 := x; y1 := y;\\
&temp := x1; x1 := y1; y1 := temp;\\
&RETURN(x1, y1);\\
&end;
\end{aligned}
$$

◄

166 Example Write a Maple procedure $ITHDIGIT(x,i)$ that takes a positive integer x and gives its i-th digit when read from right to left.

Solution: ► *The idea is the following. Consider a positive integer, for example* 23456789785765*, and let us find its four digit from the left (it is obviously* 5*, but let's forget that for a minute. The trick is to move the decimal point four units left, obtaining* 2345678978.5765*. Now we delete the integral part, obtaining* $.5765$*. We now move the decimal point one unit to the right, obtaining* 5.765*. The digit we want is the integral part of this last number, namely,* 5*. Here is the code for that set of instructions.*

```
>  ITHDIGIT:= proc(x, i) b:=x*10^(-i); n:=b-floor(b); z:=10*n; RETURN(floor(z));   end;
```

We can write the code more succinctly as

```
>  ITHDIGIT:= proc(x, i) RETURN(floor(10*(x*10^(-i)-floor(x*10^(-i)))));   end;
```

but this makes it somewhat harder to read.

◄

For our next example we need to be able to find the number of digits used when writing a given positive integer x. This can be found using the Maple command `length(x)`.

167 Example Write a Maple procedure that will output the set of digits that appear in a positive integer.

Solution: ► *The idea is to use example 166 and find every digit. Since a set does not include repetitions, we shew our output in a set.*

$$
\begin{aligned}
&SETOFDIGITS := proc(x)\\
&\quad RETURN(\{seq(ITHDIGIT(x,i), i = 1..length(x))\});\\
&end;
\end{aligned}
$$

In order to use this procedure, we must type the code of `ITHDIGIT` *prior to it.*

◄

Homework

168 Exercise Let $A = \{1,2,3,4\}$, $B = \{3,4,5,6\}$, and $f(x) = \dfrac{x}{x^2+1}$. Write Maple code that will find $f(A \setminus B) \cup f(B \setminus A)$ and $f((A \setminus B) \cup (B \setminus A))$.

169 Exercise Given a list of data $[x_1, x_2, \ldots, x_n]$, their *variance* is given by $\dfrac{\sum_{k=1}^{n}(x-\mu)^2}{n}$, where μ is the average of the x_k. Write Maple code to compute the variance of a given list.

170 Exercise *Without introducing a temporary variable*, write a procedure `SWAP2(x,y)` that swaps the values of two variables. For example, if $x = 1$ and $y = 2$, then `SWAP2(x,y).` prints $x = 2$ and $y = 1$.

171 Exercise Using example 166 and the Maple functions `sum()` and `length()`, write a procedure `SUMDIGITS(x)` that computes the sum of the digits of a given positive integer x.

172 Exercise Using example 166, write a procedure `PEELER(x)` that will "peel out" the first and last digit of a positive integer with at least three digits. Leading zeroes are ignored. For example, `PEELER(1234)` will return 23 and `PEELER(1023014)` will return 23014 (ignoring the leading 0 obtained).

2.4 Conditional Statements and the "for" Loop

A *boolean expression* is one that evaluates either `true` or `false`. We can form boolean expressions with the relation operators

`=`	equal to
`<`	less than
`<=`	less than or equal to
`>`	greater than
`>=`	greater than or equal to
`<>`	not equal to

Do not confuse the assignment statement := *with the equality checking operator* =.

The standard logic rules hold for these operations.

The conditional statement in Maple takes various forms. The shortest is

```
if (condition) then (commands) fi;.
```

Other forms are

```
if (condition) then (commands) else (commands) fi;
```

and

```
if (condition) then elif (commands) elif (commands) ... else (commands) fi;.
```

173 Example Suppose you didn't know anything about Maple's maximum **max** function. Write Maple code that will find the maximum of two numbers.

Solution: ▶ *Here is one possible answer.*

```
> MAXI := proc(x,y)
    if x >= y
    then   RETURN(x);
    else   RETURN(y);
    fi;
  end;
```

◄

174 Example Write Maple code to evaluate the following piecewise assignment rule:

$$f(x) = \begin{cases} -2 & \text{if} \quad x \leq -3 \\ x^2 & \text{if} \quad -3 < x \leq 2 \\ 2 & \text{if} \quad 2 < x < 4 \\ 1 & \text{if} \quad x > 4 \end{cases}$$

Solution: ► *Here is one possible answer.*

```
> f := proc(x)
    if   x <= -3
      then   -2
        elif   x <= 2
          then   x^2
          elif   x < 4
            then   2
              else   1
    fi;
  end;
```

Maple has a direct way of declaring a piecewise function, by means of the command `piecewise()`*.*

```
>  f:=x->piecewise(x<=-3,-2,x<=2,x^2,x<4,2,1);
```

◄

We now investigate our first looping statement. The *for* loop has the following syntax, where the `by (step)` is optional.

```
for index from (low) by (step) to (high) do (instructions) od;.
```

"for" loops are particularly useful when all data in a certain range must be examined, as in checking the maximum of list of numbers, or adding numbers in a set.

175 Example Suppose you didn't know anything about Maple's `sum` command. Write a Maple procedure to find the sum

$$1^2 - 3^2 + 5^2 - 7^2 + \cdots - 99^2 + 101^2.$$

Solution: ► *We use a temporary variable to store the partial results. We must initialise it to* **0***, otherwise Maple will deposit garbage in it.*

```
> SUMMM := proc()
    total := 0;
    for   k   from   1   by   2   to   101
    do   total := total + (-1)^((k-1)/2) * k^2   od;
  end;
```

This can be, of course, accomplished more succinctly with

```
> sum((-1)^(k+1)*(2*k-1)^2,k=1..51);
```

◄

176 Example Give Maple code that will compute the sum of all the integers in $\{1,2,3\ldots,1000\}$ which are neither divisible by 3 nor 5.

Solution: ► *We use the technique of the preceding problem.*

```
> SECTIONSUM := proc()
    total := 0;
    for k from 1 to 1000
    do if k mod 3 <> 0 and k mod 5 <> 0
    then total := total + k;
    fi; od;
    RETURN(total);
  end;
```

◄

177 Example (Linear Search) Write a Maple procedure `MEMBER(D,w)` that tests whether a given word `w` is a member of a dictionary `D`. Test the program with `L:=[abacus, number, algorithm]` and the words `algorithm` and `ossifrage`.

Solution: ► *Here is a possible answer.*

```
> MEMBER := proc(D, w)
    for k from 1 to nops(D)
    do if w = D[k] then RETURN(true)
    fi; od; false;
  end:
> L:=[abacus, number, algorithm]; MEMBER(L,algorithm); MEMBER(L,ossifrage);
```

$$L := [abacus, number, algorithm]$$

$$true$$

$$false$$

Of course, this program must be refined to guard against idiotic inputs. Also, it is particularly inefficient, since it searches word for word, and even if the word has been found, it continues searching. We will see how to improve this later on with the `while` *loop.* ◄

178 Example A Mersenne prime is a prime of the form $2^p - 1$, where p is a prime. Thus $3 = 2^2 - 1, 7 = 2^3 - 1, 31 = 2^5 - 1$ are all Mersenne primes, but $2^{11} - 1 = 23 \cdot 89$ is not a Mersenne prime. Write a Maple procedure that generates all Mersenne primes up to $2^{500} - 1$. You may avail of Maple's `isprime()` function.

Solution: ► *Here is a possible answer.*

```
> MARINMERSENNE := proc()
    for k from 1 to 500
        do if isprime(2^k-1) then print(2^k-1,"is a Mersenne prime.")
    fi; od; end:
```

```
> MARINMERSENNE();
```

$$3, \text{"is a Mersenne prime."}$$

$$7, \text{"is a Mersenne prime."}$$

$$31, \text{"is a Mersenne prime."}$$

$$127, \text{"is a Mersenne prime."}$$

$$8191, \text{"is a Mersenne prime."}$$

$$131071, \text{"is a Mersenne prime."}$$

$$524287, \text{"is a Mersenne prime."}$$

$$2147483647, \text{"is a Mersenne prime."}$$

$$2305843009213693951, \text{"is a Mersenne prime."}$$

$$618970019642690137449562111, \text{"is a Mersenne prime."}$$

$$162259276829213363391578010288127, \text{"is a Mersenne prime."}$$

$$170141183460469231731687303715884105727, \text{"is a Mersenne prime."}$$

◀

179 Example Write a procedure `MAXILIST(X)` that determines the maximum entry in a given number list L. The procedure must work from scratch, that is, using Maple's `max()` function is not allowed.

Solution: ► *Here is a possible answer. Observe since at the beginning we had no way of knowing what* `maxime` *was, we declare it to be the first element of the list. That is, we used the first member of the array as a* sentinel value.

```
> MAXILIST := proc(X)
    maxime := X[1];
    for k from 1 to nops(X)
    do if X[k] > maxime then maxime := X[k];
    fi; od;
    RETURN(maxime);
end;
> X:=[-10,-90,98,2]:  MAXILIST(X);
```

$$98$$

◀

180 Example (The Locker-room Problem) A locker room contains n lockers, numbered 1 through n. Initially all doors are open. Person number 1 enters and closes all the doors. Person number 2 enters and opens all the doors whose numbers are multiples of 2. Person number 3 enters and if a door whose number is a multiple of 3 is open then he closes it; otherwise he opens it. Person number 4 enters and changes the status (from open to closed and viceversa) of all doors whose numbers are multiples of 4, and so forth till person number n enters and changes the status of door number n. Write an algorithm to determine which lockers are closed.

Solution: ► *Here is one possible approach. We use an array of size* n *to denote the lockers so that we may modify the status of the entries. The value* **true** *will denote an open locker and the value* **false** *will denote a closed locker. We first close all the doors.*

```
> LOCKERS:= proc(n)
    X:= array([seq(false,k=1..n)]);
    for j from 2 to n
      do for k from j to n do
        if k mod j = 0 then X[k]:= not(X[k]); fi; od; od;
    for k from 1 to n do if not(X[k])
          then print(k,"is open."); fi; od;
end;
>  LOCKERS(100);
```

$$1, \text{"is open."}$$

$$4, \text{"is open."}$$

$$9, \text{"is open."}$$

$$16, \text{"is open."}$$

$$25, \text{"is open."}$$

$$36, \text{"is open."}$$

$$49, \text{"is open."}$$

$$64, \text{"is open."}$$

$$81, \text{"is open."}$$

$$100, \text{"is open."}$$

Notice that if d *divides* n *so does* $\frac{n}{d}$. *Thus we can pair up every the different divisors of* n, *and have an even number of divisors as long as we do not have* $d = \frac{n}{d}$. *This means that the integers with an even number of divisors will have all doors open, and those with an odd number of divisors will all all doors closed. This last event happens when* $d = \frac{n}{d} \implies n = d^2$, *that is, when* n *is a square.*

◄

Homework

181 Exercise Suppose you didn't know anything about Maple's absolute value `abs()` function. Write a Maple procedure `AbsVal(x)` that will find the absolute value of a real number x.

182 Exercise Using Maple's `ithprime()` function, write a procedure that writes the first N primes.

183 Exercise Nest the `MAXI` procedure of example 173 into a new procedure `MAXI3(x,y,z)` that finds the maximum of three real numbers.

184 Exercise A *twin prime* is a prime p such that $p+2$ is also a prime. Write a Maple procedure to count all the twin primes between 1 and 1000000. You may use Maple's `isprime()` function. [2]

185 Exercise For $n \geq 1$, set $!n = \sum_{k=0}^{n-1} k!$. Duro Kurepa conjectured that $\gcd(!n, n!) = 2$ for all $n \geq 2$. This has been verified for all $n < 1000000$. Using Maple's `gcd(a,b)` function write a procedure to verify this up to $n = 150$. (For larger values, you may get a compilation error depending on your processor.)

2.5 The "while" Loop

The `while` loop has syntax `while (condition-true) do (statements) od;`.

186 Example It is known that the harmonic series $\sum_{k\geq 1} \frac{1}{k}$ diverges. Find the smallest N for which

$$\sum_{1\leq k\leq N} \frac{1}{k} \geq 10.$$

Solution: ► *Here is one possible answer. We use a "while" loop to detect the very first time that the sum exceeds* 10.

```
> HARMONIC := proc(n)
   sum := 0;  k := 0;
   while sum <= n
       do sum := sum + 1/(k+1);  k := k+1  od;
     RETURN(k);
end;
>  HARMONIC(10);
```

$$12367$$

In Calculus II you learn that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \log n + \gamma + \mathcal{O}\left(\frac{1}{n}\right)$. *Here* $\gamma \approx 0.57721566490153286\ldots$ *is the Euler-Mascheroni constant.*[3] *Hence we need* $\log n \approx 10 - \gamma \implies n \approx e^{10-.577} \approx 12620$, *not far from the value Maple found.*

◄

187 Example By availing of the Maple `isprime()` function, write a procedure that determines the first prime greater than $1{,}000{,}000{,}001$.

Solution: ► *Here is one possible way.*

```
> FIRSTPRIME := proc()
    for k from 1000000001 by 2
        while not(isprime(k)) do od;
        RETURN(k)
end;
```

Notice the flexibility of Maple's `for` *loop allowing a* `while` *loop as an upper bound.*

Maple can accomplish this with just one line.

```
>  nextprime(1000000001);
```

◄

[2] It is not known whether the number of twin primes is infinite. Viggo Brunn proved, however, that the infinite series $\sum_{p \text{ is a twin prime}} \frac{1}{p}$ converges.

[3] Another open problem, it not known whether γ is irrational.

188 Example The numbers

$$1, 2, 3, \ldots, 2003$$

are written on a blackboard, in increasing order. Then the first, the fourth, the seventh, etc. are erased, leaving the numbers

$$2, 3, 5, 6, 8, 9, 11, 12, 14, \ldots$$

on the board. This process is repeated, leaving the numbers

$$3, 5, 8, 9, \ldots.$$

The process continues until one number remains on the board and is finally erased. What is the last number to be erased?

Solution: ► *We first write a procedure* `SHRINKLIST()` *that is one iteration of the instructions. For example, if one is given the list* $X := [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$, *then* `SHRINKLIST(L)` *returns* $[2, 3, 5, 6, 8, 9]$. *We then keep shrinking the array until we find the number we want.* `SHRINKLIST()` *creates a new list* L *with the non-deleted elements. Here is the procedure* `SHRINKLIST()`.

```
> SHRINKLIST := proc(A)
L := [ ]; newindex := 1;
    for k from 1 to nops(A)
        do if k mod 3 <> 1
        then C[newindex] := A[k]; L := [op(L), C[newindex]];
        newindex := newindex + 1; fi; od;
eval(L);
end;
```

We now complete the process by shrinking the initial list until it has only one element.

```
> COMPUTELAST := proc(X)
Y := X; while nops(Y) > 1
    do Y := SHRINKLIST(Y); od;
RETURN(Y[1]);
end;
```

Upon invoking `COMPUTELAST([seq(k,k=1..2003)])`, *we see that the last integer left is* 1598.

Here is how to solve this problem without programming. Let J_n *be the first number remaining after* n *erasures, so* $J_0 = 1$, $J_1 = 2$, $J_3 = 3$, $J_4 = 5$, *etc. We prove by induction that*

$$J_{n+1} = \frac{3}{2} J_n \text{ if } J_n \text{ is even,}$$

and

$$J_{n+1} = \frac{3}{2}(J_n + 1) - 1 \text{ if } J_n \text{ is odd.}$$

Assume first that $J_n = 2N$. *Consider the number* $3N$. *There are initially* N *smaller numbers* $\equiv 1 \mod 3$. *So after the first erasure, it will lie in* $2N$*-th place. Hence, it will lie in first place after* $n+1$ *erasures. Assume now that* $J_n = 2N+1$. *Consider* $3N+2$. *There are initially* $N+1$ *smaller numbers* $\equiv 1 \mod 3$. *So after the first erasure, it will lie in* $2N+1$*-st place. Hence, it will lie in first place after* $n+1$ *erasures. That completes the induction. We may now calculate successively the members of the sequence:* $1, 2, 3$, $5, 8, 12$, $18, 27, 41, 62$, $93, 140, 210$, $315, 473, 710$, $1065, 1598, 2397$. *Hence* 1598 *is the last surviving number from* $1, 2, \ldots, 2003$.

◄

189 Example A *palindrome* is a strictly positive integer whose decimal expansion is symmetric and does not end in 0. For example, 2, 11, 3010103, 19988991 are all palindromes. Write a Maple procedure `ISPALINDROME(x)` that determines whether the positive integer x is palindrome.

Solution: ► *Trying to solve this problem by purely arithmetic functions one runs into the trouble that Maple does not make any distinction between, say, the integer* **11** *and the integer* **011**. *In order to respect repetitions and order, we first convert the integer into a list. The algorithm below is self explanatory, and we are using the algorithm* `ITHDIGIT` *from example 166.*

```
> MAKEMEINTOLIST := proc(x)
L := [ ];
for k from 1 to length(x)
    do L := [op(L), ITHDIGIT(x, length(x) - k + 1)]; od;
eval(L);
end;
```

Now, we can simply determine whether x *is a palindrome by comparing* `L[k]` *with* `L[nops(L)-k+1]`.

```
> ISPALINDROME := proc(x)
L := MAKEMEINTOLIST(x);
for k from 1 to length(x)/2
    do if L[k] <> L[nops(L) - k + 1] then RETURN(false); fi; od;
true;
end;
```

◄

190 Example An array $X := (x_1, x_2, \ldots, x_n)$ is given. Write a procedure that reverses the elements of X, that is, that returns $(x_n, x_{n-1}, \ldots, x_1)$.

Solution: ► *The trick is to swap* (x_1, x_n), (x_2, x_{n-1}), *etc.*

```
> REVERSELIST := proc(X)
    Y := X; left := 1; right := nops(X);
        while(left < right) do
    temp := Y[left]; Y[left] := Y[right]; Y[right] := temp;
        left := left + 1; right := right - 1; od;
            eval(Y);
end;
> REVERSELIST([1,2,5]);
```

$$[5, 2, 1]$$

◄

Homework

191 Exercise (Digit Reversing) Write a Maple procedure `REVERSEDIGITS` that prints the digits of a positive integer in reverse order. Thus `REVERSEDIGITS(123)` will print **321**. The program interprets, say, **01230** as **1230** and so you should have `REVERSEDIGITS(1230)` return **321**.

192 Exercise By strictly arithmetic means, that is, without using lists and without using example 166, write a Maple procedure `FIRSTISLAST(x)` that checks whether the first and the last digit of the integer $x > 0$ (with at least two digits), are equal. The program interprets, say, **01230** as **1230** and so you should have `FIRSTISLAST(01230)` return false.

193 Exercise Using Maple's `rand(low..high)` function for producing a random number from `low` to `high`, simulate the toss of a die n times.

194 Exercise Using example 189, find the sum of all palindromes between M and N, with $M < N$.

195 Exercise (Goldbach's Conjecture) It is an unsolved problem to shew that every even integer $n \geq 6$ can be written as the sum of two odd primes. Using Maple's `isprime()` function, write a Maple program to verify Goldabach's conjecture for any even integer ≤ 1000000.

196 Exercise (Postage Problem) Suppose that you have two types of postage stamps: one costing a cents and the other b cents. We say that a postage of h cents is *realisable* if there are positive integral solutions x, y to the equation $ax + by = h$. Write a Maple procedure `POSTAGE(a,b,h)` in which you input three positive integers a, b, h and tell whether the postage of h cents is realisable with stamps costing a and b cents.

197 Exercise (Circle Problem) Given a positive integer n write a Maple program that counts the number of solutions to $x^2 + y^2 \leq n$ where x, y are both positive integers.

198 Exercise Write a Maple procedure that gives the Roman numeral representation of any integer between 1 and 3999.

199 Exercise A list

$$X := (x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_{m+n})$$

is given. Write a procedure `SWITCHLIST(X,m,n)` that, for given subscripts m and n, it will return

$$(x_{m+1}, x_{m+2}, \ldots, x_{m+n}, x_1, x_2, \ldots, x_m).$$

200 Exercise An array $X = (x_1, x_2, \ldots, x_n)$ is given, sorted such that $x_1 \leq x_2 \leq \cdots \leq x_n$. Count the number of different x_k.

201 Exercise Two given lists $X := (x_1, \ldots x_k)$ and $Y := (y_1, \ldots y_l)$ are sorted so that $x_1 < \cdots < x_k$ and $y_1 < \cdots y_l$. Find how many elements in common they have, that is, find the cardinality of their intersection.

202 Exercise What is the smallest positive integral power of 7 whose first three digits (from left to right) are 222? In general, write a Maple procedure that given strictly positive integers a and x will compute the smallest positive integral power of x that will begin with a. For $a = 222$, the least power is $k = 327$. The program seems to take a long time to compute various values.

2.6 Iteration and Recursion

A *recursive procedure* is one where future steps are computed and rely on entirely on previously computed steps. An *iterative procedure* is one which is obtained by repetition of a code fragment. These definitions are imperfect, but we hope they will become clearer with some examples.

203 Example (Fibonacci Numbers) The *Fibonacci Numbers* are defined recursively by

$$f_0 = 0, \quad f_1 = 1, \quad f_{n+1} = f_n + f_{n-1}, \qquad n \geq 1,$$

so the sequence goes

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 \ldots$$

Write a Maple program that computes the n-th Fibonacci number.

Solution: ► *Here is an iterative solution.*

```
> FIBONEI := proc(n)
if n <= 1 then  f := n;
else  fold := 0;  fnew := 1;
for  k  from  2  to  n
do  f := fold + fnew;  fold := fnew;  fnew := f;
od;  fi;
RETURN(f);
end;
```

Here is a recursive solution.

```
> FIBONEII := proc(n)
if n <= 1 then n;
else FIBONEII(n-2) + FIBONEII(n-1);
fi;
end;
```

It is worth to compare the running time between the two programs. For this, use Maple's `time()` *command.*

```
>   time(FIBONEI(20));
>   time(FIBONEII(20));
```

Observe that the recursive program is much slower. In fact, if I try to compute `FIBONEIi(200)`*, my computer takes several minutes. This is because each time* `FIBONEIi()` *is called, Maple has to recalculate the preceding values, without remembering them. If you use the option* `remember`*, then the programm runs much faster.*

```
> FIBONEIII := proc(n)
option remember;
if n <= 1 then n;
else FIBONEIII(n-2) + FIBONEIII(n-1);
fi;
end;
>   time(FIBONEIII(2000));
>   time(FIBONEI(2000));
```

◄

204 Example (Horner's Method) Write an iterative algorithm $Horner(p, x0)$ to evaluate a polynomial

$$p := x \mapsto a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

at $x = x_0$.

Solution: ► *Observe that we may successively compute*

$$a_n,\ a_n x_0 + a_{n-1},\ x_0(x_0 a_n + a_{n-1}) + a_{n-2},\ x_0(x_0(x_0 a_n + a_{n-1}) + a_{n-2}) + a_{n-3},\ \ldots$$

each time multiplying the preceding result by x_0 and adding a constant. We enter the coefficients of the polynomial in a list $p := [a_0, a_1, \ldots, a_n]$, and carry out the instructions just described. Observe that $n+1 = nops(p)$ and that $a_k = p[k+1]$. Here is the code.

```
> HORNER := proc(p, x0)
total := 0;
for k from 1 to nops(p)
do total := total * x0 + p[nops(p) - k + 1]; od;
end;
```

◄

205 Example (Collatz Conjecture) Consider the function $f : \mathbb{N} \to \mathbb{N}$,

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even,} \\ 3n+1 & \text{if } n \text{ is odd.} \end{cases}$$

If one considers the sequence

$$n, \qquad f(n), \quad f(f(n)), \quad f(f(f(n))), \ldots,$$

it is not known whether this sequence will ultimately end with a 1. Write a Maple program that computes this sequence until it halts with a 1 (if at all).

Solution: ► *Here is a possible solution.*

```
> COLLATZ := proc(n) x := n;
for k from 1 while x > 1 do print(x);
if x mod 2 = 1 then x := 3*x+1; else x := x/2; fi;
od;
end;
```

◄

Homework

206 Exercise Write two procedures, one iterative and the other recursive, for computing $n!$ for an integer $n \geq 0$. Your procedure cannot involve the operator $!$.

2.7 Some Classic Algorithms

We now examine some classic algorithms. Among them we will find Eratosthenes' Sieve, Euclid's Algorithm and several sorting algorithms. Maple has many of these algorithms as built-in functions, but the question arises: how is Maple able to perform these feats? How are its functions written? Our purpose is to learn to reinvent the wheel, but without trying to go overboard and get distracted by too many details. We omit an important topic, to prove whether our algorithms are *correct*. The interested reader may consult [CLRS] or [Knu] for this topic.

207 Example (Eratosthenes' Sieve) Let $n > 0$ be a composite integer. Then we may factor n as $n = ab$ with positive integers $1 < a \leq b < n$. Let p be the smallest prime factor dividing n. Then $p^2 \leq ab = n$, and so n has a prime factor $p \leq \sqrt{n}$. This means that if a positive integer has no divisor less than or equal its square root, then it is a prime.

For example, to test whether 103 is prime, we divide 103 by every positive integer between 2 and $\lfloor\sqrt{103}\rfloor = 10$. Since 103 is not divisible by any integer in the interval $[2;10]$ we conclude that 103 is prime.

Write a Maple program that tests for primality of a positive integer.

Solution: ► *Here is a possible approach.*

```
TISPRIME := proc(x)
    k := 2; flag := true;
    if x = 1 then print("1 is a unit.");
    flag := false;
    else while k <= floor(sqrt(x)) and flag
do if x mod k = 0
then flag := false; fi; k := k+1; od;
fi;
    if(flag) then print(x,"is prime")
    elif x > 1
    then print(x,"is divisible by",k-1);
    fi;
end;
```

Maple, of course, has its own internal function `isprime()` *to test whether an integer is prime.*

```
> isprime(60637);
```

◄

208 Definition If a and b are two strictly positive integers, then their *greatest common divisor*, denoted by $\gcd(a,b)$ is the largest positive integer dividing both a and b.

For example, $\gcd(20,30)=10$, $\gcd(44,45)=1$ and if $p \neq q$ are primes then $\gcd(p,q)=1$.

Recall that by the Division Algorithm, for all positive integers a and $b>0$, then we can find unique integers q,r called the *quotient* and the *remainder*, respectively, such that

$$a = qb + r, \qquad 0 \leq r < b.$$

For example, if $a=1004, b=75$ then

$$1004 = 13\cdot 75 + 29,$$

whence $q=13$ and $r=29$.

Let a,b be positive integers. After using the Division Algorithm repeatedly, we find the sequence of equalities

$$\begin{array}{lcll} a & = & bq_1 + r_2, & 0 < r_2 < b, \\ b & = & r_2 q_2 + r_3 & 0 < r_3 < r_2, \\ r_2 & = & r_3 q_3 + r_4 & 0 < r_4 < r_3, \\ \vdots & \vdots & \vdots & \vdots \\ r_{n-2} & = & r_{n-1}q_{n-1} + r_n & 0 < r_n < r_{n-1}, \\ r_{n-1} & = & r_n q_n. & \end{array} \tag{2.1}$$

The sequence of remainders will eventually reach a r_{n+1} which will be zero, since $b, r_2, r_3, \ldots$ is a monotonically decreasing sequence of integers, and cannot contain more than b positive terms.

The Euclidean Algorithm rests on the fact that $\gcd(a,b)=\gcd(b,r_2)=\gcd(r_2,r_3)=\cdots=\gcd(r_{n-1},r_n)=r_n$.

We illustrate it with $a=1004, b=75$ then

$$\begin{array}{lcl} 1004 & = & 13\cdot 75 + 29, \\ 75 & = & 2\cdot 29 + 17, \\ 29 & = & 1\cdot 17 + 12, \\ 17 & = & 1\cdot 12 + 5, \\ 12 & = & 2\cdot 5 + 2, \\ 5 & = & 2\cdot 2 + 1, \\ 2 & = & 2\cdot 1, \end{array}$$

whence $\gcd(1004,75)=1$.

209 Example (Euclidean Algorithm) Write a Maple procedure that takes two strictly positive integers and returns their greatest common divisor using the Euclidean Algorithm. You may only use the Maple `mod` function to find remainders and various multiplications or divisions to find quotients.

Solution: ► *Here is a possible solution.*

$EUCLIDALGO := proc(a,b)$
$x := a; y := b;$
$while\ (x\ <>\ 0\ and\ y\ <>\ 0)\ do\ if\ x >= y\ then\ x := x\ mod\ y$
$else\ y := y\ mod\ x\ fi;\ od;$
$if\ x = 0\ then\ RETURN(y)\ else\ RETURN(x)\ fi;$
$end;$

Maple, of course, has its own internal function `gcd(a, b)` *to find the greatest divisor of two numbers.*

◄

210 Example (Positive Integral Powers) Write a Maple procedure that will compute a^n, where a is a given real number and n is a given positive integer.

Solution: ► *: Here is an approach which essentially reduces computing an n-th power to squaring. We successively square x getting a sequence*

$$x \to x^2 \to x^4 \to x^8 \to \cdots \to x^{2^k},$$

and we stop when $2^k \le n < 2^{k+1}$. For example, if $n = 11$ we compute $x \to x^2 \to x^4 \to x^8$. We now write $11 = 8+2+1$ and so $x^{11} = x^8 x^2 x$.

$POWER := proc(x,n)$
$product := 1;\ \ c := x;\ \ k := n;$
$while\ k <> 0\ do\ if\ k\ mod\ 2\ =\ 0$
$then\ k := k/2;\ \ c := c * c;$
$else\ k := k-1;\ \ product := product * c;\ \ fi;\ \ od;$
$RETURN(product);$
$end;$

◄

We now investigate some sorting algorithms. Sorting algorithms are ubiquitous in applications, for example, alphabetising a list of names, or arranging a sequence of scores monotonically.

211 Example (Bubblesort) We now sort a list $L := (x_1, x_2, \ldots, x_n)$ of numbers into an increasing sequence. We proceed naively as follows: we compare two items at a time and swap them if they are misplaced. The pass through the list is repeated until no swaps are needed, thereby sorting the list. We utilise imbricated two `for` loops, the first running with index i, $1 \le i \le \text{nops(X)} - 1$ and the second running with index j, $1 \le j \le \text{nops(X)} - i$. For example, to sort the list $[3,4,5,2,1]$:

1. The outer loop has four runs: $1 \le i \le 4$.
2. The list is $[3,4,5,2,1]$. For $i = 1$, we start with 3 and 4. As they are rightly sorted we do nothing. We now compare 4 and 5. Since they are rightly sorted, we do nothing. We continue and compare 5 and 2. Since they are wrongly sorted, we swap them, obtaining the new array $[3,4,2,5,1]$. We compare now 5 and 1, and we swap them since they are wrongly sorted. We obtain the array $[3,4,2,1,5]$. Notice that this moves the largest element to the last position.
3. The list is now $[3,4,2,1,5]$. For $i = 2$, we again start with 3 and 4. As they are rightly sorted we do nothing. We now compare 4 and 5. Since they are rightly sorted, we do nothing. We continue and compare 4 and 2. Since they are wrongly sorted, we swap them, obtaining the new array $[3,2,4,1,5]$. We compare now 4 and 1, and we swap them since they are wrongly sorted. We obtain the array $[3,2,1,4,5]$. Notice that this moves the second largest element to the ante-penultimate position.
4. The list is now $[3,2,1,4,5]$. For $i = 3$, we compare 3 and 2. As they are wrongly sorted we swap them, obtaining $[2,3,1,4,5]$. We now compare 3 and 1. Since they are wrongly sorted, we swap them, obtaining $[2,1,3,4,5]$.

5. For $i = 4$, the list is now $[2,1,3,4,5]$. We compare 2 and 1 and swap them, obtaining the sorted list. $[1,2,3,4,5]$.

The steps of the outer for loops are thus

$$[3,4,5,2,1] \to [3,4,2,1,5] \to [3,2,1,4,5] \to [2,1,3,4,5] \to [1,2,3,4,5].$$

Solution: ► *Here is a Maple implementation.*

```
> BUBBLE := proc(X)
Y := X;
for i from 1 to (nops(Y) - 1)
do for j from 1 to (nops(Y) - i)
do if Y[j] > Y[j+1] then temp := Y[j]; Y[j] := Y[j+1]; Y[j+1] := temp; fi; od; od;
eval(Y);
end;
```

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212 Example (Quicksort) Quicksort is a "divide-and-conquer" algorithm for sorting data. One chooses any one number, x say, from the list in question and then shoves the numbers in the list which are greater than x into one end of the list, and the numbers which are smaller than x into the other end of the list. This partitions the array into two smaller arrays and then one proceeds to carry out the same instructions into these two smaller arrays, etc.

Solution: ► *Here is the implementation, directly quoted from [WM]:*

```
> partition := proc(m,n) i := m;
j := n; x := A[j]; while i < j do
if A[i] > x then A[j] := A[i]; j := j - 1; A[i] := A[j]; else i := i + 1 fi; od; A[j] := x;
p := j;
end:
> QUICKIE := proc(A,m,n)
if m < n then partition(m,n); QUICKIE(A,m,p - 1);
QUICKIE(A,p + 1,n); fi;
eval(A);
end;
```

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Homework

213 Exercise Without using Maple's `isprime()`, `ifactor()` functions, etc., write a procedure that prints the factorisation of a given integer $n > 0$ into primes.

3 Orders of Infinity

3.1 Big Oh and Vinogradov's Notation

Why bother? It is clear that the sequences $\{n\}_{n=1}^{+\infty}$ and $\{n^2\}_{n=1}^{+\infty}$ both tend to $+\infty$ as $n \to +\infty$. We would like now to refine this statement and compare one with the other. In other words, we will examine their speed towards $+\infty$.

Throughout we only consider sequences of real numbers.

214 Definition We write $a_n = \mathcal{O}(b_n)$ if $\exists C > 0$, $\exists N > 0$ such that $\forall n \geq N$ we have $|a_n| \leq C|b_n|$. We then say that a_n is *Big Oh* of b_n, or that a_n *is of order at most* b_n as $n \to +\infty$. Observe that this means that $\left|\frac{a_n}{b_n}\right|$ is bounded for sufficiently large n. The notation $a_n << b_n$, due to Vinogradov, is often used as a synonym of $a_n = \mathcal{O}(b_n)$.

☞ *A sequence $\{a_n\}_{n=1}^{+\infty}$ is bounded if and only if $a_n << 1$.*

An easy criterion to identify Big Oh relations is the following.

215 Theorem If $\lim_{n\to+\infty} \frac{a_n}{b_n} = c \in \mathbb{R}$, then $a_n << b_n$.

Proof: *Since a convergent sequence is bounded, the sequence $\left\{\frac{a_n}{b_n}\right\}_{n=+1}^{+\infty}$ is bounded: so for sufficiently large n, $\left|\frac{a_n}{b_n}\right| < C$ for some constant $C > 0$. This proves the theorem.* ❑

☞ *The = in the relation $a_n = \mathcal{O}(b_n)$ is not a true equal sign. For example $n^2 = \mathcal{O}\left(n^3\right)$ since $\lim_{n\to+\infty} \frac{n^2}{n^3} = 0$ and so $n^2 << n^3$ by Theorem 215. On the other hand, $\lim_{n\to+\infty} \frac{n^3}{n^2} = +\infty$ so that for sufficiently large n, and for all $M > 0$, $n^3 > Mn^2$, meaning that $n^3 \neq \mathcal{O}\left(n^2\right)$. Thus the Big Oh relation is not symmetric.*[1]

216 Theorem (Lexicographic Order of Powers) Let $(\alpha, \beta) \in \mathbb{R}$ and consider the sequences $\{n^\alpha\}_{n=1}^{+\infty}$ and $\left\{n^\beta\right\}_{n=1}^{+\infty}$. Then $n^\alpha << n^\beta \iff \alpha \leq \beta$.

Proof: *If $\alpha \leq \beta$ then $\lim_{n\to+\infty} \frac{n^\alpha}{n^\beta}$ is either 1 (when $\alpha = \beta$) or 0 (when $\alpha < \beta$), hence $n^\alpha << n^\beta$ by Theorem 215.*

If $n^\alpha << n^\beta$ then for sufficiently large n, $n^\alpha \leq Cn^\beta$ for some constant $C > 0$. If $\alpha > \beta$ then this would mean that for all large n we would have $n^{\alpha-\beta} \leq C$, which is absurd, since for a strictly positive exponent $\alpha - \beta$, $n^{\alpha-\beta} \to +\infty$ as $n \to +\infty$.❑

217 Example As $n \to +\infty$,

$$n^{1/10} << n^{1/3} << n << n^{10/9} << n^2,$$

for example.

[1] One should more properly write $a_n \in \mathcal{O}(b_n)$, as $\mathcal{O}(b_n)$ is the set of sequences growing to infinity no faster than b_n, but one keeps the = sign for historical reasons.

218 Theorem If $c \in \mathbb{R} \setminus \{0\}$ then $\mathcal{O}(ca_n) = \mathcal{O}(a_n)$, that is, the set of sequences of order at most ca_n is the same set at those of order at most a_n.

Proof: *We prove that* $b_n = \mathcal{O}(ca_n) \iff b_n = \mathcal{O}(a_n)$. *If* $b_n = \mathcal{O}(ca_n)$ *the there are constants* $C > 0$ *and* $N > 0$ *such that* $|b_n| \leq C|ca_n|$ *whenever* $n \geq N$. *Therefore,* $|b_n| \leq C'|a_n|$ *whenever* $n \geq N$, *where* $C' = C|c|$, *meaning that* $b_n = \mathcal{O}(a_n)$. *Similarly, if* $b_n = \mathcal{O}(a_n)$ *the there are constants* $C_1 > 0$ *and* $N_1 > 0$ *such that* $|b_n| \leq C_1|a_n|$ *whenever* $n \geq N_1$. *Since* $c \neq 0$ *this is equivalent to* $|b_n| \leq \frac{C_1}{c}(c|a_n|) = C''(c|a_n|)$ *whenever* $n \geq N_1$, *where* $C'' = \frac{C_1}{c}$, *meaning that* $b_n = \mathcal{O}(ca_n)$. *Therefore,* $\mathcal{O}(a_n) = \mathcal{O}(ca_n)$. ❑

219 Example As $n \to +\infty$,

$$\mathcal{O}(n^3) = \mathcal{O}\left(\frac{n^3}{3}\right) = \mathcal{O}(4n^3).$$

220 Theorem (Sum Rule) Let $a_n = \mathcal{O}(x_n)$ and $b_n = \mathcal{O}(y_n)$. Then $a_n + b_n = O(\max(|x_n|, |y_n|))$.

Proof: *There exist strictly positive constants* C_1, N_1, C_2, N_2 *such that*

$$n \geq N_1, \implies |a_n| \leq C_1|x_n| \qquad \text{and} \qquad n \geq N_2, \implies |b_n| \leq C_2|y_n|.$$

Let $N' = \max(N_1, N_2)$. *Then for* $n \geq N$, *by the Triangle inequality*

$$|a_n + b_n| \leq |a_n| + |b_n| \leq C_1|x_n| + C_2|y_n|.$$

Let $C' = \max(C_1, C_2)$. *Then*

$$|a_n + b_n| \leq C'(|x_n| + |y_n|) \leq 2C' \max(|x_n|, |y_n|),$$

whence the theorem follows. ❑

221 Corollary Let $a_n = k_0 n^m + k_1 n^{m-1} + k_2 n^{m-2} + \cdots + k_{m-1} n + k_n$ be a polynomial of degree m in n with real number coefficients. The $a_n = \mathcal{O}(n^m)$, that is, a_n is of order at most its leading term.

Proof: *By the Sum Rule Theorem 220 the leading term dominates.*❑

222 Theorem (Transitivity Rule) If $a_n = O(b_n)$ and $b_n = O(c_n)$, then $a_n = \mathcal{O}(c_n)$.

Proof: *There are strictly positive constants* C_1, C_2, N_1, N_2 *such that*

$$n \geq N_1, \implies |a_n| \leq C_1|b_n| \qquad \text{and} \qquad n \geq N_2, \implies |b_n| \leq C_2|c_n|.$$

If $n \geq \max(N_1, N_2)$, *then* $|a_n| \leq C_1|b_n| \leq C_1C_2|c_n| = C|c_n|$, *with* $C = C_1C_2$. *This gives* $a_n = \mathcal{O}(c_n)$. ❑

223 Example By Corollary 221, $5n^4 - 2n^2 + 100n - 8 = \mathcal{O}(5n^4)$. By Theorem 218, $\mathcal{O}(5n^4) = \mathcal{O}(n^4)$. Hence

$$5n^4 - 2n^2 + 100n - 8 = \mathcal{O}(n^4).$$

224 Theorem (Multiplication Rule) If $a_n = O(x_n)$ and $b_n = O(y_n)$, then $a_n b_n = \mathcal{O}(x_n y_n)$.

Proof: *There are strictly positive constants* C_1, C_2, N_1, N_2 *such that*

$$n \geq N_1, \implies |a_n| \leq C_1|x_n| \qquad \text{and} \qquad n \geq N_2, \implies |b_n| \leq C_2|y_n|.$$

If $n \geq \max(N_1, N_2)$, *then* $|a_n b_n| \leq C_1C_2|x_n y_n| = C|x_n y_n|$, *with* $C = C_1C_2$. *This gives* $a_n b_n = \mathcal{O}(x_n y_n)$. ❑

225 Theorem (Lexicographic Order of Exponentials) Let $(a,b) \in \mathbb{R}$, $a > 1$, $b > 1$, and consider the sequences $\{a^n\}_{n=1}^{+\infty}$ and $\{b^n\}_{n=1}^{+\infty}$. Then $a^n << b^n \iff a \leq b$.

Proof: *Recall that if $r \in \mathbb{R}$, then $r^n \to 0$ if $|r| < 1$ and if $|r| > 1$ then $\{r^n\}_{n=1}^{+\infty}$ diverges. Put now $r = \frac{a}{b}$ and use Theorem 215.* ❑

226 Example $\frac{1}{2^n} << 1 << 2^n << e^n << 3^n$.

227 Lemma Let $a \in \mathbb{R}$, $a > 1$, $k \in \mathbb{N} \setminus \{0\}$. Then $n^k << a^n$.

Proof: *Using L'Hôpital's Rule k times, $\lim_{n\to+\infty} \frac{n^k}{a^n} = 0$. Now apply Theorem 215.* ❑

228 Theorem ("Exponentials are faster than powers") Let $a \in \mathbb{R}$, $a > 1$, $\alpha \in \mathbb{R}$. Then $n^\alpha << a^n$.

Proof: *Put $k = \max(1, \lfloor\alpha\rfloor + 1)$. Then by Theorem 216, $n^\alpha << n^k$. By Lemma 227, $n^k << a^n$, and by the Transitivity of Big Oh (Theorem 222), $n^\alpha << n^k << a^n$.* ❑

229 Example

$$n^{100} << e^n.$$

230 Theorem ("Logarithms are slower than powers") Let $(\alpha, \beta) \in \mathbb{R}^2$, $\alpha > 0$. Then $(\log n)^\beta << n^\alpha$.

Proof: *If $\beta \leq 0$, then $(\log n)^\beta << 1$ and the assertion is evident, so assume $\beta > 0$. For $x > 0$, then $\log x < x$. Putting $x = n^{\alpha/\beta}$, we get*

$$\log n^{\alpha/\beta} < n^{\alpha/\beta} \implies \log n < \frac{\beta n^{\alpha/\beta}}{\alpha} \implies (\log n)^\beta < \frac{\beta^\beta n^\alpha}{\alpha^\beta},$$

whence $(\log n)^\beta << n^\alpha$. ❑

By the Multiplication Rule (Theorem 224) and Theorems 216, 228, 230, in order to compare two expressions of the type $a^n n^b (\log)^c$ and $u^n n^v (\log)^w$ we simply look at the lexicographic order of the exponents, keeping in mind that logarithms are slower than powers, which are slower than exponentials.

231 Example In increasing order of growth we have

$$\frac{1}{e^n} << \frac{1}{2^n} << \frac{1}{n^2} = \frac{1}{\log n} << 1 << (\log\log n)^{10} << \sqrt{\log n} << \frac{n}{\log n} << n << n\log n << e^n.$$

232 Example Decide which one grows faster as $n \to +\infty$: $n^{\log n}$ or $(\log n)^n$.

Solution: ► *Since $n^{\log n} = e^{(\log n)^2}$ and $(\log n)^n = e^{n \log\log n}$, and since $(\log n)^2 << n \log\log n$, we conclude that $n^{\log n} << (\log n)^n$.* ◄

We now define two more fairly common symbols in asymptotic analysis.

233 Definition We write $a_n = o(b_n)$ if $\frac{a_n}{b_n} \to 0$ as $n \to +\infty$, and say that a_n is *small oh* of b_n, or that a_n *grows slower* than b_n as $n \to +\infty$.

234 Definition A sequence $\{a_n\}_{n=1}^{+\infty}$ is said to be *infinitesimal* if $a_n = o(1)$, that is, if $a_n \to 0$ as $n \to +\infty$.

☞ *We know from above that for* $a > 1$ $\lim_{n\to+\infty} \frac{n^\alpha}{a^n} = 0$, *and so* $n^\alpha = o(a^n)$. *Also, for* $\gamma > 0$, $\lim_{n\to+\infty} \frac{(\log n)^\beta}{n^\gamma} = 0$, *and so* $(\log n)^\beta = o(n^\gamma)$.

235 Definition We write $a_n \sim b_n$ if $\frac{a_n}{b_n} \to 1$ as $n \to +\infty$, and say that a_n is *asymptotic* to b_n.

Asymptotic sequences are thus those that grow at the same rate as the index increases.

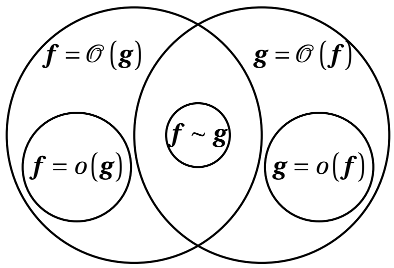

Figure 3.1: Diagram of O relations.

236 Example The sequences $\{n^2 - n\sin n\}_{n=1}^{+\infty}$, $\{n^2 + n - 1\}_{n=1}^{+\infty}$ are asymptotic since

$$\frac{n^2 - n\sin n}{n^2 + n - 1} = \frac{1 - \frac{\sin n}{n}}{1 + \frac{1}{n} - \frac{1}{n^2}} \to 1,$$

as $n \to +\infty$.

237 Theorem Let $\{a_n\}_{n=1}^{+\infty}$ and $\{b_n\}_{n=1}^{+\infty}$ be two properly diverging sequences. Then

$$a_n \sim b_n \iff a_n = b_n(1 + o(1)).$$

Proof: *Since the limit is* $1 > 0$, *either both diverge to* $+\infty$ *or both to* $-\infty$. *Assume the former, and so, eventually,* b_n *will be strictly positive. Now,*

$$\begin{aligned}
\lim_{n\to+\infty} \frac{a_n}{b_n} = 1 \quad &\iff \quad \forall \varepsilon > 0, \exists N > 0, 1 - \varepsilon < \frac{a_n}{b_n} < 1 + \varepsilon \\
&\iff \quad b_n - b_n\varepsilon < a_n < b_n + b_n\varepsilon \\
&\iff \quad |a_n - b_n| < b_n\varepsilon \\
&\iff \quad a_n - b_n = o(b_n).
\end{aligned}$$

❑

The relationship between the three symbols is displayed in figure 3.1.

Homework

238 Exercise Prove that $\mathscr{O}(\mathscr{O}(a_n)) = \mathscr{O}(a_n)$.

239 Exercise Let $k \in \mathbb{R}$ be a constant. Prove that $k + \mathscr{O}(a_n) = \mathscr{O}(k + a_n) = \mathscr{O}(a_n)$.

240 Exercise Let $k \in \mathbb{R}$, $k > 0$, be a constant. Prove that $(a_n + b_k)^k << a_n^k + b_n^k$.

241 Exercise For a sequence of real numbers $\{a_n\}_{n=1}^{+\infty}$ it is known that $a_n = \mathscr{O}\left(n^2\right)$ and $a_n = o(n^2)$. Which of the two statements conveys more information?

242 Exercise True or false: $a_n = \mathscr{O}(n) \implies a_n = o(n)$.

243 Exercise True or false: $a_n = o(n) \implies a_n = \mathscr{O}(n)$.

244 Exercise True or false: $a_n = o\left(n^2\right) \implies a_n = \mathscr{O}(n)$.

245 Exercise True or false: $a_n = o(n) \implies a_n = \mathscr{O}\left(n^2\right)$.

3.2 Some Asymptotic Estimates

We are mainly interested in providing asymptotic estimates for sums. In the case when a closed formula for the sum is known, the problem is half solved. If the terms of a sum are monotonic, then one may apply a method akin to the integral test.

246 Example Since $1 + 2 + \cdots + n = \dfrac{n^2}{2} + \dfrac{n}{2}$, we have $1 + 2 + \cdots + n \sim \dfrac{n^2}{2}$.

247 Example (Harmonic Sum) Prove that $1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} \sim \log n$.

Solution: ► *Using the fact that* $x \mapsto \dfrac{1}{x}$ *is decreasing for* $x > 0$, *if* $k > 0$ *is an integer, for* $x \in]k; k+1[$,

$$\frac{1}{k+1} < \frac{1}{x} < \frac{1}{k} \implies \int_k^{k+1} \frac{\mathrm{d}x}{k+1} < \int_k^{k+1} \frac{\mathrm{d}x}{x} < \int_k^{k+1} \frac{\mathrm{d}x}{k} \implies \frac{1}{k+1} < \int_k^{k+1} \frac{\mathrm{d}x}{x} < \frac{1}{k}.$$

Letting k *run from* 1 *to* $n-1$ *on the first inequality we deduce,*

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^n \frac{\mathrm{d}x}{x} \implies 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \int_1^n \frac{\mathrm{d}x}{x} = 1 + \log n \implies \frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}{\log n} < 1 + \frac{1}{\log n}.$$

Letting k *run from* 1 *to* $n-1$ *on the second inequality,*

$$\int_1^n \frac{\mathrm{d}x}{x} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \implies \log n + \frac{1}{n} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \implies 1 + \frac{1}{n \log n} < \frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}{\log n}.$$

The assertion now follows by the Sandwich Theorem. ◄

248 Example Using Calculus it can be proved that $x \mapsto x \log x$ is increasing for $x > e^{-1}$. Recall that using an integration by parts, $\displaystyle\int_1^n \log x \, \mathrm{d}x = n \log n - n + 1$. Use this to find an asymptotic estimate for $\displaystyle\sum_{k=1}^n \log k$.

Solution: ► *We use the same method as in example 247. If* $k > 0$ *is an integer, for* $x \in]k; k+1[$,

$$\log k < \log x < \log(k+1) \implies \int_k^{k+1} \log k \mathrm{d}x < \int_k^{k+1} \log x \mathrm{d}x < \int_k^{k+1} \log(k+1) \mathrm{d}x \implies \log k < \int_k^{k+1} \log x \mathrm{d}x < \log(k+1).$$

Letting k *run from* 1 *to* $n-1$ *on the first inequality we deduce,*

$$\begin{aligned} \log 1 + \log 2 + \cdots + \log(n-1) < \int_1^n \log x \quad &\implies \quad \log 1 + \log 2 + \cdots + \log(n-1) < \int_1^n \log x \mathrm{d}x = n \log n - n + 1 \\ &\implies \quad \log 1 + \log 2 + \cdots + \log n < \log n + n \log n - n + 1. \end{aligned}$$

Letting k run from 1 to $n-1$ on the second inequality,

$$\int_1^n \log x \mathrm{d}x < \log 2 + \log 3 + \cdots + \log n \quad \implies \quad n\log n < \log 2 + \log 3 + \cdots + \log n$$

$$\implies \quad n\log n - n + 1 = n\log n - n + 1 + \log 1 < \log 1 + \log 2 + \log 3 + \cdots + \log n.$$

We deduce that

$$1 - \frac{1}{\log n} + \frac{1}{n\log n} < \frac{\log 1 + \log 2 + \cdots + \log n}{n\log n} < 1 + \frac{1}{n} - \frac{1}{\log n} + \frac{1}{n\log n}.$$

The Sandwich Theorem now gives $\sum_{1\leq k\leq n} \log k \sim n\log n$. ◄

249 Example Prove that for sufficiently large n,

$$e\frac{n^n}{e^n} < n! < e\frac{n^{n+1}}{e^n}.$$

Solution: ► *From example 248,*

$$n\log n - n + 1 < \log n! < n\log n - n + \log n + 1,$$

which gives upon exponentiation,

$$e\frac{n^n}{e^n} < n! < e\frac{n^{n+1}}{e^n}.$$

◄

The true order of magnitude of $n!$ is given by Stirling's formula:

$$n! \sim \frac{n^n}{e^n}\sqrt{2\pi n}.$$

Homework

250 Exercise Let f_n denote the nth Fibonacci number. Shew that $f_n = \mathcal{O}\left(1.62^n\right)$.

251 Exercise Prove that $e^n << n!$.

252 Exercise Prove that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \sim 2\sqrt{n}$.

253 Exercise From the fact that $x \mapsto \log x$ is a concave function, deduce that

$$x \in]k; k+1[\implies \log k + \log(k+1) < 2\log x.$$

Use this to improve the upper bound in example 248.

3.3 Analysis of Algorithms

In this section we will provide rough estimates for the *time* that takes out to carry out some algorithms. The problem at hand is the following: given an input of size n (however that "size" is measured), which we will assume grows indefinitely towards $+\infty$, we would like to know how the memory requirements and the running time for a computer to process it, in fact, we would like to find a certain function f and say that the algorithms *complexity* is $\mathcal{O}\left(f(n)\right)$.

254 Example Suppose it takes a digital camera 10^{-6} of a second to process a pixel. Estimate how much time will it take it to handle a 1 megapixel (that is, one million pixels) image if the algorithm it uses is of complexity $\mathcal{O}(n)$, $\mathcal{O}\left(n\log n\right)$, or $\mathcal{O}\left(n^2\right)$, where n is the number of pixels.

Solution: ► *In the case of the linear algorithm, the camera takes about* $10^{-6}10^{6}=1$ *second. If the algorithm is of order* $n\log n$*, the camera takes about* $10^{-6}(10^{6})\log 10^{6}\approx 13$ *seconds. If the algorithm is of complexity* n^2*, the camera will take about* $10^{-6}10^{12}=10^{6}$ *seconds. Now, a week is*

$$7\times 24\times 60\times 60=604800$$

seconds, so it would take the camera approximately about 11 *days to process such a picture!* ◄

255 Definition A *bit* is a binary unit: either a 0 or a 1. The *bit complexity* of an algorithm is the number of steps that it takes to process a given input measured in bits.

256 Example (Bit Complexity of Ordinary Addition) Two positive integers m and n are to be added. Find the order of bit operations required to carry out their sum.

Solution: ► *Assume without loss of generality that* $m\leq n$*. First we convert* m *and* n *into bits:* m *has* $\lfloor\lfloor\log_2 m\rfloor\rfloor+1$ *bits and* n *has* $\lfloor\lfloor\log_2 n\rfloor\rfloor+1$*. We line up the bits and add them. There are at most* $\lfloor\lfloor\log_2 n\rfloor\rfloor+1$ *sums performed and at most* $\lfloor\lfloor\log_2 m\rfloor\rfloor+1$ *carries. Hence, there are about* $\mathcal{O}\left(\lfloor\lfloor\log_2 n\rfloor\rfloor+1+\lfloor\lfloor\log_2 m\rfloor\rfloor+1\right)=\mathcal{O}\left(\log n\right)$ *bit operations.* ◄

257 Example (Bit Complexity of Ordinary Multiplication) Two positive integers m and n are to be multiplied. Find the order of bit operations required to carry out their product.

Solution: ► *Assume without loss of generality that* $m\leq n$*. Again, we first we convert* m *and* n *into bits:* m *has* $\lfloor\lfloor\log_2 m\rfloor\rfloor+1$ *bits and* n *has* $\lfloor\lfloor\log_2 n\rfloor\rfloor+1$*. We multiply bit by bit requiring* $(\lfloor\lfloor\log_2 n\rfloor\rfloor+1)(\lfloor\lfloor\log_2 m\rfloor\rfloor+1)=\mathcal{O}\left((\log n)^2\right)$ *partial multiplications. After the partial multiplications we need at most* $\mathcal{O}\left(\lfloor\lfloor\log_2 n\rfloor\rfloor+1+\lfloor\lfloor\log_2 m\rfloor\rfloor+1\right)=\mathcal{O}\left(\log n\right)$ *additions of at most* $\mathcal{O}\left(\lfloor\lfloor\log_2 n\rfloor\rfloor+1\right)$ *bits, that is,* $\mathcal{O}\left((\log n)^2\right)$ *additions. Hence, ordinary multiplication requires* $\mathcal{O}\left((\log n)^2+(\log n)^2\right)=\mathcal{O}\left(\log^2 n\right)$ *bit operations.* ◄

Most algorithms that take just a `for` loop are easy to analyse: the number of operations they take to perform is about the length of the loop. Thus if we have a for loop of the form

```
> S1; for k from 1 to n do S2; od;
```

then this fraction of the algorithm has computational time t_1+nt_2 where t_1 and t_2 are, respectively, the computational times of the statements $S1$ and $S2$.

The test in a conditional statement has usually a bit complexity of $\mathcal{O}(1)$, which must be added to its branchings `then` or `else`.

258 Example Let K be a constant. Find the bit complexity of the fragment

```
> for k from 1 to K do O(1) od;
```

Solution: ► *In this case the complexity of the fragment is* $K\mathcal{O}(1)=\mathcal{O}(K)=\mathcal{O}(1)$*, since* K *is a constant.* ◄

259 Example Find the bit complexity of the fragment

```
> for k from 1 to n do O(1) od;
```

Solution: ► *In this case the complexity of the fragment is* $n\mathcal{O}(1)=\mathcal{O}(n)$*.* ◄

260 Example Find the bit complexity of the imbricated loop

```
> for k from 1 to n do for j from 1 to n do  O(1) od;  od;
```

Solution: ► *The inner* `for` *loop has complexity* $n\mathcal{O}(1) = \mathcal{O}(n)$. *The outer* `for` *loop is simply adding these complexities, and hence the fragment has complexity* $n\mathcal{O}(n) = \mathcal{O}(n^2)$. *Alternatively, there are* $\sum_{1\le k\le n}\sum_{1\le j\le n}\mathcal{O}(1) = n^2\mathcal{O}(1) = \mathcal{O}(n^2)$ *bit operations.* ◄

261 Example Find the bit complexity of the imbricated loop

```
> for k from 1 to n do for j from 1 to i do  O(1) od;  od;
```

Solution: ► *There are* $\sum_{1\le k\le n}\sum_{1\le j\le i}\mathcal{O}(1) = \sum_{1\le k\le n} i\mathcal{O}(1) = \frac{n(n+1)}{2}\mathcal{O}(1) = \mathcal{O}(n^2)$ *operations.* ◄

262 Example Find the bit complexity of the fragment

```
> c:=1; while (c<n) do  O(1); c:=2*c; od;
```

Solution: ► *After* i *iterations, the value of* c *will be* 2^i. *We need* $2^i < n \implies i < \log_2 n$. *Thus the number of iterations and the complexity of the loop is* $\mathcal{O}(\log_2 n) = \mathcal{O}(\log n)$. ◄

263 Example Find the bit complexity of the fragment

```
> c:=n; while (c>1) do  O(1); c:=c/2; od;
```

Solution: ► *After* i *iterations, the value of* c *will be* $\frac{n}{2^i}$. *We need* $\frac{n}{2^i} > 1 \implies i < \log_2 n$. *Thus the number of iterations and the complexity of the loop is* $\mathcal{O}(\log_2 n) = \mathcal{O}(\log n)$. ◄

Sometimes we are simply interested in the *number of operations* (additions, multiplications, etc.) necessary to carry out a task. In such cases, we over-estimate by looking at the worst case scenario.

264 Example What is the worst-case running time of the following program?

$> a := proc(n)$
$x := 0;$
$for\ \ i\ \ from\ \ 1\ \ to\ \ n-1\ \ do$
$for\ \ j\ \ from\ \ i+1\ \ to\ \ n\ \ do$
$for\ \ k\ \ from\ \ 1\ \ to\ \ do$
$x := x+1;\ \ od;\ \ od;\ \ od;$
$RETURN(x);$
$end;$

Solution: ► *Each of the* `for` *loop takes about* $\mathcal{O}(n)$ *operations, hence the worst running time is about* $\mathcal{O}(n^3)$. ◄

265 Example (Eratosthenes Sieve) Calculate the number of operations of Eratosthenes sieve of example 207.

Solution: ► *For given* $n > 0$ *Observe that we loop over* $\sqrt{n}$ *potential divisors. For each divisor* k, *we cross out* $\frac{n}{k}$ *numbers. The number of operations carried out is*

$$\sum_{1\le k\le\sqrt{n}}\frac{n}{k} = n\sum_{1\le k\le\sqrt{n}}\frac{1}{k} \sim n\log\sqrt{n} = \mathcal{O}(n\log n),$$

where we have used the result of example 247. ◄

Homework

266 Exercise What is the complexity of the algorithm for finding the maximum of a list of example 179?

267 Exercise What is the complexity of the algorithm for finding the linear search in an unsorted dictionary of example 177?

268 Exercise What is the complexity of the algorithm for finding the nth power of x of example 210?

269 Exercise What is the worst case complexity of the bubblesort algorithm of example 211?

270 Exercise What is the worst case complexity of the quicksort algorithm of example 212?

4 Answers and Hints

11 Hint: What is $500000500000 + x$?

12 We compute the sum of all integers from 1 to 1000 and weed out the sum of the multiples of 3 and the sum of the multiples of 5, but put back the multiples of 15, which we have counted twice. Put

$$A_n = 1 + 2 + 3 + \cdots + n,$$

$$B = 3 + 6 + 9 + \cdots + 999 = 3A_{333},$$
$$C = 5 + 10 + 15 + \cdots + 1000 = 5A_{200},$$
$$D = 15 + 30 + 45 + \cdots + 990 = 15A_{66}.$$

The desired sum is

$$\begin{aligned} A_{1000} - B - C + D &= A_{1000} - 3A_{333} - 5A_{200} + 15A_{66} \\ &= 500500 - 3 \cdot 55611 - 5 \cdot 20100 + 15 \cdot 2211 \\ &= 266332. \end{aligned}$$

13 We want the sum of the integers of the form $6r + 2, r = 0, 1, \ldots, 16.$ But this is

$$\sum_{r=0}^{16} (6r + 2) = 6 \sum_{r=0}^{16} r + \sum_{r=0}^{16} 2 = 6 \frac{16(17)}{2} + 2(17) = 850.$$

17 49500000.

18 $\dfrac{(2n+1)(-1)^{n+1} + 1}{4}$.

19 $\dfrac{n(n+1)(-1)^{n+1}}{2}$.

20 Use the same method as in theorem 1: put

$$S = 3 + 3^2 + \cdots + 3^n.$$

Then

$$3S = 3^2 + 3^3 + \cdots + 3^n + 3^{n+1}.$$

Subtracting,

$$3S - S = (3^2 + 3^3 + \cdots + 3^n + 3^{n+1}) - (3 + 3^2 + \cdots + 3^n) = 3^{n+1} - 3.$$

The answer is $\dfrac{3^{n+1} - 3}{2}$.

21 By the binomial theorem, $0 = (1-1)^n = \sum_{0 \le k \le n} \binom{n}{k} (-1)^k$.

22 By the binomial theorem, $4^n = (1+3)^n = \sum_{0 \le k \le n} \binom{n}{k} 3^k$, and so $\sum_{1 \le k \le n} \binom{n}{k} 3^k = 4^n - 1$.

23 We have

$$\sum_{1 \le i \le n} \sum_{1 \le k \le n} 1 = \sum_{1 \le i \le n} n = n^2.$$

24 We have

$$\sum_{1 \le i \le n} \sum_{1 \le k \le i} 1 = \sum_{1 \le i \le n} i = \frac{n(n+1)}{2}.$$

25 We have

$$\sum_{1\le i\le n}\sum_{1\le k\le i} k = \sum_{1\le i\le n}\frac{i(i+1)}{2} = \frac{n(n+1)(n+2)}{6}.$$

26 We have

$$\sum_{1\le i\le n}\sum_{1\le k\le n} ik = \left(\sum_{1\le i\le n} i\right)\left(\sum_{1\le k\le n} k\right) = \frac{n^2(n+1)^2}{4}.$$

27

1. $2^{63} = 9223372036854775808$,
2. $2^{64} - 1 = 18446744073709551614$,
3. 1.2×10^{15} kg, or 1200 billion tonnes
4. 3500 years

28 Put $S = 1 + x + x^2 + \cdots + x^{80}$. Then

$$S - xS = (1 + x + x^2 + \cdots + x^{80}) - (x + x^2 + x^3 + \cdots + x^{80} + x^{81}) = 1 - x^{81},$$

or $S(1-x) = 1 - x^{81}$. Hence

$$1 + x + x^2 + \cdots + x^{80} = \frac{x^{81}-1}{x-1}.$$

Therefore

$$\frac{x^{81}-1}{x-1} = \frac{x^{81}-1}{x^{27}-1}\cdot\frac{x^{27}-1}{x^{9}-1}\cdot\frac{x^{9}-1}{x^{3}-1}\cdot\frac{x^{3}-1}{x-1}.$$

Thus

$$1 + x + x^2 + \cdots + x^{80} = (x^{54} + x^{27} + 1)(x^{18} + x^9 + 1)(x^6 + x^3 + 1)(x^2 + x + 1).$$

30 Using the identity $x^2 - y^2 = (x-y)(x+y)$ and letting P be the sought product:

$$\begin{aligned}
(2-1)P &= (2-1)(2+1)\cdot\left(2^2+1\right)\cdot\left(2^{2^2}+1\right)\cdot\left(2^{2^3}+1\right)\cdots\left(2^{2^{99}}+1\right)\\
&= \left(2^2-1\right)\cdot\left(2^2+1\right)\cdot\left(2^{2^2}+1\right)\cdot\left(2^{2^3}+1\right)\cdots\left(2^{2^{99}}+1\right)\\
&= \left(2^{2^2}-1\right)\cdot\left(2^{2^2}+1\right)\cdot\left(2^{2^3}+1\right)\cdots\left(2^{2^{99}}+1\right)\\
&= \left(2^{2^3}-1\right)\cdot\left(2^{2^3}+1\right)\cdot\left(2^{2^4}+1\right)\cdots\left(2^{2^{99}}+1\right)\\
&\ \vdots\\
&= (2^{2^{99}}-1)(2^{2^{99}}+1)\\
&= 2^{2^{100}}-1,
\end{aligned}$$

whence

$$P = 2^{2^{100}} - 1.$$

31 Hints: Using $\log_a b = \dfrac{\log_c a}{\log_c b}$, transform into a telescoping product. $2^{10} = 1024$.

32 Divide the interval into dyadic (powers of 2) blocks, and note that $x \mapsto \lfloor \log_2 x \rfloor$ is constant there. Thus

$$\begin{aligned}
\sum_{k=1}^{1000} \lfloor \log_2 k \rfloor &= \sum_{m=1}^{9} \sum_{2^{m-1} < n < 2^m} \lfloor \log_2 n \rfloor + \sum_{k=512}^{1000} \lfloor \log_2 k \rfloor \\
&= \sum_{m=1}^{9} \sum_{2^{m-1} < n < 2^m} (m-1) + \sum_{k=512}^{1000} 9 \\
&= \sum_{m=1}^{9} (m-1)2^{m-1} + 489(9) \\
&= 0 \cdot 2^0 + 1 \cdot 2^1 + +2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + 5 \cdot 2^5 + 6 \cdot 2^6 + 7 \cdot 2^7 + 8 \cdot 2^8 + 4401 \\
&= 0 + 2 + 8 + 24 + 64 + 160 + 384 + 896 + 2048 + 4401 \\
&= 7987.
\end{aligned}$$

33 From the hint: $k \cdot k! = (k+1)! - k!$ and we get the telescoping sum

$$\sum_{1 \le k \le n} k \cdot k! = \sum_{1 \le k \le n} (k+1)! - k! = (2! - 1!) + (3! - 2!) + (4! - 3!) + \cdots ((n+1)! - n!) = (n+1)! - 1!.$$

34 Put $f(x) = (1+x)^n = \sum_{0 \le k \le n} \binom{n}{k} x^k$. Then

$$f'(x) = n(1+x)^{n-1} = \sum_{0 \le k \le n} k \binom{n}{k} x^{k-1} = \sum_{1 \le k \le n} k \binom{n}{k} x^{k-1},$$

since the term $k = 0$ vanishes. The result follows upon taking $x = 1$.

35 Put $f(x) = (1+x)^n = \sum_{0 \le k \le n} \binom{n}{k} x^k$. Then

$$f'(x) = n(1+x)^{n-1} = \sum_{0 \le k \le n} k \binom{n}{k} x^{k-1} = \sum_{1 \le k \le n} k \binom{n}{k} x^{k-1},$$

since the term $k = 0$ vanishes. Put now

$$g(x) = x f'(x) = nx(1+x)^{n-1} = \sum_{1 \le k \le n} k \binom{n}{k} x^k,$$

and so

$$g'(x) = n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2} = \sum_{1 \le k \le n} k^2 \binom{n}{k} x^{k-1},$$

The result follows upon taking $x = 1$.

37 Put

$$p(x) = (1 - x^2 + x^4)^{109}(2 - 6x + 5x^9)^{1996}.$$

Observe that $p(x)$ is a polynomial of degree $4 \cdot 109 + 9 \cdot 1996 = 18400.$ Thus $p(x)$ has the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{18400} x^{18400}.$$

The sum of all the coefficients of $p(x)$ is

$$p(1) = a_0 + a_1 + a_2 + \cdots + a_{18400},$$

which is also $p(1) = (1 - 1^2 + 1^4)^{109}(2 - 6 + 5)^{1996} = 1.$ The desired sum is thus 1.

38 Put

$$p(x) = (1 - x^2 + x^4)^{2003} = a_0 + a_1 x + a_2 x^2 + \cdots + a_{8012} x^{8012}.$$

Then

❶ $a_0 = p(0) = (1 - 0^2 + 0^4)^{2003} = 1.$

❷ $a_0 + a_1 + a_2 + \cdots + a_{8012} = p(1) = (1 - 1^2 + 1^4)^{2003} = 1.$

❸

$$\begin{aligned} a_0 - a_1 + a_2 - a_3 + \cdots - a_{8011} + a_{8012} &= p(-1) \\ &= (1 - (-1)^2 + (-1)^4)^{2003} \\ &= 1. \end{aligned}$$

❹ The required sum is $\dfrac{p(1)+p(-1)}{2} = 1$.

❺ The required sum is $\dfrac{p(1)-p(-1)}{2} = 0$.

39 We have

$$\begin{array}{lllll} f(2) & = & (-1)^2 1 - 2f(1) & = & 1 - 2f(1) \\ f(3) & = & (-1)^3 2 - 2f(2) & = & -2 - 2f(2) \\ f(4) & = & (-1)^4 3 - 2f(3) & = & 3 - 2f(3) \\ f(5) & = & (-1)^5 4 - 2f(4) & = & -4 - 2f(4) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f(999) & = & (-1)^{999} 998 - 2f(998) & = & -998 - 2f(998) \\ f(1000) & = & (-1)^{1000} 999 - 2f(999) & = & 999 - 2f(999) \\ f(1001) & = & (-1)^{1001} 1000 - 2f(1000) & = & -1000 - 2f(1000) \end{array}$$

Adding columnwise,

$$f(2) + f(3) + \cdots + f(1001) = 1 - 2 + 3 - \cdots + 999 - 1000 - 2(f(1) + f(2) + \cdot + f(1000)).$$

This gives

$$2f(1) + 3(f(2) + f(3) + \cdots + f(1000)) + f(1001) = -500.$$

Since $f(1) = f(1001)$ we have $2f(1) + f(1001) = 3f(1)$. Therefore

$$f(1) + f(2) + \cdots + f(1000) = -\frac{500}{3}.$$

40 The quantity on the sinistral side is

$$\begin{aligned} &\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) \\ &\quad -2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right) \\ &= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) \\ &\quad -2 \cdot \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right) \\ &= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) \\ &\quad -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right) \\ &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}, \end{aligned}$$

as we wanted to shew.

41 If $x = 123456789$, then $(123456789)^2 - (123456787)\cdot(123456791) = x^2 - (x-2)(x+2) = 4$.

42 If $a = 10^3, b = 2$ then

$$1002004008016032 = a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5 = \frac{a^6 - b^6}{a-b}.$$

This last expression factors as

$$\begin{aligned}\frac{a^6-b^6}{a-b} &= (a+b)(a^2+ab+b^2)(a^2-ab+b^2)\\ &= 1002\cdot 1002004\cdot 998004\\ &= 4\cdot 4\cdot 1002\cdot 250501\cdot k,\end{aligned}$$

where $k < 250000$. Therefore $p = 250501$.

43 Shew first that $\csc 2x = \cot x - \cot 2x$. Use telescoping cancellation.

44 Multiplying both sides by $\sin\frac{\pi}{7}$ and using $\sin 2x = 2\sin x\cos x$ we obtain

$$\begin{aligned}\sin\frac{\pi}{7}P &= (\sin\frac{\pi}{7}\cos\frac{\pi}{7})\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{4\pi}{7}\\ &= \frac{1}{2}(\sin\frac{2\pi}{7}\cos\frac{2\pi}{7})\cdot\cos\frac{4\pi}{7}\\ &= \frac{1}{4}(\sin\frac{4\pi}{7}\cos\frac{4\pi}{7})\\ &= \frac{1}{8}\sin\frac{8\pi}{7}.\end{aligned}$$

As $\sin\frac{\pi}{7} = -\sin\frac{8\pi}{7}$, we deduce that

$$P = -\frac{1}{8}.$$

45 Let

$$A = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{9999}{10000}$$

and

$$B = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{10000}{10001}.$$

Clearly, $x^2 - 1 < x^2$ for all real numbers x. This implies that

$$\frac{x-1}{x} < \frac{x}{x+1}$$

whenever these four quantities are positive. Hence

$$\begin{array}{ccc} 1/2 & < & 2/3\\ 3/4 & < & 4/5\\ 5/6 & < & 6/7\\ \vdots & \vdots & \vdots\\ 9999/10000 & < & 10000/10001\end{array}$$

As all the numbers involved are positive, we multiply both columns to obtain

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{9999}{10000} < \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{10000}{10001},$$

or $A < B$. This yields $A^2 = A \cdot A < A \cdot B$. Now

$$A \cdot B = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{7}{8} \cdots \frac{9999}{10000} \cdot \frac{10000}{10001} = \frac{1}{10001},$$

and consequently, $A^2 < A \cdot B = 1/10001$. We deduce that $A < 1/\sqrt{10001} < 1/100$.

49 For $j = k$, $a_k b_j - a_j b_k = 0$, so we may relax the inequality in the last sum. We have

$$\begin{aligned}
\sum_{1 \le k < j \le n} (a_k b_j - a_j b_k)^2 &= \sum_{1 \le k \le j \le n} (a_k^2 b_j^2 - 2a_k b_k a_j b_j + a_j^2 b_k^2) \\
&= \sum_{1 \le k \le j \le n} a_k^2 b_j^2 - 2 \sum_{1 \le k \le j \le n} a_k b_k a_j b_j + \sum_{1 \le k \le j \le n} a_j^2 b_k^2 \\
&= \sum_{k=1}^{n} \sum_{j=1}^{n} a_k^2 b_j^2 - \left(\sum_{k=1}^{n} a_k b_k \right)^2,
\end{aligned}$$

proving the theorem.

50 Let the the sum of integers be $S = (l+1) + (l+2) + (l+n)$. Using Gauss' trick we obtain $S = \dfrac{n(2l+n+1)}{2}$. As $S = 1000$, $2000 = n(2l+n+1)$. Now $2000 = n^2 + 2ln + n > n^2$, whence $n \le \lfloor \sqrt{2000} \rfloor = 44$. Moreover, n and $2l+n+1$ are divisors of 2000 and are of opposite parity. Since $2000 = 2^4 5^3$, the odd factors of 2000 are 1, 5, 25, and 125. We then see that the problem has the following solutions:

$$n = 1,\ l = 999,$$

$$n = 5,\ l = 197,$$

$$n = 16,\ l = 54,$$

$$n = 25,\ l = 27.$$

57 Its x coordinate is

$$\frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \cdots = \frac{\frac{1}{2}}{1 - \frac{-1}{4}} = \frac{2}{5}.$$

Its y coordinate is

$$1 - \frac{1}{4} + \frac{1}{16} - \cdots = \frac{1}{1 - \frac{-1}{4}} = \frac{4}{5}.$$

Therefore, the fly ends up in $\left(\frac{2}{5}, \frac{4}{5}\right)$.

58 From the MacLaurin expansion for $x \mapsto e^x$,

$$f(x) = xe^x = \sum_{n \ge 0} \frac{x^{n+1}}{n!}.$$

Then

$$f'(x) = xe^x + e^x = \sum_{n \ge 0} \frac{(n+1)x^n}{n!}.$$

Multiplying by x,

$$xf'(x) = x^2 e^x + xe^x = \sum_{n \ge 0} \frac{(n+1)x^{n+1}}{n!}.$$

Differentiating this last equality,

$$xf''(x) + f'(x) = 2xe^x + x^2 e^x + xe^x + e^x = \sum_{n \ge 0} \frac{(n+1)^2 x^n}{n!}.$$

Letting $x \to 1$, we obtain

$$\sum_{n \ge 0} \frac{(n+1)^2}{n!} = 2e + e + e + e = 5e.$$

59 For $|x| < 1$,

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

Differentiating,

$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2} \implies \sum_{n=1}^{+\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Letting $x = \frac{1}{2}$,

$$\sum_{n=1}^{+\infty} \frac{n}{2^{n-1}} = 4.$$

60 For $|x| < 1$,

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

Differentiating,

$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}.$$

Multiplying by x,

$$x + 2x^2 + 3x^3 + \cdots = \frac{x}{(1-x)^2}.$$

Differentiating again,

$$1 + 4x + 9x^2 + \cdots = \frac{1+x}{(1-x)^3} \implies \sum_{n=1}^{+\infty} n^2 x^{n-1} = \frac{1+x}{(1-x)^3}$$

Letting $x = \frac{1}{2}$,

$$\sum_{n=1}^{+\infty} \frac{n^2}{2^{n-1}} = 12.$$

61 We divide the sum into decimal blocks. There are 9^k k-digit integers in the interval $[10^k; 10^{k+1}[$ that do not have a 0 in their decimal representation. Thus

$$\sum_{n \in \mathscr{S}} \frac{1}{n} = \sum_{k=0}^{+\infty} \sum_{n \in [10^k; 10^{k+1}[\cap \mathscr{S}} \frac{1}{n} \leq \sum_{k=0}^{+\infty} 9^k \left(\frac{1}{10^k}\right) = 10.$$

62 Since $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$, observe that $\arctan \dfrac{1}{n^2+n+1} = \arctan(n+1) - \arctan n$. Hence the series telescopes to $\lim_{n \to +\infty} \arctan(n+1) - \arctan 1 = \dfrac{\pi}{4}$.

64 Observe that

$$\frac{1}{4n^2-1} = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}.$$

Hence

$$\sum_{n=1}^{+\infty} \frac{1}{4n^2-1} = \left(\frac{1}{2(1)} - \frac{1}{2(3)}\right) + \left(\frac{1}{2(3)} - \frac{1}{2(5)}\right) + \left(\frac{1}{2(5)} - \frac{1}{2(7)}\right) + \cdots = \frac{1}{2(1)} = \frac{1}{2}.$$

67 By unique factorisation of the integers, the desired sum is

$$\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right)\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\right) = \frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{3}} = 3.$$

68 We have, using Abel's Theorem

$$\begin{aligned} \frac{\pi}{4} &= \int_0^1 \frac{\mathrm{d}x}{1+x^2} \\ &= \int_0^1 \left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right) \mathrm{d}x \\ &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots, \end{aligned}$$

as wanted. Note: this series was known to Leibniz, for which he exclaimed that *Deus numero impare gaudet*, "God delights in odd numbers," quoting Virgil.

70 Observe that
$$\frac{y}{1-y^2} = \frac{1}{1-y} - \frac{1}{1-y^2}.$$

72 By (1.2)
$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}} = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{1+\frac{k^2}{n^2}}} = \int_0^1\frac{\mathrm{d}x}{\sqrt{1+x^2}} = \log(x+\sqrt{1+x^2})\Big|_0^1 = \log(1+\sqrt{2}).$$

73 We have
$$\prod_{k=2}^{n}\frac{k^3-1}{k^3+1} = \prod_{k=2}^{n}\frac{k-1}{k+1}\prod_{k=2}^{n}\frac{k^2+k+1}{k^2-k+1}.$$
Now
$$\prod_{k=2}^{n}\frac{k-1}{k+1} = \frac{(n-1)!}{\frac{(n+1)!}{2}} = \frac{2}{n(n+1)}.$$
By observing that $(k+1)^2-(k+1)+1 = k^2+k+1$, we gather that
$$\prod_{k=2}^{n}\frac{k^2-k+1}{k^2+k+1} = \frac{3^2+3+1}{2^2-2+1}\cdot\frac{4^2+4+1}{3^2+3+1}\cdot\frac{5^2+5+1}{4^2+4+1}\cdots\frac{n^2+n+1}{(n-1)^2+(n-1)+1} = \frac{n^2+n+1}{3}.$$
Thus
$$\prod_{k=2}^{n}\frac{k^3-1}{k^3+1} = \frac{2}{3}\cdot\frac{n^2+n+1}{n(n+1)} \to \frac{2}{3},$$
as $n\to+\infty$.

94 Observe that $(1+i)^2 = 1+2i+i^2 = 2i$ and so $(1+i)^{2004} = 2^{1002}i^{1002} = -2^{1002}$. Also, $(1-i)^2 = 1-2i+i^2 = -2i$ and so $(1-i)^{2000} = 2^{1000}i^{1000} = 2^{1000}$. Hence
$$\frac{(1+i)^{2004}}{(1-i)^{2000}} = \frac{-2^{1002}}{2^{1000}} = -4$$

95 Observe that $n+(n+1)i+(n+2)i^2+(n+3)i^3 = n+ni+i-n-2-ni-3i = -2-2i$. Thus grouping every four terms,
$$\begin{aligned}
1+2i+3i^2+4i^3+5i^4+\cdots+2007i^{2006} &= (1+2i+3i^2+4i^3)+(5i^4+6i^5+7i^6+8i^7)+\cdots+(2001i^{2000}+2002i^{2001}+2003i^{2002}+2004i^{2003})+ \\
&= \underbrace{(-2-2i)+(-2-2i)+\cdots+(-2-2i)}_{501 \text{ terms}}+2005+2006i-2007 \\
&= -1002-1002i+2005+2006i-2007 \\
&= -1004-1004i.
\end{aligned}$$

96 Using the binomial theorem and Euler's formula,
$$\begin{aligned}
32\cos^6 2x &= \left(e^{2ix}+e^{-2ix}\right)^6 \\
&= \binom{6}{0}e^{12ix}+\binom{6}{1}e^{10ix}e^{-2ix}+\binom{6}{2}e^{8ix}e^{-4ix}+\binom{6}{3}e^{6ix}e^{-6ix}+\binom{6}{4}e^{4ix}e^{-8ix}+\binom{6}{5}e^{2ix}e^{-10ix}+\binom{6}{6}e^{-12ix} \\
&= e^{12ix}+6e^{8ix}+15e^{4ix}+20+15e^{-4ix}+6e^{-8ix}+e^{-12ix} \\
&= (e^{12ix}+e^{-12ix})+6(e^{8ix}+e^{-8ix})+15(e^{4ix}+e^{-4ix})+20 \\
&= 2\cos 12x+12\cos 8x+30\cos 4x+20,
\end{aligned}$$
from where we deduce the result.

97 From

$$\cos 3x = 4\cos^3 x - 3\cos x, \qquad \sin 3x = 3\sin x - 4\sin^3 x,$$

we gather, upon using the double angle and the sum identities,

$$\begin{aligned}
\tan 3x &= \frac{3\sin x - 4\sin^3 x}{4\cos^3 x - 3\cos x} \\
&= \tan x\left(\frac{3 - 4\sin^2 x}{4\cos^2 x - 3}\right) \\
&= \tan x\left(\frac{3 - 4\sin^2 x}{1 - 4\sin^2 x}\right) \\
&= \tan x\left(1 + \frac{2}{1 - 4\sin^2 x}\right) \\
&= \tan x + \frac{2\sin x}{\cos x - 4\sin^2 x\cos x}. \\
&= \tan x + \frac{2\sin x}{\cos x - 2\sin x\sin 2x} \\
&= \tan x + \frac{2\sin x}{\cos x - 2\left(\frac{\cos x}{2} - \frac{\cos 3x}{2}\right)} \\
&= \tan x + \frac{2\sin x}{\cos 3x}.
\end{aligned}$$

Finally, upon letting $x = \frac{\pi}{9}$ we gather,

$$\sqrt{3} = \tan\frac{\pi}{3} = \tan\frac{\pi}{9} + \frac{2\sin\frac{\pi}{9}}{\cos\frac{\pi}{3}} = \tan\frac{\pi}{9} + 4\sin\frac{\pi}{9},$$

as it was to be shewn.

98 Let $f(x) = (1 + x + x^2)^n$.

1. Clearly $a_0 + a_1 + a_2 + a_3 + a_4 + \cdots = f(1) = 3^n$.

2. We have

$$\begin{aligned}
f(1) &= a_0 + a_1 + a_2 + a_3 + \cdots \\
f(-1) &= a_0 - a_1 + a_2 - a_3 + \cdots
\end{aligned}$$

   Summing these two rows,

$$f(1) + f(-1) = 2a_0 + 2a_2 + 2a_4 + \cdots,$$

   whence

$$a_0 + a_2 + a_4 + \cdots = \frac{1}{2}(f(1) + f(-1)) = \frac{1}{2}(3^n + 1).$$

3. We see that

$$f(1) - f(-1) = 2a_1 + 2a_3 + 2a_5 + \cdots$$

   Therefore

$$a_1 + a_3 + a_5 + \cdots = \frac{1}{2}(f(1) - f(-1)) = \frac{1}{2}(3^n - 1).$$

4. Since we want the sum of every fourth term, we consider the fourth roots of unity, that is, the complex numbers with $x^4 = 1$. These are $\pm 1, \pm i$. Now consider the equalities

$$\begin{aligned}
f(1) &= a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + \cdots \\
f(-1) &= a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8 - a_9 + \cdots \\
f(i) &= a_0 + ia_1 - a_2 - ia_3 + a_4 + ia_5 - a_6 - ia_7 + a_8 + ia_9 + \cdots \\
f(-i) &= a_0 - ia_1 - a_2 + ia_3 + a_4 - ia_5 - a_6 + ia_7 + a_8 - ia_9 + \cdots
\end{aligned}$$

Summing these four rows,

$$f(1)+f(-1)+f(i)+f(-i)=4a_0+4a_4+4a_8+\cdots,$$

whence

$$a_0+a_4+a_8+\cdots=\frac{1}{4}(f(1)+f(-1)+f(i)+f(-i))=\frac{1}{4}(3^n+1+i^n+(-i)^n).$$

5. Consider the equalities

$$\begin{array}{rcl} f(1) & = & a_0+a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8+\cdots \\ -f(-1) & = & -a_0+a_1-a_2+a_3-a_4+a_5-a_6+a_7-a_8+\cdots \\ -if(i) & = & -ia_0+a_1+ia_2-a_3-ia_4+a_5+ia_6-a_7-ia_8+\cdots \\ if(-i) & = & ia_0+a_1-ia_2-a_3+ia_4+a_5-ia_6-a_7+ia_8+\cdots \end{array}$$

Adding

$$f(1)-f(-1)-if(i)+if(i)=4a_1+4a_5+4a_9+\cdots,$$

whence

$$a_1+a_5+a_9+\cdots=\frac{1}{4}\left(3^n-1-i^{n+1}-(-i)^{n+1}\right).$$

99 Since we want every third term starting with the zeroth one, we consider the cube roots of unity, that is, $\omega^3=1$. These are $\omega=-1/2-\sqrt{3}/2, \omega^2=-1/2+\sqrt{3}/2$ and $\omega^3=1$. If $\omega\neq 1$, then $1+\omega+\omega^2=0$. If $\omega=1, 1+\omega+\omega^2=3$. Thus if k is not a multiple of 3, $1^k+\omega^k+\omega^{2k}=0$, and if k is a multiple of 3, then $1^k+\omega^k+\omega^{2k}=3$. By the Binomial Theorem we then have

$$\begin{array}{rcl} (1+1)^{1995}+(1+\omega)^{1995} & & \\ +(1+\omega^2)^{1995} & = & \sum_{k\leq 1995}(1^k+\omega^k+\omega^{2k})\binom{1995}{k} \\ & = & \sum_{k\leq 665}3\binom{1995}{3k}. \end{array}$$

But $(1+\omega)^{1995}=(-\omega^2)^{1995}=-1$, and $(1+\omega^2)^{1995}=(-w)^{1995}=-1$. Hence

$$\sum_{k\leq 665}\binom{1995}{3k}=\frac{1}{3}(2^{1995}-2).$$

114 Let a_n be this number. Clearly $a_1=2$. The nth line is cut by he previous $n-1$ lines at $n-1$ points, adding n new regions to the previously existing a_{n-1}. Hence

$$a_n=a_{n-1}+n,\ a_1=2.$$

We use the same method as in example 107 to solve this recurrence. write

$$\begin{array}{rcl} a_2 & = & a_1+2, \\ a_3 & = & a_2+3, \\ a_4 & = & a_3+4, \\ \vdots & \vdots & \vdots \\ a_{n-1} & = & a_{n-2}+(n-1), \\ a_n & = & a_{n-1}+n, \end{array}$$

Add these equalities and cancel common terms on the left and right,

$$a_2+a_3+a_4+\cdots+a_{n-1}+a_n=a_1+a_2+a_3+a_4+\cdots+a_{n-1}+(2+3+\cdots+n)\implies a_n=a_1+\left(\frac{n(n+1)}{2}-1\right)=\frac{n^2+n+2}{2},$$

upon using Corollary 3. A Maple sequence for solving this recurrence is

```
> rsolve({a(k)=a(k-1)+k, a(1)=2}, a(n));
```

115 Observe that

$$a_n - a_{n-1} = \left(1 + \sum_{k=1}^{n-1} a_k\right) - \left(1 + \sum_{k=1}^{n-2} a_k\right) = a_{n-1}.$$

This means that $a_n = 2a_{n-1}$ and so

$$\begin{array}{lcl} a_n & = & 2a_{n-1} \\ a_{n-1} & = & 2a_{n-2} \\ \vdots & \vdots & \vdots \\ a_2 & = & 2a_1 \end{array}$$

Multiplying all these equalities,

$$a_n a_{n-1} \cdots a_2 = 2^{n-1} a_{n-1} a_{n-2} \cdots a_1 \implies a_n = 2^{n-1} a_1 = 2^{n-1}.$$

116 $x_n = 3^n + n^2$.

117 $x_n = 2^n + 3^n$.

118 Let $n = 2, 2^2, \ldots 2^k$. Then

$$\begin{array}{lcl} a_2 & = & 2a_1 + 6(2) - 1 \\ a_4 & = & 2a_2 + 6(4) - 1 \\ a_8 & = & 2a_4 + 6(8) - 1 \\ \vdots & \vdots & \vdots \\ a_{2^{k-1}} & = & 2a_{2^{k-2}} + 6(2^{k-1}) - 1 \\ a_{2^k} & = & 2a_{2^{k-1}} + 6(2^k) - 1 \end{array}$$

Multiplying successively each equation by $2^{k-1}, 2^{k-2}, \ldots, 2, 1$, obtaining,

$$\begin{array}{lcl} 2^{k-1} a_2 & = & 2^k a_1 + 6(2) \cdot 2^{k-1} - 2^{k-1} \\ 2^{k-2} a_4 & = & 2^{k-1} a_2 + 6(4) \cdot 2^{k-2} - 2^{k-2} \\ 2^{k-3} a_8 & = & 2^{k-2} a_4 + 6(8) \cdot 2^{k-3} - 2^{k-3} \\ \vdots & \vdots & \vdots \\ 2a_{2^{k-1}} & = & 2^2 a_{2^{k-2}} + 6(2^{k-1}) \cdot 2 - 2 \\ a_{2^k} & = & 2a_{2^{k-1}} + 6(2^k) - 1 \end{array}$$

Adding and cancelling,

$$a_{2^k} = 2^k a_1 + 6k \cdot 2^k - (1 + 2 + 2^2 + \cdots + 2^{k-1}) = 2^k + 6 \cdot k \cdot 2^k - 2^k + 1 = 6k2^k + 1,$$

where we have used Theorem 1. Now let $n \geq 1$ be an integer. If $2^k = n$ then $k = \log_2 n$ and

$$a_n = 6n(\log_2 n) + 1.$$

119 $x_n = 2(9^n) + 7n.$

120 We have

$$\begin{array}{lll} x_0 & = & 7 \\ x_1 & = & x_0 + 1 \\ x_2 & = & x_1 + 2 \\ x_3 & = & x_2 + 3 \\ \vdots & \vdots & \vdots \\ x_n & = & x_{n-1} + n \end{array}$$

Adding both columns,

$$x_0 + x_1 + x_2 + \cdots + x_n = 7 + x_0 + x_2 + \cdots + x_{n-1} + (1 + 2 + 3 + \cdots + n).$$

Cancelling and using the fact that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$,

$$x_n = 7 + \frac{n(n+1)}{2}.$$

121 Observe that

$$\begin{array}{lll} a_n & = & 2a_{n-1} + n - 1 \\ a_{n-1} & = & 2a_{n-2} + n - 2 \\ a_{n-2} & = & 2a_{n-3} + n - 3 \\ \vdots & \vdots & \vdots \\ a_3 & = & 2a_2 + 1 \\ a_2 & = & 2a_1 + 1 \end{array}$$

Starting from the top, multiply successively by $2, 2^2, \ldots, 2^{n-1}$, obtaining,

$$\begin{array}{lll} 2a_n & = & 2^2 a_{n-1} + 2(n-1) \\ 2^2 a_{n-1} & = & 2^3 a_{n-2} + 2^2(n-2) \\ 2^3 a_{n-2} & = & 2^4 a_{n-3} + 2^3(n-3) \\ \vdots & \vdots & \vdots \\ 2^{n-2} a_3 & = & 2^{n-1} a_2 + 2^{n-2} \cdot 2 \\ 2^{n-1} a_2 & = & 2^n a_1 + 2^{n-1} \cdot 1 \end{array}$$

Adding and cancelling,

$$2a_n = 2^n a_1 + \sum_{k=1}^{n-1} k 2^{n-k} = 2^n + 2^n \sum_{k=1}^{n-1} \frac{k}{2^k} = 2^n + 2^n \left(-\frac{2n}{2^n} - \frac{2}{2^n} + 2 \right) = 3 \cdot 2^n - 2n - 2,$$

where we have used Corollary 2. Finally,

$$a_n = 3 \cdot 2^{n-1} - n - 1.$$

122 Observe that $a_m(j+1)+1=(a_m(j))^2+2a_m(j)+1=(a_m(j)+1)^2$. Put $v_j=a_m(j)+1$. Then $v_{j+1}=v_j^2$, and $\ln v_{j+1}=2\ln v_j$; Put $y_j=\ln v_j$. Then $y_{j+1}=2y_j$; and hence $2^n y_0=y_n$ or $2^n\ln v_0=\ln v_n$ or $v_n=(v_0)^{2^n}=(1+\frac{d}{2^m})^{2^n}$ or $a_m(n)+1=(1+\frac{d}{2^m})^{2^n}$. Thus $a_n(n)=(\frac{d}{2^n}+1)^{2^n}-1\to e^d-1$ as $n\to\infty$.

123 Let $v_n=\log u_n$. Then $v_n=\log u_n=\log u_{n-1}^{1/2}=\frac{1}{2}\log u_{n-1}=\frac{v_{n-1}}{2}$. As $v_n=v_{n-1}/2$, we have $v_n=v_0/2^n$, that is, $\log u_n=(\log u_0)/2^n$. Therefore, $u_n=3^{1/2^n}$.

124 Let $x_n,y_n,n=0,1,2,\ldots$ denote the fraction of water in urns I and II respectively at stage n. Observe that $x_n+y_n=1$ and that

$$x_0=1;y_0=0$$

$$\begin{aligned}
x_1&=x_0-\frac{1}{2}x_0=\frac{1}{2};y_1=y_1+\frac{1}{2}x_0=\frac{1}{2}\\
x_2&=x_1+\frac{1}{3}y_1=\frac{2}{3};y_2=y_1-\frac{1}{3}y_1=\frac{1}{3}\\
x_3&=x_2-\frac{1}{4}x_2=\frac{1}{2};y_1=y_1+\frac{1}{4}x_2=\frac{1}{2}\\
x_4&=x_3+\frac{1}{5}y_3=\frac{3}{5};y_1=y_1-\frac{1}{5}y_3=\frac{2}{5}\\
x_5&=x_4-\frac{1}{6}x_4=\frac{1}{2};y_1=y_1+\frac{1}{6}x_4=\frac{1}{2}\\
x_6&=x_5+\frac{1}{7}y_5=\frac{4}{7};y_1=y_1-\frac{1}{7}y_5=\frac{3}{7}\\
x_7&=x_6-\frac{1}{8}x_6=\frac{1}{2};y_1=y_1+\frac{1}{8}x_6=\frac{1}{2}\\
x_8&=x_7+\frac{1}{9}y_7=\frac{5}{9};y_1=y_1-\frac{1}{9}y_7=\frac{4}{9}
\end{aligned}$$

A pattern emerges (which may be proved by induction) that at each odd stage n we have $x_n=y_n=\frac{1}{2}$ and that at each even stage we have (if $n=2k$) $x_{2k}=\frac{k+1}{2k+1}$, $y_{2k}=\frac{k}{2k+1}$. Since $\frac{1978}{2}=989$ we have $x_{1978}=\frac{990}{1979}$.

127 Consider iterates of $f(x)=\frac{N-1}{N}(x-Mp)$, where x is the initial amount of coconuts. Then $x=tN^{N+1}-Mp(N-1)$, where t is the smallest positive integer that makes x positive.

128 Number the envelopes $1,2,3,\cdots,n$. We condition on the last envelope. Two events might happen. Either n and $r(1\le r\le n-1)$ trade places or they do not.

In the first case, the two letters r and n are misplaced. Our task is just to misplace the other $n-2$ letters, $(1,2,\cdots,r-1,r+1,\cdots,n-1)$ in the slots $(1,2,\cdots,r-1,r+1,\cdots,n-1)$. This can be done in D_{n-2} ways. Since r can be chosen in $n-1$ ways, the first case can happen in $(n-1)D_{n-2}$ ways.

In the second case, let us say that letter r, $(1\le r\le n-1)$ moves to the n-th position but n moves not to the r-th position. Since r has been misplaced, we can just ignore it. Since n is not going to the r-th position, we may relabel n as r. We now have $n-1$ numbers to misplace, and this can be done in D_{n-1} ways.
As r can be chosen in $n-1$ ways, the total number of ways for the second case is $(n-1)D_{n-1}$. Thus $D_n=(n-1)D_{n-2}+(n-1)D_{n-1}$.

141 The required sequence is

```
>  (123456789)^2-(123456787)*(123456791);
                              4
```

142 The required command line is

```
>  gcd(a, b)*lcm(a,b);
```

143 The required sequence is

$$((10\string^4+324)*(22\string^4+324)$$
$$*(34\string^4+324)*(46\string^4+324)$$
$$*(58\string^4+324))/((4\string^4+324)$$
$$*(16\string^4+324)*(28\string^4+324)$$
$$*(40\string^4+324)*(52\string^4+324));$$

$$373$$

Using Sophie Germain's trick,

$$a^4+4b^4 = a^4+4a^2b^2+4b^4 = (a^2+2b^2)^2-(2ab)^2 = (a^2-2ab+2b^2)(a^2+2ab+2b^2),$$

and so with $b=3^4$, we gather that

$$a^4+324 = (a(a+6)-18)(a(a-6)+18),$$

meaning that most factors cancel out, leaving just

$$\frac{58\cdot 64+18}{-2\cdot 4+18} = \frac{3730}{10} = 373.$$

144 Put $u=\sqrt{1+\sqrt{1+\sqrt{x}}}$, then $x=(u^2-2)^2u^4$ and $\mathrm{d}x = (4u^3(u^2-2)^2+4u^5(u^2-2))\mathrm{d}u$. Hence

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{1+\sqrt{1+\sqrt{x}}}} &= \int \frac{(4u^3(u^2-2)^2+4u^5(u^2-2))\mathrm{d}u}{u} \\
&= 4\int u^2(u^2-2)^2\mathrm{d}u + 4\int u^4(u^2-2)\mathrm{d}u \\
&= 4\int (u^6-4u^4+4u^2)\mathrm{d}u + 4\int (u^6-2u^4)\mathrm{d}u \\
&= 8\int u^6\mathrm{d}u - 24\int u^4\mathrm{d}u + 16\int u^2\mathrm{d}u \\
&= \frac{8}{7}u^7 - \frac{24}{5}u^5 + \frac{16}{3}u^3 + C \\
&= \frac{8}{7}(\sqrt{1+\sqrt{1+\sqrt{x}}})^7 - \frac{24}{5}(\sqrt{1+\sqrt{1+\sqrt{x}}})^5 + \frac{16}{3}(\sqrt{1+\sqrt{1+\sqrt{x}}})^3 + C.
\end{aligned}$$

The required command line is

```
> int(1/sqrt(1+sqrt(1+sqrt(x))),x);
```

$$1/2\sqrt{2}x\,\mathit{hypergeom}\left([2,1/4,3/4],[3,3/2],-\sqrt{x}\right)$$

Note: Maple X expresses the answer in terms of hypergeometric functions, and hence, our solution is perhaps better.

145 The command lines appear below.

```
> int(max(abs(x-1),x^2+2),x=-1..2);
```

$$9$$

146 Put $u=\sqrt{\tan x}$ and so $u^2=\tan x$, $2u\mathrm{d}u = \sec^2 x\mathrm{d}x = (\tan^2 x+1)\mathrm{d}x = (u^4+1)\mathrm{d}x$. Hence the integral becomes

$$\int \sqrt{\tan x}\mathrm{d}x = 2\int \frac{u^2}{u^4+1}\mathrm{d}u.$$

To decompose the above fraction into partial fractions observe (Sophie Germain's trick) that $u^4+1 = u^4+2u^2+1-2u^2 = (u^2+u\sqrt{2}+1)(u^2-u\sqrt{2}+1)$ and hence

$$\begin{aligned}
\int \sqrt{\tan x}\mathrm{d}x &= 2\int \frac{u^2}{u^4+1}\mathrm{d}u \\
&= -\frac{\sqrt{2}}{2}\int \frac{u}{u^2+u\sqrt{2}+1}\mathrm{d}u + \frac{\sqrt{2}}{2}\int \frac{u}{u^2-u\sqrt{2}+1}\mathrm{d}u \\
&= -\frac{\sqrt{2}}{4}\log(u^2+u\sqrt{2}+1) + \frac{\sqrt{2}}{4}\log(u^2-u\sqrt{2}+1) + \frac{\sqrt{2}}{2}\arctan(\sqrt{2}u+1) - \frac{\sqrt{2}}{2}\arctan(-\sqrt{2}u+1) + C \\
&= -\frac{\sqrt{2}}{4}\log(\tan x+\sqrt{2\tan x}+1) + \frac{\sqrt{2}}{4}\log(\tan x-\sqrt{2\tan x}+1) \\
&\quad + \frac{\sqrt{2}}{2}\arctan(\sqrt{2\tan x}+1) - \frac{\sqrt{2}}{2}\arctan(-\sqrt{2\tan x}+1) + C
\end{aligned}$$

The required Maple sequence is

```
> int(sqrt(tan(x)),x);
```

$$1/2\,\frac{\sqrt{2}\sqrt{\tan(x)}\cos(x)\arccos(\cos(x)-\sin(x))}{\sqrt{\cos(x)\sin(x)}}-1/2\sqrt{2}\ln\left(\cos(x)+\sqrt{2}\sqrt{\tan(x)}\cos(x)+\sin(x)\right)$$

147 The required sequence is

```
> (1+I)^2004/(1-I)^2000;
```

$$-4$$

148 The command line is

```
> ifactor(1002004008016032);
```

$$(2)^5(3)^2(7)(109)^2(167)(250501)$$

149 The required command lines are

```
> factor((x + y)^5 - x^5 - y^5);
```

$$5xy(x+y)(y^2+xy+x^2)$$

```
> factor((x + y)^7 - x^7 - y^7);
```

$$7xy(x+y)(y^2+xy+x^2)^2$$

150 Here is one possible answer

```
> is((a^2 + b^2)*(c^2+ d^2)= (a*c + b*d)^2 + (a*d - b*c)^2);
```

$$true$$

151 Here is one possible answer

```
> sum(k*I^(k-1), k=1..2007);
```

$$-1004+1004I$$

152 Here is a possible way.

```
> simplify(sum(floor(log[2](k)),k=1..1000));
```

$$7987$$

153 The following Maple routine finds the exact value.

```
> convert(cos(Pi/5),radical);
```

$$\frac{1}{4}\sqrt{5}+\frac{1}{4}$$

Consider a regular pentagon $ABCDE$. Let x be the length of any one of its sides. Recall that the Golden Section τ satisfies

$$\tau>0,\quad \frac{1}{\tau}=\frac{\tau}{1+\tau}\implies \tau=\frac{1+\sqrt{5}}{2}.$$

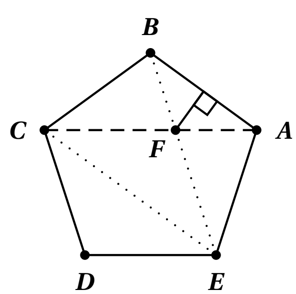

Figure 4.1: Problem 153.

Let F be the point of intersection of the line segment $[AC]$ and $[BE]$. Since $[AC]\parallel[DE]$, $\widehat{FCE}=\widehat{CED}$ and thus $\triangle FCD\equiv\triangle DEC$. Hence $FC=CD=x$. Observe that $\triangle FAB$ is isosceles and similar to $\triangle FCE$. Letting $t=AF$ and observing that $CE=CA=x+t$, we have,

$$\frac{FA}{FC}=\frac{BA}{CE}\implies\frac{t}{x}=\frac{x}{t+x}\implies\frac{1}{\frac{x}{t}}=\frac{\frac{x}{t}}{1+\frac{x}{t}}\implies\frac{x}{t}=\tau.$$

Since $\widehat{FCE} = \widehat{CED}$ and $\widehat{BCA} = \widehat{FCE}$, we have $\widehat{BCA} = \widehat{FCE} = \widehat{CED} = \frac{1}{3}\cdot\frac{3\pi}{5} = \frac{\pi}{5}$. This means that $\widehat{FCE} = \frac{3\pi}{5}$ and hence $\widehat{ABF} = \widehat{FAB} = \frac{\pi}{5}$. Erecting a perpendicular from F to $[AB]$, we deduce from $\triangle FAB$,

$$\cos\frac{\pi}{5} = \frac{\frac{x}{2}}{t} = \frac{x}{2t} = \frac{\tau}{2} = \frac{1+\sqrt{5}}{4}.$$

159 Let $A := \{2,4,6,\ldots,100\}$, $B := \{3,6,9,\ldots,99\}$. We want the number of elements in $X \setminus (A \cup B)$. The following Maple code calculates this. We have suppressed the outputs in order to economise space.

```
> X:={seq(k, k=1..100)};
> A:={seq(2*k,k=1..50)};)
> B:={seq(3*k,k=1..33)};)
> nops(X minus (A union B));
> X minus (A union B);
```

160 Let $A := \{1^2,2^2,3^2,\ldots,31^2\}$ (observe $\lfloor\sqrt{1000}\rfloor = 31$), $B := \{1^3,2^3,\ldots,10^3\}$ (observe $\lfloor\sqrt[3]{1000}\rfloor = 10$), and $C := \{1^5,2^5,3^5\}$ (observe $\lfloor\sqrt[5]{1000}\rfloor = 3$). We want the number of elements in $X \setminus (A \cup B \cup C)$. The following Maple code calculates this. We have suppressed the outputs in order to economise space.

```
> X:={seq(k, k=1..1000)};
> A:={seq(k^2,k=1..31)};)
> B:={seq(k^3,k=1..10)};)
> C:={seq(k^5,k=1..3)};)
> nops(X minus (A union B union C));
> X minus (A union B union C);
```

161 Here is a possible answer. The code will not do anything unless a list X is declared prior to it.

```
> sum(X[k], k=1..nops(X));
```

168 One may use the following code. We omit the Maple output.

```
> A:={1,2,3,4};
> B:={3,4,5,6};
> map(x->f(x), A minus B) union  map(x->f(x), B minus A);
> map(x->f(x), (A minus B) union (B minus A)) ;
```

169 One may use the following code. We omit the Maple output.

```
> MU:=proc(X) sum(X[i], i=1..nops(X))/nops(X) end;
> VARIANCE:= proc(X) sum((X[i]-mu(X))^2, i=1..nops(X))/nops(X) end;
```

170 Here is one way.

$> SWAP2 := proc(x,y)$
$x1 := x+y; y1 := x; x1 := x1 - y1;$
$RETURN(x1,y1);$
$end;$

171 Here is one way.

```
> SUMDIGITS:= proc(x) RETURN(sum(ITHDIGIT(x,i),i=1..length(x))); end;
```

172 Here is one way. Observe that $a - (a \mod 10)$ deletes the last digit of a replacing it with a zero, and so, $(a - a \mod b)/10$ deletes the last digit of a. Furthermore, the integer `ITHDIGIT(b, length(n))*10^(length(b)-1)` has as many digits as b and has the same leftmost digit of b. Thus `b-ITHDIGIT(b, length(b))*10^(length(b)-1)` deletes the first digit of b. We need to apply these two operations in sequence.

$PEELER := proc(x)$
$a := x;\ \ b := (a - (a\ \ mod\ \ 10))/10;$
$RETURN(b - ITHDIGIT(b, length(b)) * 10\text{^}(length(b)-1));$
$end;$

181 Here is one possible answer.

```
> ABSVAL:= proc(x,y) if x>=0 then RETURN(x) else RETURN(-x) fi; end;
```

182 Here is a possible answer.

```
> PRIMES:= proc(N) for k from 1 to N do print(ithprime(k)) od;  end;
```

183 Here is one possible answer.

$MAXI3 := proc(x, y, z)$
$MAXI := proc(a, b)\ \ if\, a >= b\ \ then\ \ RETURN(a);\ \ else\ \ RETURN(b);\ \ fi\ end;$
$if\ \ MAXI(x, y) >= z\ \ then\ \ MAXI(x, y)$
$else\ \ z\ \ fi;$
$end;$

184 Here is one possible answer.

$TWINPRIMES := proc()count := 0;$
$for\ \ k\ \ from\ \ 1\ \ to\ \ 1000000$
$do\ \ if\ \ isprime(k)\ \ and\ \ isprime(k+2)\ \ then\ \ count := count + 1;$
$fi;\ \ od;$
$RETURN(count);$
$end;$

185 Here is a possible Maple™ procedure.

```
>  KUREPA:= proc(A)  for a from 1 to A do if gcd(sum(k!, k=0..a-1), a!)  <> 2 then print('a
'=a)  fi; od; end;
```

Take $A \leq 150$.

191 Here is a possible answer.

$> REVERSEDIGITS := proc(n)$
$b := n;\ \ new := 0;$
$whileb <> 0\ \ do\ \ r := bmod10;\ \ b := floor(b/10);$
$new := new * 10 + r;\ \ od;$
$RETURN(new);$
$end;$

192 Here is a possible answer. The last digit of x is $x \mod 10$. Its first digit is $\lfloor x/10^{\text{length}(x)-1} \rfloor$.

$> FIRSTISLAST := proc(x)$
$if\ \ (x\ \ mod\ \ 10)\ \ = floor(x/10\char`^(length(x) - 1))$
$then\ \ RETURN(true)$
$else\ \ RETURN(false)\ \ fi;\ \ end;$

193 Here is a possible answer.

```
>  DIETOSS:=proc(n)  die:=rand(1..6); k:=1;while(k<=n)  do k:=k+1 ; print(die()); od; end;
```

194 Here is a possible answer.

$> SUMPALINDROMES := proc(M, N)$
$total := 0;$
$for\ \ k\ \ from\ \ M\ \ to\ \ N\ \ do$
$if\ \ ISPALINDROME(k)\ \ then\ \ total := total + k;\ \ fi;\ \ od;$
$RETURN(total);$
$end;$

195 Here is a possible answer.

$> GOLDBACH := proc(n)$
$for\ \ k\ \ from\ \ 3\ \ to\ \ (n-3)$
$do\ \ if\ \ isprime(k)\ \ and\ \ isprime(n-k)\ \ then\ \ print(n, "=", k, "+", n-k)\ \ fi;\ \ od;$
$end;$

196 Here is a possible answer.

```
> POSTAGE := proc(a,b,h)
realisable := false;  x := -1;
    while(x <= h/a  and  not(realisable))  do  x := x+1;  y := -1;
    while(y <= h/b  and  not(realisable))  do  y := y+1;
if  h = a*x+b*y  then  realisable := true;  fi;
  od;  od;
print(h,"is",a,"*",x,"+",b,"*",y);
end;
```

197 Here is a possible answer.

```
> CIRCLEPROBLEM := proc(n)
a := 0;  s := 0;
while(a*a <= n)  do  b := 0;  t := 0;
while(a*a+b*b <= n)  do  b := b+1;  t := t+1;  od;
a := a+1;  s := s+t;  od;
RETURN(s);
end;
```

198 Our algorithm works as follows: the maximum number of consecutive repetitions in a roman numeral is three, and so every number in the given range can be formed with one of the strings in $[M,CM,D,CD,C,XC,L,XL,X,IX,V,IV,I]$.

```
> ROMAN := proc(n)
romannumeral := [ ];  hindunumber := n;
a := [1000,900,500,400,100,90,50,40,10,9,5,4,1];
r := [M,CM,D,CD,C,XC,L,XL,X,IX,V,IV,I];
for  k  from  1  to  nops(a)
do  while  hindunumber >= a[k]
do  hindunumber := hindunumber - a[k];
romannumeral := [op(romannumeral),r[k]];  od;  od;
RETURN(romannumeral[ ]);
end;
```

We can do this more efficiently with Maple's `convert` command.

```
>  convert(1966,roman);
```

199 We use the procedure REVERSELIST from example 190. We first revert the portions $(x_1,x_2,\dots,x_m)$ to $(x_m,x_{m+1},\dots,x_1)$ and $(x_{m+1},x_{m+2},\dots,x_{m+n})$ to $(x_{m+n},x_{m+n-1},\dots,x_{m+1})$. We concatenate them to

$$(x_m,x_{m+1},\dots,x_1,x_{m+n},x_{m+n-1},\dots,x_{m+1}),$$

and we revert this last array to

$$(x_{m+1},x_{m+2},\dots,x_{m+n},x_1,x_2,\dots,x_m)$$

which is what we wanted.

```
> SWITCHLIST := proc(X,m,n)
Y := X;
L1 := REVERSELIST(X[1..m]);
L2 := REVERSELIST(X[m+1..m+n]);
L := REVERSELIST([op(L1),op(L2)]);
RETURN(L);
end;
>  SWITCHLIST([1,2,3,4,5,a,b,c,d,e,f,g,h,i,j],5,10);
```

$$[a,b,c,d,e,f,g,h,i,j,1,2,3,4,5]$$

200 Here is a possible answer.

```
> DIFFERENT := proc(X)
i := 1;  dif := 1;
while  i <> nops(X)
do  i := i+1;  if  X[i] <> X[i-1]  then  dif := dif+1;  fi;  od;
end;
```

201 Here is a possible solution.

$$
\begin{aligned}
&\boldsymbol{LISTCOMMONERS := proc(X,Y)}\\
&\boldsymbol{k1 := 0;\ \ l1 := 0;\ \ n := 0;}\\
&\boldsymbol{while\ \ (k1 <> nops(X))\ \ and\ \ (l1 <> l)}\\
&\boldsymbol{do\ \ if\ \ X[k1+1] < Y[l1+1]}\\
&\boldsymbol{then\ \ k1 := k1+1;}\\
&\boldsymbol{elif X[k1+1] > Y[l1+1]}\\
&\boldsymbol{thenl1 := l1+1;}\\
&\boldsymbol{else\ \ k1 := k1+1;\ \ l1 := l1+1;\ \ n := n+1;\ \ fi;\ \ od;}\\
&\boldsymbol{RETURN(n);}\\
&\boldsymbol{end;}
\end{aligned}
$$

202 Here is a possible solution.

```
>  a:=proc(a,x)  k:=1; while
>  floor(x^k/10^(length(x^k)-length(a))) <> a do k:=k+1; od; RETURN(k);
>  end;
```

206 Here is an iterative one.

$$
\begin{aligned}
&\boldsymbol{fact1 := proc(n) f := 1;}\\
&\boldsymbol{if n <= 1\ \ then\ \ f;}\\
&\quad\boldsymbol{else\ \ for\ \ k\ \ from\ \ 1\ \ to\ \ n\ \ do\ \ f := k*f;\ \ od;\ \ fi;}\\
&\boldsymbol{RETURN(f);}\\
&\boldsymbol{end;}
\end{aligned}
$$

Here is a recursive one.

$$
\begin{aligned}
&\boldsymbol{fact2 := proc(n)}\\
&\boldsymbol{option\ \ remember;}\\
&\boldsymbol{if\ \ n <= 1\ \ then\ \ 1}\\
&\boldsymbol{else\ \ n * fact2(n-1)\ \ fi;}\\
&\boldsymbol{end;}
\end{aligned}
$$

By typing

```
>  time(fact1(200)); time(fact2(200));
```

we see that the iterative version is somewhat faster.

213 Here is one possible way. We recall that a composite integer n must have a prime factor $\le \sqrt{n}$.

$$
\begin{aligned}
&\boldsymbol{PrimeFactors := proc(n)}\\
&\boldsymbol{k := n;\ \ t := 2;}\\
&\boldsymbol{while\ \ not\ \ k = 1}\\
&\quad\boldsymbol{do\ \ if\ \ k\ \ mod\ \ t = 0\ \ then\ \ k := k/t;\ \ print(t);}\\
&\qquad\boldsymbol{elif\ \ t*t > k\ \ then\ \ t := k;}\\
&\qquad\quad\boldsymbol{else\ \ t := t+1;\ \ fi;\ \ od;}\\
&\boldsymbol{end;}
\end{aligned}
$$

241 $a_n = o\left(n^2\right)$ does, since this says that $\lim_{n\to+\infty} \frac{a_n}{n^2} = 0$, whereas $a_n = \mathcal{O}\left(n^2\right)$ says that $\frac{a_n}{n^2}$ is bounded by some positive constant.

242 False. Take $a_n = 2n$, for example. Then $a_n << n$, $\frac{a_n}{n} = 2$, and so $\frac{a_n}{n} \not\to 0$.

243 True. $\frac{a_n}{n} \to 0$ and so by Theorem 215, $a_n << n$.

244 False. Take $a_n = n^{3/2}$. Then $\frac{a_n}{n^2} \to 0$ but $a_n \neq \mathcal{O}(n)$.

245 True. $\frac{a_n}{n} \to 0$ and so by Theorem 215, $a_n << n$. Since $n << n^2$, the assertion follows by transitivity.

251 For $n \ge 3$,

$$\underbrace{e\cdot e\cdots e}_{n \text{ times}} \le e\cdot e 3\cdot 4\cdots n = \frac{e^2 n!}{2} \implies e^n = \mathcal{O}(n!).$$

252 Use the fact that $x \mapsto \dfrac{1}{\sqrt{x}}$ decreases for $x > 0$. Then

$$\frac{1}{\sqrt{k+1}} < \int_k^{k+1} \frac{dx}{\sqrt{x}} < \frac{1}{\sqrt{k}}$$

gives

$$\sum_{k=2}^{n} \frac{1}{\sqrt{k}} < \int_1^{n} \frac{dx}{\sqrt{x}} < \sum_{k=1}^{n-1} \frac{1}{\sqrt{k}},$$

which implies that

$$2\sqrt{n} - 2 + \frac{1}{\sqrt{n}} < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} - 1,$$

from where the required result is easily deduced.

266 $\mathscr{O}(n)$, where n is the size of the list.

267 $\mathscr{O}(n)$, where n is the size of the dictionary.

268 $\mathscr{O}(\log n)$.

269 $\mathscr{O}\left(n^2\right)$.

270 $\mathscr{O}\left(n^2\right)$.

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