2D Turbulence

Governing equations

The governing equations for 2d incompressible turbulence are the Navier–Stokes equations.

$$\boxed{\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial \phi}{\partial x} + \nu \nabla^2 u\\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial \phi}{\partial y} + \nu \nabla^2 v\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\end{aligned}}$$

where u = x-component of the velocity
v = y-component of the velocity
ν = viscosity
ϕ = pressure divided by constant density.

► There exists a function ψ such that:

$$\boxed{u = -\frac{\partial \psi}{\partial y}} \quad \text{and} \quad \boxed{v = \frac{\partial \psi}{\partial x}}$$

which we call the <u>stream function.</u> ◄

► We also define $\boxed{\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}$

which we call the <u>vorticity</u>. ◄

<u>Definitions:</u> Let A, B be two arbitrary fields. We define:

$$\boxed{\begin{aligned}
H(A,B) &= \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\\
J(A,B) &= \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}
\end{aligned}}$$

With this definition: $\zeta = H(u, v)$.

Properties of the H operator.

The H operator corresponds to taking the "curl" in 2d except that it yields a scalar. We state properties that we need:

Prop: H commutes with ∇^2 and $\partial/\partial t$:

$$\frac{\partial}{\partial t} H(A,B) = H\left(\frac{\partial A}{\partial t}, \frac{\partial B}{\partial t}\right)$$

$$\nabla^2 H(A,B) = H(\nabla^2 A, \nabla^2 B)$$

Proof is trivial.

Prop: H has a limited linearity property:

$$H(\lambda_1 A_1 + \lambda_2 A_2, \lambda_1 B_1 + \lambda_2 B_2) = \lambda_1 H(A_1, A_2) + \lambda_2 H(B_1, B_2).$$

Proof is trivial.

Prop.: H applied on the gradient of a scalar field equals 0

$$H\nabla\varphi = 0.$$

Proof

$$H\nabla\varphi = H\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial x}\right) =$$

$$= \frac{\partial^2\varphi}{\partial x\,\partial y} - \frac{\partial^2\varphi}{\partial x\,\partial y} = 0 \qquad \square.$$

Thm: Let $\mathcal{U} = u u_x + v u_y$ and $\mathcal{V} = u v_x + v v_y$.
Then $H(\mathcal{U},\mathcal{V}) = J(\psi, \zeta) = u\zeta_x + v\zeta_y$.

Proof

$$H(\mathcal{U},\mathcal{V}) = H(u u_x + v u_y,\ u v_x + v v_y) =$$

$$= (u v_x + v v_y)_x - (u u_x + v u_y)_y =$$

$$= u_x v_x + u v_{xx} + v_x v_y + v v_{xy} - u_y u_x - u u_{xy} - v_y u_y - v u_{yy} =$$

$$= u(v_{xx} - u_{xy}) + v(v_{xy} - u_{yy}) + (v_x - u_y)(u_x + v_y) =$$

$$= u(v_x - u_y)_x + v(v_x - u_y)_y + (v_x - u_y)\cdot 0 =$$

(because $u_x + v_y = 0$).

$$= u\zeta_x + v\zeta_y.$$

Since $u = -\psi_y$ and $v = \psi_x$, it follows that

$$H(\mathcal{U}, \mathcal{V}) = u\zeta_x + v\zeta_y = -\psi_y \zeta_x + \psi_x \zeta_y =$$
$$= \psi_x \zeta_y - \psi_y \zeta_x = J(\psi, \zeta). \qquad \square.$$

▼ <u>Vorticity-streamfunction formulation.</u>

The vorticity-streamfunction formulation of the Navier-Stokes equations is:

$$\boxed{\begin{aligned} &\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = \nu \nabla^2 \zeta. \\ &\nabla^2 \psi = \zeta \end{aligned}}$$

<u>Proof</u>

Let $\mathcal{U} = u u_x + v u_y$ and $\mathcal{V} = u v_x + v v_y$. Then:

$$\frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial t} H(u, v) = H\left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right) =$$
$$= H\left(\nu \nabla^2 u - \varphi_x - \mathcal{U},\ \nu \nabla^2 v - \varphi_y - \mathcal{V}\right) =$$
$$= H(\nu \nabla^2 u, \nu \nabla^2 v) - H(\varphi_x, \varphi_y) - H(\mathcal{U}, \mathcal{V}) =$$
$$= \nu \nabla^2 H(u, v) - 0 - J(\psi, \zeta) = \nu \nabla^2 \zeta - J(\psi, \zeta) \Rightarrow$$

$$\Rightarrow \frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = \nu \nabla^2 \zeta$$

Also $\zeta = v_x - u_y = (\psi_x)_x - (-\psi_y)_y = \psi_{xx} + \psi_{yy} = \nabla^2 \psi \qquad \square$

↳ Let c be a constant. Then

$$\nabla^2(\psi + c) = \cancel{\psi}\nabla^2 \psi$$
$$J(\psi + c, \zeta) = J(\psi, \zeta)$$

therefore the streamfunction ψ lacks uniqueness by a constant.

Turbulence as a dynamical system

When $\nu = 0$, the resulting solution of the Navier-Stokes equations is called <u>inviscid flow</u>. As it turns out, the solution as $\nu \to 0$ does not converge to inviscid flow. Instead, we get <u>turbulent flow</u>. Turbulent flow is <u>chaotic</u>; a small error in the initial condition ω propagates as a rapidly increasing error at a later point in time. This is why we formulate turbulence as a dynamical system:

$\underline{\Omega}$ = the set of all initial conditions ω for ψ that satisfy the boundary conditions.

$\tilde{\psi}(t,x,y,\omega)$ = streamfunction at point (t,x,y) assuming initial condition $\omega \in \underline{\Omega}$.

$\tilde{\zeta}(t,x,y,\omega)$ = vorticity at point (t,x,y) assuming initial condition $\omega \in \underline{\Omega}$.

Similarly we define $\tilde{u}(t,x,y,\omega)$ and $\tilde{v}(t,x,y,\omega)$. The tildes indicate the actual fields. Removing the tildes will mean that we have removed the probabilistic average:

$$\psi(t,x,y,\omega) = \tilde{\psi}(t,x,y,\omega) - \int_{\underline{\Omega}} \tilde{\psi}(t,x,y,\omega_0)\, d\mu(\omega_0).$$

We use similar definitions for $\zeta(t,x,y,\omega)$, $u(t,x,y,\omega)$, and $v(t,x,y,\omega)$. A consequence is that

$$\overline{\psi(t,x,y)} = 0 \quad \text{and} \quad \overline{\zeta(t,x,y)} = 0 \quad \text{and}$$
$$\overline{u(t,x,y)} = \overline{v(t,x,y)} = 0$$

where the bar is probabilistic average (ω has been averaged out, so we do not list it).

Since the elements of $\underline{\Omega}$ are snapshots $\psi_0(x,y)$ we define our group G as the set of time-shifts as predicted by the Navier-Stokes equations such that:

$$\forall g_\tau \in G : g_\tau \omega = g_\tau \tilde{\psi}_0(x,y) \equiv \tilde{\psi}(\tau, x, y, \omega)$$

We assume the existence and uniqueness of a solution to the Navier-Stokes equation, to allow G to be well-defined, but this assumption is just a conjecture.
We also assume the following conjectures:
1) There is a unique measure μ such that the $(\Omega, \mathcal{M}, \mu, G)$ is a dynamical system.
2) We assume that the ergodic theorem applies. on $(\Omega, \mathcal{M}, \mu, G)$
By construction, $\tilde{\psi}$ is stationary:

$$\tilde{\psi}(t+\tau, x, y, \omega) = \tilde{\psi}(t, x, y, g_\tau \omega), \ \forall g_\tau \in G.$$

It follows then, as we have shown, that the fluctuation ψ is also stationary:

$$\psi(t+\tau, x, y, \omega) = \psi(t, x, y, g_\tau \omega), \ \forall g_\tau \in G$$

These results very easily follow for u, v, and ζ as well. Then we can apply the ergodic theorem on ψ, u, v, ζ and get relations between the time average and probabilistic average.

Symmetry groups for Navier-Stokes equations.

The following groups are symmetry groups of the Navier-Stokes equations; if ψ is a solution then $g\psi$ is also a solution:
1) Space translations

$$g_\rho : \psi(t, \vec{r}) \longrightarrow \psi(t, \vec{r} + \vec{\rho}), \quad \vec{\rho} \in \mathbb{R}^2$$

2) Rotations

$$g_A : \psi(t, \vec{r}) \longrightarrow \psi(t, A\vec{r}), \quad A \in SO(\mathbb{R}^2).$$

Our abstract discussion about symmetry groups applies here if we set $V = \mathbb{R}^2$. With 3d fluid flow, we must instead set $V = \mathbb{R}^3$. The interesting question is whether the Navier-Stokes equations define a dynamical system that is symmetric and ergodic with respect to rotations and translation. We give the following definitions:

Def: If $(\Omega, \mathcal{M}, \mu, G)$ is symmetric and ergodic under translation then we call the corresponding flow homogeneous.

Def: If $(\Omega, \mathcal{M}, \mu, G)$ is symmetric and ergodic under rotation then we call the corresponding flow isotropic.

We make the following hypothesis for fully developed turbulence:

Conjecture: Fully developed turbulence is homogeneous and isotropic.

The correlation function

Assume homogeneity and isotropy. Then the correlation function for the fluctuations of ψ the streamfunction, which we define as:

$$\boxed{B(\vec{r}_1, \vec{r}_2) = \int_{\Omega} \psi(t, \vec{r}_1, \vartheta)\, \psi(t, \vec{r}_2, \vartheta)\, d\mu(\vartheta).}$$

depends only on $\rho = \|\vec{r}_2 - \vec{r}_1\|$. To see why, recall our general discussion about symmetry. Homogeneity and isotropy imply that

$$B(\vec{r}_1, \vec{r}_2) = B(A\vec{r}_1 + \vec{\rho}, A\vec{r}_2 + \vec{\rho}) \quad , \forall A \in SO(\mathbb{R}^2), \forall \vec{\rho} \in \mathbb{R}^2$$

If we define an equivalence relation:

$$(\vec{a}_1, \vec{a}_2) \sim (\vec{b}_1, \vec{b}_2) \Leftrightarrow \exists A \in SO(\mathbb{R}^2), \exists \vec{\rho} \in \mathbb{R}^2 : \begin{cases} \vec{a}_1 = A\vec{b}_1 + \vec{\rho} \\ \vec{a}_2 = A\vec{b}_2 + \vec{\rho} \end{cases}$$

Then the equivalence classes are:

$$C_\rho = \{ (\vec{a}_1, \vec{a}_2) \mid \|\vec{a}_2 - \vec{a}_1\| = \rho \} \quad , \rho \in [0, +\infty)$$

So they are isomorphic to $[0, +\infty)$ and we can define a function $B(\rho)$ such that

$$\boxed{\|\vec{r}_2 - \vec{r}_1\| = \rho \rightarrow B(\vec{r}_1, \vec{r}_2) = B(\rho).}$$

Using arguments similar to the ones that we have used on the time correlation function, it can be shown that:

1) $|B(\rho)| \leq B(0)$
2) $B'(0) = 0$

We may also extend the definitions about the integral time scale and associated differential time-scale and obtain a definition for the integral length scale and the associated differential length scale

$$\boxed{L(\psi) = \frac{1}{B(0)} \int_0^{+\infty} B(\rho)\, d\rho} \quad \text{and} \quad \boxed{\lambda(\psi) = \left[-\frac{2B(0)}{B''(0)} \right]^{1/2}}$$

$\downarrow$ integral length scale $\qquad$ $\downarrow$ associated differential length scale

In a finite simulation of 2d turbulence with periodic boundary conditions, the extent of the domain of the simulation must be greater than $L(\psi)$ by at least an order of magnitude, and the resolution must resolve scales at the order of $\lambda(\psi)$.

The ergodic theorem

Because ψ is stationary, it follows from the ergodic theorem that we can exchange probabilistic average for time-average:

$$B(\vec{r}_1, \vec{r}_2) = \lim_{T \to +\infty} \left[\frac{1}{T} \int_0^T \psi(t, \vec{r}_1, \vartheta)\, \psi(t, r_2, \vartheta)\, dt \right], \forall \vartheta \in \Omega - \omega$$

for almost all $\vartheta \in \Omega$.

If we assume homogeneity, and let H be the symmetry group of translations in space, then the ergodic theorem applies on the system $(\Omega, \mathfrak{m}, \mu, H)$, and we get that we may also exchange probabilistic average for spatial average:

$$\boxed{B(\vec{r}, \vec{r}+\vec{\rho}) = \lim_{L \to +\infty} \left[\frac{1}{L^3} \int_{[-L/2, L/2]^3} \psi(t, \vec{r}, \vartheta)\, \psi(t, \vec{r}+\vec{\rho}, \vartheta)\, d\vec{r} \right], \forall \vartheta \in \Omega - \omega}$$

By differentiating under the sign of the integrals, these results extend to correlation functions that involve velocity and vorticity as well.

Another consequence is that if we set $\vec{r}_1 = \vec{r}_2$, we get the probabilistic variance of ψ which is equal (by the ergodicity) to both the spatial and the time variance of ψ:

$$\boxed{\langle \psi^2 \rangle = \int_\Omega \psi^2(t, \vec{r}, \vartheta)\, d\mu(\vartheta) = B(0).}$$

Finally, note that despite the notation $B(\rho)$, the correct way to think about ~~also~~ the correlation function is as a function of <u>both</u> ρ and t. It is only the $\vartheta \in \Omega$ dependence that is formally averaged out when we <u>define</u> $B(\rho)$. However, this time dependence becomes important only with $t \sim T_I$ or $t \gg T_I$, that is only at time scales much greater than the integral time scale. One way to interpret this is to consider it as an instance of <u>multiple scale analysis.</u> $B(\rho)$ "is" then a time average over the small scales $t \sim T_I$ statistics, except that these statistics vary in a coherent way at $t \gg T_I$.

The reason why we emphasize this is because knowing $\partial B/\partial t$ is the key to deriving the evolution of the energy and enstrophy spectra.

The streamfunction spectrum

Let $B(\rho)$ be the correlation function for a given homogeneous and isotropic field. We define the stream function spectrum using the following theorem:

Thm: The Fourier transform $\tilde{B}(\vec{k})$ defined by

$$\tilde{B}(\vec{k}) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} B(\|\vec{r}\|)\exp(-i\vec{k}\cdot\vec{r})\,d\vec{r}$$

depends only on the modulus $k = \|\vec{k}\|$ and is equal to

$$\boxed{\tilde{B}(\vec{k}) = \frac{1}{2\pi}\int_0^{+\infty} \rho B(\rho)\, J_0(k\rho)\, d\rho}$$

Proof

We expand the double integral in polar coordinates.

Let $\vec{k} = (k\cos\varphi, k\sin\varphi)$

$\vec{r} = (r\cos\vartheta, r\sin\vartheta)$

Then $\vec{k}\cdot\vec{r} = (k\cos\varphi)(r\cos\vartheta) + (k\sin\varphi)(r\sin\vartheta) =$

$= kr(\cos\varphi\cos\vartheta + \sin\varphi\sin\vartheta) = kr\cos(\vartheta-\varphi)$

It follows that

$$\tilde{B}(\vec{k}) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} B(\|\vec{r}\|)\exp(-i\vec{k}\cdot\vec{r})\,d\vec{r} =$$

$$= \frac{1}{4\pi^2}\int_0^{+\infty} d\rho \int_0^{2\pi} d\vartheta\, \rho B(\rho)\exp(-ik\rho\cos(\vartheta-\varphi)) =$$

$$= \frac{1}{4\pi^2}\int_0^{+\infty} \rho B(\rho)\left\{\int_0^{2\pi}\exp(-ik\rho\cos(\vartheta-\varphi))\,d\vartheta\right\} d\rho \overset{\text{Lemma}}{=}$$

$$= \frac{1}{4\pi^2}\int_0^{+\infty} \rho B(\rho)\,[2\pi J_0(k\rho)]\,d\rho = \frac{1}{2\pi}\int_0^{+\infty}\rho B(\rho) J_0(k\rho)\,d\rho \qquad \square$$

Now it remains to show that:

Lemma: $$\boxed{I = \int_0^{2\pi}\exp[-ik\rho\cos(\vartheta-\varphi)]\,d\vartheta = 2\pi J_0(k\rho)}$$

Proof

$$I = \int_0^{2\pi} \exp(-ik\rho\cos(\vartheta-\varphi))\,d\vartheta = \int_0^{2\pi} \exp(-ik\rho\cos\vartheta)\,d\vartheta \quad \text{(periodicity)} =$$

$$= \int_0^{2\pi} \cos(-k\rho\cos\vartheta)\,d\vartheta + i\int_0^{2\pi} \sin(-k\rho\cos\vartheta)\,d\vartheta =$$

$$= \int_0^{2\pi} \cos(k\rho\cos\vartheta)\,d\vartheta \quad \text{(because the } [0,\pi/2]\cup[3\pi/2,2\pi] \text{ contribution cancels out the } [\pi/2,3\pi/2] \text{ contribution)}$$

$$= 4\int_0^{\pi/2} \cos(k\rho\cos\vartheta)\,d\vartheta \quad \text{(the contributions for the intervals } [\pi/2,\pi], [\pi,3\pi/2] \text{ and } [3\pi/2,2\pi] \text{ are all equal)}$$

$$= 4\left(\frac{\pi}{2} J_0(k\rho)\right) = 2\pi J_0(k\rho) \quad \square$$

Now we define the streamfunction spectrum as follows:

Def: The streamfunction spectrum $b(k)$ is the surface integral of $\tilde{B}(\vec{k})$ over the circle $|\vec{k}| = k$:

$$b(k) = \int_{|\vec{k}|=k} \tilde{B}(\vec{k})\,ds$$

↳ Since $\tilde{B}(\vec{k}) = \text{const}$ on $|\vec{k}| = k$, it follows that

$$b(k) = \tilde{B}(\vec{k}) \int_{|\vec{k}|=k} ds = 2\pi k \tilde{B}(\vec{k}) = 2\pi k \cdot \frac{1}{2\pi} \int_0^{+\infty} \rho B(\rho) J_0(k\rho)\,d\rho =$$

$$= \int_0^{+\infty} k\rho B(\rho) J_0(k\rho)\,d\rho$$

so we have that:

$$\boxed{b(k) = \int_0^{+\infty} k\rho\, B(\rho) J_0(k\rho)\,d\rho}$$

Now, we can show that $B(\rho)$ can be expanded in terms of $b(k)$ and Bessel functions $J_0(k\rho)$:

Thm : $$\boxed{B(\rho) = \int_0^{+\infty} b(k)\, J_0(k\rho)\, dk}$$

Proof

$$B(\rho) = \int_{\mathbb{R}^2} \tilde{B}(\vec{k}) \exp(i\vec{k}\cdot\vec{r})\, d\vec{k} = \int_{\mathbb{R}^2} \frac{b(k)}{2\pi k} \exp(i\vec{k}\cdot\vec{r})\, d\vec{k} =$$

$$= \int_0^{+\infty} dk \int_0^{2\pi} d\varphi\; k\, \frac{b(k)}{2\pi k} \exp(ik\rho\cos(\vartheta-\varphi)) =$$

$$= \int_0^{+\infty} \frac{b(k)}{2\pi} \left\{ \int_0^{2\pi} \exp(ik\rho\cos(\vartheta-\varphi))\, d\varphi \right\} dk =$$

$$= \int_0^{+\infty} \frac{b(k)}{2\pi}\, 2\pi J_0(k\rho)\, dk = \int_0^{+\infty} b(k)\, J_0(k\rho)\, dk \qquad \square$$

So, in effect the 2d Fourier expansion reduces to an expansion of Bessel functions, with $b(k)$ being the "weights". A more physical interpretation of $b(k)$ is given by the following result:

Thm : The probabilistic variance $\overline{\psi^2}$ of the streamfunction is given by: $$\boxed{\overline{\psi^2} = \int_0^{+\infty} b(k)\, dk}$$

Proof

$$\overline{\psi^2} = \int_{\Omega} \psi^2(t,\vec{r},\omega)\, d\mu(\omega) = \int_{\Omega} \psi(t,\vec{r},\omega)\, \psi(t,\vec{r}+\vec{0},\omega)\, d\mu(\omega) =$$

$$= B(0) = \int_0^{+\infty} b(k)\, J_0(k\cdot 0)\, dk = J_0(0) \int_0^{+\infty} b(k)\, dk = \int_0^{+\infty} b(k)\, dk \qquad \square$$

▶ So, one way of thinking of $b(k)$ as a spectrum is by saying that $b(k)$ is the contribution of the k wavenumbers to the statistical variance of the fluctuations of the streamfunction. ◀

Our subsequent discussion has two parts:
a) How can we estimate $\mathcal{E}(k)$ in a simulation.
b) What can we evaluate from $\mathcal{E}(k)$?
Both topics are very interesting.

Estimating the streamfunction spectrum

Let's fix time to a specific value t, and write the Fourier expansion of the fluctuation $\psi(\vec{r})$ as:

$$\psi(\vec{r}) = \int_{\mathbb{R}^2} \Psi(\vec{k}) \exp(i\vec{k}\cdot\vec{r})\, d\vec{k}$$

We begin our analysis by attempting to evaluate

$$\overline{\Psi(\vec{k})\Psi(\vec{\ell})} = \int_{\Omega} \Psi(\vec{k},\omega)\Psi(\vec{\ell},\omega)\, d\mu(\omega).$$

First, we state a few Lemmas:

__Lemma__: Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be some real function. Then:

$$\boxed{\int_{\mathbb{R}^2} \varphi(\|\vec{r}\|) \exp(-i\vec{k}\cdot\vec{r})\, d\vec{r} = \int_0^{+\infty} 2\pi\rho\, \varphi(\rho)\, J_0(\|\vec{k}\|\rho)\, d\rho.}$$

__Proof__: We have already proven this statement on page 9 □

__Lemma__:
$$\boxed{\int_{\mathbb{R}^2} \exp(-i\vec{k}\cdot\vec{r})\, d\vec{r} = 4\pi^2 \delta(\vec{k})}$$

__Proof__

$$\int_{\mathbb{R}^2} \delta(\vec{k}) \exp(i\vec{k}\cdot\vec{r})\, d\vec{k} = \exp(i\vec{0}\cdot\vec{r}) = e^0 = 1 \rightarrow$$

$$\Rightarrow \frac{1}{4\pi^2} \int_{\mathbb{R}^2} 1 \cdot \exp(-i\vec{k}\cdot\vec{r})\, d\vec{r} = \delta(\vec{k}) \rightarrow$$

$$\Rightarrow \int_{\mathbb{R}^2} \exp(-i\vec{k}\cdot\vec{r})\, d\vec{r} = 4\pi^2 \delta(\vec{k}).$$ □

Lemma: (The addition lemma on 2d Fourier transform)

Let $\rho(\vec{r}_1,\vec{r}_2) = \|\vec{r}_1 - \vec{r}_2\|$ be the distance between two points $\vec{r}_1, \vec{r}_2 \in \mathbb{R}^2$. Then:

$$\boxed{\int_{\mathbb{R}^2} J_0(k_0\rho(\vec{r}_1,\vec{r}_2)) \exp(-i\vec{k}\cdot\vec{r}_1)\,d\vec{r}_1 = 2\pi \frac{\delta(\|\vec{k}\|-k_0)}{k_0} \exp(-i\vec{k}\cdot\vec{r}_2)}$$

Proof

The proof is constructive, as follows:

$$\int_0^{+\infty} \delta(k-k_0) J_0(k\rho)\,dk = J_0(k_0\rho) \Rightarrow$$

$$\Rightarrow \delta(k-k_0) = \int_0^{+\infty} k\rho\, J_0(k_0\rho) J_0(k\rho)\,d\rho = \int_0^{+\infty} k_0\rho\, J_0(k_0\rho) J_0(k\rho)\,d\rho$$

$$= \frac{k_0}{2\pi}\int_0^{+\infty} 2\pi\rho\, J_0(k_0\rho) J_0(k\rho)\,d\rho = \quad \text{Lemma, for any } \|\vec{k}\| = k.$$

$$= \frac{k_0}{2\pi}\int_{\mathbb{R}^2} J_0(k_0\|\vec{r}\|) \exp(-i\vec{k}\cdot\vec{r})\,d\vec{r} = \quad \text{substitution } \vec{r}_1 = \vec{r} + \vec{r}_2$$

$$= \frac{k_0}{2\pi}\int_{\mathbb{R}^2} J_0(k_0\|\vec{r}_1 - \vec{r}_2\|) \exp(-i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2))\,d\vec{r}_1$$

$$= \frac{k_0}{2\pi} \exp(i\vec{k}\cdot\vec{r}_2) \int_{\mathbb{R}^2} J_0(k_0\rho(\vec{r}_1,\vec{r}_2)) \exp(-i\vec{k}\cdot\vec{r}_1)\,d\vec{r}_1 \quad , \forall \vec{k} : \|\vec{k}\| = k$$

therefore

$$\int_{\mathbb{R}^2} J_0(k_0\rho(\vec{r}_1,\vec{r}_2)) \exp(-i\vec{k}\cdot\vec{r}_1)\,d\vec{r}_1 = 2\pi\frac{\delta(\|\vec{k}\|-k_0)}{k_0} \exp(-i\vec{k}\cdot\vec{r}_2). \quad \square$$

Using these results we now show the following:

<u>Thm</u> : The probabilistic correlation between $\Psi(\vec{p},\vartheta)$ and $\Psi^*(\vec{q},\vartheta)$ in homogeneous and isotropic flow is given by :

$$\boxed{\overline{\Psi(\vec{p})\,\Psi^*(\vec{q})} = \frac{1}{2\pi}\,\frac{b(\|\vec{p}\|)}{\|\vec{p}\|}\;\delta(\vec{p}-\vec{q}).}$$

<u>Proof</u>

Recall that $\Psi(\vec{p}) = \dfrac{1}{4\pi^2}\displaystyle\int_{\mathbb{R}^2} \psi(\vec{r}_1)\exp(-i\vec{p}\cdot\vec{r}_1)\,d\vec{r}_1$

and $\overset{*}{\Psi}(\vec{q}) = \dfrac{1}{4\pi^2}\displaystyle\int_{\mathbb{R}^2} \psi(\vec{r}_2)\exp(+i\vec{q}\cdot\vec{r}_2)\,d\vec{r}_2$

and $\displaystyle\int_{\underline{\Omega}} \psi(\vec{r}_1,\vartheta)\,\psi(\vec{r}_2,\vartheta)\,d\mu(\vartheta) = B(\|\vec{r}_1-\vec{r}_2\|).$

It follows that:

$$\overline{\Psi(\vec{p})\Psi^*(\vec{q})} = \int_{\underline{\Omega}} \Psi(\vec{p},\vartheta)\,\Psi(\vec{q},\vartheta)\,d\mu(\vartheta) =$$

$$= \frac{1}{(4\pi^2)^2}\int_{\underline{\Omega}} d\mu(\vartheta)\int_{\mathbb{R}^2} d\vec{r}_1\int_{\mathbb{R}^2} d\vec{r}_2\left[\psi(\vec{r}_1,\vartheta)\,\psi(\vec{r}_2,\vartheta)\exp\left[i\vec{q}\cdot\vec{r}_2 - i\vec{p}\cdot\vec{r}_1\right]\right]$$

$$= \frac{1}{(4\pi^2)^2}\iint_{\mathbb{R}^4} d\vec{r}_1\,d\vec{r}_2\left\{\left[\int_{\underline{\Omega}}\psi(\vec{r}_1,\vartheta)\,\psi(\vec{r}_2,\vartheta)\,d\mu(\vartheta)\right]\exp\left[i\vec{q}\cdot\vec{r}_2 - i\vec{p}\cdot\vec{r}_1\right]\right\}$$

$$= \frac{1}{(4\pi^2)^2}\iint_{\mathbb{R}^4} d\vec{r}_1\,d\vec{r}_2\left[B(\rho(\vec{r}_1,\vec{r}_2))\exp\left[i\vec{q}\cdot\vec{r}_2 - i\vec{p}\cdot\vec{r}_1\right]\right] =$$

$$= \frac{1}{(4\pi^2)^2}\iint_{\mathbb{R}^4} d\vec{r}_1\,d\vec{r}_2\left[\int_0^{+\infty} b(k)\,J_0(k\rho(\vec{r}_1,\vec{r}_2))\,dk\right]\exp\left[i\vec{q}\cdot\vec{r}_2 - i\vec{p}\cdot\vec{r}_1\right] =$$

$$= \frac{1}{(4\pi^2)^2}\int_0^{+\infty} b(k)\left\{\iint_{\mathbb{R}^4} d\vec{r}_1\,d\vec{r}_2\left[J_0(k\rho(\vec{r}_1,\vec{r}_2))\exp\left[i\vec{q}\cdot\vec{r}_2 - i\vec{p}\cdot\vec{r}_1\right]\right]\right\}dk.$$

Now pause and evaluate the integral in the braces:

$$I = \iint_{\mathbb{R}^4} d\vec{r}_1\,d\vec{r}_2\left[J_0(k\rho(\vec{r}_1,\vec{r}_2))\exp\left[i\vec{q}\cdot\vec{r}_2 - i\vec{p}\cdot\vec{r}_1\right]\right] =$$

$$= \int_{\mathbb{R}^2} d\vec{r}_2 \exp[i\vec{q}\cdot\vec{r}_2] \left\{ \int_{\mathbb{R}^2} J_0(k\rho(\vec{r}_1,\vec{r}_2)) \exp[-i\vec{p}\cdot\vec{r}_1]\, d\vec{r}_1 \right\} =$$

addition lemma

$$= \int_{\mathbb{R}^2} d\vec{r}_2 \exp[i\vec{q}\cdot\vec{r}_2] \left\{ 2\pi \frac{\delta(\|\vec{p}\|-k)}{k} \exp[-i\vec{p}\cdot\vec{r}_2] \right\} =$$

$$= 2\pi \frac{\delta(\|\vec{p}\|-k)}{k} \int_{\mathbb{R}^2} d\vec{r}_2 \exp[i(\vec{q}-\vec{p})\cdot\vec{r}_2] =$$

$$= \frac{(2\pi)^3}{k} \delta(\|\vec{p}\|-k)\, \delta(\vec{p}-\vec{q}).$$

Substitute to the original integral:

$$\overline{\psi(\vec{p})\psi^*(\vec{q})} = \frac{1}{(2\pi)^4} \int_0^{+\infty} b(k) \left\{ \frac{(2\pi)^3}{k} \delta(\|\vec{p}\|-k)\,\delta(\vec{p}-\vec{q}) \right\} dk =$$

$$= \frac{\delta(\vec{p}-\vec{q})}{2\pi} \int_0^{+\infty} \frac{b(k)}{k} \delta(\|\vec{p}\|-k)\, dk =$$

$$= \frac{b(\|\vec{p}\|)}{2\pi\|\vec{p}\|} \delta(\vec{p}-\vec{q}). \qquad \square$$

The two immediate corollaries of this result are:

<u>Corralary</u>: The Fourier components of the streamfunction fluctuations are uncorrelated:

$$\vec{p} \neq \vec{q} \Rightarrow \overline{\psi(\vec{p})\psi(\vec{q})} = 0$$

when the flow is homogeneous and isotropic.

<u>Corollary</u>: The spectrum $b(k)$ is proportional to the probabilistic average of $|\psi(\vec{p})|^2$ for every $\vec{p} \in \mathbb{R}^2$ such that $\|\vec{p}\| = k$

$$\boxed{b(k) = 2\pi k \left[\overline{|\psi(\vec{p})|^2}\right]}, \ \forall \vec{p} \in \mathbb{R}^2 : \|\vec{p}\| = k.$$

Now recall our original problem; estimating $b(k)$ in a numerical simulation. The above corollary can provide us with a method for estimating $b(k)$, provided that we exchange the probability average that is implied by the overbar with a spatial average over all wavenumbers with $\|\vec{p}\| = k$. So in doing this,

note that we are invoking an ergodic hypothesis!!
Let $C_K = \{\vec{p} \in \mathbb{R}^2 \mid \|\vec{p}\| = K\}$ be a circle with radius K. Then:

$$\mathcal{E}(K) = 2\pi K \int_{\Omega} |\Psi(\vec{p},\vartheta)|^2 \, d\mu(\vartheta) \cong 2\pi K \frac{1}{|C_K|} \int_{C_K} |\Psi(\vec{p},\vartheta)|^2 \, d\vec{p} =$$

$$= \int_{C_K} |\Psi(\vec{p},\vartheta)|^2 \, d\vec{p}$$

because $|C_K| = 2\pi K$ is the length of the circle. The integral should be interpreted as a line integral. We have then an analytic estimate for $\mathcal{E}(K)$:

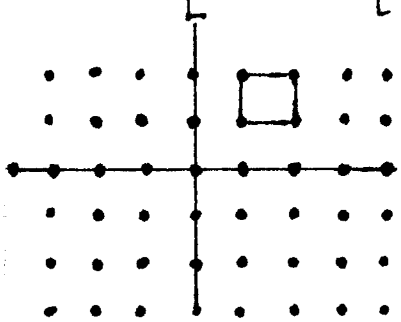
$$\boxed{\mathcal{E}(K) \cong \int_{C_K} |\Psi(\vec{p},\vartheta)|^2 \, dp}$$

A problem with this result is that we can not apply it directly in a numerical simulation without making any further approximations. Suppose that in a simulation the domain is a square with side L and periodic boundary conditions. Let $S_L = [-L/2, L/2] \times [-L/2, L/2]$ be the domain. Then the streamfunction $\psi(\vec{r})$ has a Fourier series expansion:

$$\boxed{\psi(\vec{r}) = \sum_{\vec{p} \in A_L} \hat{\psi}_p \exp(i\vec{p}\cdot\vec{r})}$$

where A_L is a lattice of points on the $\mathbb{R}^2$ plane given by:

$$A_L = \frac{2\pi}{L}\mathbb{Z}^2 = \left\{ \frac{2\pi(i,j)}{L} \;\middle|\; i \in \mathbb{Z} \wedge j \in \mathbb{Z} \right\}$$



The points of A_L divide the $\mathbb{R}^2$ plane into squares with side $2\pi/L$. As $L \to +\infty$, the points become more and more dense. And

$$\lim_{L \to +\infty} A_L = \mathbb{R}^2$$

The series coefficients can be obtained from $\psi(\vec{r})$ from the following relation:

$$\boxed{\hat{\psi}_p = \frac{1}{4\pi^2} \int_{S_L} \psi(\vec{r}) \exp(i\vec{p}\cdot\vec{r}) . \, d\vec{r}.}$$

<u>Prop</u> : The 2d Fourier transform of
$$\psi(\vec{r}) = \sum_{\vec{p}\in A_L} \hat{\psi}_p \exp(i\vec{p}\cdot\vec{r})$$
is given by:
$$\boxed{\Psi_L(\vec{k}) = \sum_{p\in A_L} \hat{\psi}_p \,\delta(\vec{k}-\vec{p})}$$

<u>Proof</u>

$$\Psi_L(\vec{k}) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \psi(\vec{r})\exp(-i\vec{k}\cdot\vec{r})\,d\vec{r} =$$
$$= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\left[\sum_{\vec{p}\in A_L}\hat{\psi}_p \exp(i\vec{p}\cdot\vec{r})\right]\exp(-i\vec{k}\cdot\vec{r})\,d\vec{r} =$$
$$= \frac{1}{4\pi^2}\sum_{\vec{p}\in A_L}\hat{\psi}_p\int_{\mathbb{R}^2}\exp[i(\vec{p}-\vec{k})\cdot\vec{r}]\,d\vec{r} =$$
$$= \frac{1}{4\pi^2}\sum_{\vec{p}\in A_L}\hat{\psi}_p\, 4\pi^2\delta(\vec{k}-\vec{p}) = \sum_{\vec{p}\in A_L}\hat{\psi}_p\,\delta(\vec{k}-\vec{p}). \qquad \square$$

Technically, our general discussion of homogeneous and isotropic flow does not apply on the field defined by $\Psi_L(\vec{k})$, because it fails to be isotropic (the lattice of wavenumbers A_L is not invariant under rotation). Therefore, estimating $b(k)$ on a literal interpretation of the simulation field is not possible, unless we adopt the following viewpoint: we assume that we are approximating a homogeneous and isotropic field $\psi(\vec{r})$, $\vec{r}\in\mathbb{R}^2$ of infinite extent which however is known only in the bounded domain S_L. Then the following is an exact statement:

$$\psi(\vec{r}) = \sum_{\vec{p}\in A_L}\hat{\psi}_p \exp[i\vec{p}\cdot\vec{r}] \quad , \quad \underline{\underline{\forall \vec{r}\in S_L}}$$

and we take it as the definition of the coefficients $\hat{\psi}_p$ that correspond to domain size L. Since in a simulation we only have access to these coefficients, we would like to approximate $b(k)$ in terms of $\hat{\psi}_p$.

► <u>notation</u> : Let $\hat{\psi}_p(n)$ be the coefficient for wavenumber $\vec{p}$ that corresponds to length $L = 2^n \equiv L_n$. We assume, of course, that $\vec{p} \in A_{L_n}$. Also, let $\Psi_L(\vec{k})$ be the transform of $\psi(\vec{r})$ periodized over S_L, as before:

$$\Psi_L(\vec{k}) = \sum_{p \in A_L} \hat{\psi}_p \, \delta(\vec{k} - \vec{p}).$$

Now we show how $\hat{\psi}_p$ relates with the transform $\Psi(\vec{k})$ of the actual (but unknown) function $\psi(\vec{r})$:

<u>Prop</u>: Let $\vec{p} \in A_L$ for some L, and consider a sequence of refinements $L_n = L \cdot 2^n$. Let $\psi_p(n)$ be the Fourier coefficient that corresponds to length L_n. Then:

$$\lim_{n \to +\infty} \hat{\psi}_p(n) = \Psi(\vec{p}).$$

<u>Proof</u>

Note that

$$\lim_{n \to +\infty} S_{L_n} = \lim_{n \to +\infty} \left[-L_n/2, L_n/2\right]^2 = (-\infty, +\infty)^2 = \mathbb{R}^2$$

therefore:

$$\lim_{n \to +\infty} \hat{\psi}_p(n) = \lim_{n \to +\infty} \frac{1}{4n^2} \int_{S_{L_n}} \psi(\vec{r}) \exp(-i\vec{p} \cdot \vec{r}) \, d\vec{r} =$$

$$= \frac{1}{4n^2} \int_{\mathbb{R}^2} \psi(\vec{r}) \exp(-i\vec{p} \cdot \vec{r}) \, d\vec{r} = \Psi(\vec{p}) \qquad \square$$

<u>Thm</u>: Let $L_n = L \cdot 2^n$ and $\psi_p(n)$ be the Fourier coefficients that correspond to length L_n. Also, let $D(k, \varepsilon)$ be a ring in wavenumber-space defined by:

$$D(k, \varepsilon) = \{ \vec{p} \in \mathbb{R}^2 \mid k \leq \|\vec{p}\| \leq k + \varepsilon \}.$$

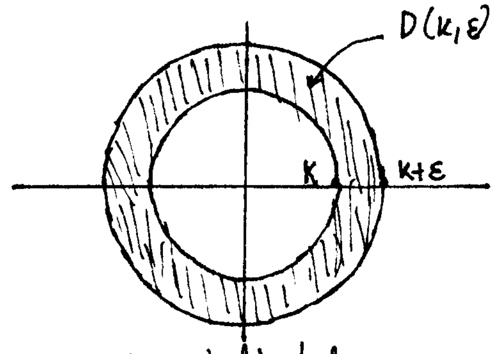
Then, the streamfunction spectrum $b(k)$ can be approximated by:

$$b(k) \cong \int_{C_k} |\Psi(\vec{p})|^2 \, d\vec{p} = \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \left[\frac{4n^2}{\varepsilon L_n^2} \sum_{\vec{p} \in A_L \cap D(k, \varepsilon)} |\hat{\psi}_p|^2 \right]$$

<u>Proof</u>

$$b(k) = \lim_{\varepsilon \to 0} b(k + \varepsilon) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_k^{k+\varepsilon} b(\kappa) \, d\kappa \cong$$

$$\cong \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{\kappa}^{\kappa+\varepsilon}\int_{C_{\kappa'}} |\Psi(\vec{p})|^2 d\vec{p}\, d\kappa' =$$



$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{D(\kappa,\varepsilon)} |\Psi(\vec{p})|^2 d\vec{p} \qquad (1)$$

where the last integral is a surface integral (not a line integral) over the ring region indicated.
To evaluate the integral we apply the Riemann definitions using as a sequence of partition ~~requirements~~ refinements the partitions of $\mathbb{R}^2$ induced by A_L for $L\to+\infty$. We get that

$$\int_{D(\kappa,\varepsilon)} |\Psi(\vec{p})|^2 d\vec{p} = \lim_{L\to+\infty}\left[\sum_{\vec{p}\in A_L\cap D(\kappa,\varepsilon)} |\Psi(\vec{p})|^2 \Delta p_L\right] =$$

$$= \lim_{L\to+\infty}\left[\sum_{\vec{p}\in A_L\cap D(\kappa,\varepsilon)} |\Psi(\vec{p})|^2 (4n^2/L^2)\right] =$$

$$= \lim_{L\to+\infty}\left[\frac{4n^2}{L^2}\sum_{\vec{p}\in A_L\cap D(\kappa,\varepsilon)} |\Psi(\vec{p})|^2\right] \qquad (2)$$

However, because $\lim_{n\to+\infty}\hat{\psi}_p(n) = \Psi(\vec{p})$, it follows that

$$\int_{D(\kappa,\varepsilon)} |\Psi(\vec{p})|^2 d\vec{p} = \lim_{L\to+\infty}\left[\frac{4n^2}{L^2}\sum_{\vec{p}\in A_L\cap D(\kappa,\varepsilon)} |\hat{\psi}_p|^2\right] \Rightarrow$$

$$\Rightarrow b(\kappa) \cong \lim_{\varepsilon\to 0}\lim_{L\to+\infty}\left[\frac{4n^2}{\varepsilon L^2}\sum_{\vec{p}\in A_L\cap D(\kappa,\varepsilon)} |\hat{\psi}_p|^2\right]. \qquad \square$$

This theorem is the main result. It tells us that $b(\kappa)$ can be estimated by averaging the modulus of all the coefficients $\hat{\psi}_p$ that correspond to wavenumbers with length almost equal to κ.
The parameter ε must be tuned by trial and error such that there is a good trade-off between computational error (large ε) and statistical error (small number of terms to average over).
The important statement in this result is that the correct way to average out the accumulated sum is by multiplying it with $4n^2/(\varepsilon L^2)$.

▼ Length scales and the streamfunction spectrum

The streamfunction spectrum $b(k)$ can be used to estimate the integral length scale $L(\psi)$ and the associated differential length scale. Recall that

$$\boxed{L(\psi) = \frac{1}{B(0)} \int_0^{+\infty} B(\rho)\,d\rho} \quad \text{and} \quad \boxed{\lambda(\psi) = \left[-\frac{2B(0)}{B''(0)}\right]^{1/2}}$$

From the definition of $J_n(x)$:

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x\sin\theta - n\theta)\,d\theta$$

it is possible to prove the following properties:

1) $[J_0(x)]' = -J_1(x)$
2) $[J_n(x)]' = (J_{n-1}(x) - J_{n+1}(x))/2 \quad , n>0$
3) $\int_0^{+\infty} J_n(ax)\,dx = \frac{1}{a} \quad , n\geq 0, \quad a>0$

Using these properties we can show that:

Thm :

$$\boxed{\begin{aligned} & B(0) = \int_0^{+\infty} b(k)\,dk \ , \qquad B''(0) = -\frac{1}{2}\int_0^{+\infty} k^2 b(k)\,dk \\ & \int_0^{+\infty} B(\rho)\,d\rho = \int_0^{+\infty} \frac{b(k)}{k}\,dk. \end{aligned}}$$

Proof

Since $J_0(0) = \frac{1}{\pi}\int_0^{\pi} \cos(0\sin\theta - 0\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi} \cos 0\,d\theta = \frac{1}{\pi}\cdot\pi = 1 \Rightarrow$

$$\Rightarrow B(0) = \int_0^{+\infty} b(k)\,J_0(0k)\,dk = \int_0^{+\infty} b(k)\,dk.$$

Also

$$\int_0^{+\infty} B(\rho)\,d\rho = \int_0^{+\infty}\left[\int_0^{+\infty} b(k)\,J_0(k\rho)\,dk\right]d\rho = \int_0^{+\infty} b(k)\left[\int_0^{+\infty} J_0(k\rho)\,d\rho\right]dk =$$

$$= \int_0^{+\infty} \frac{b(k)}{k}\,dk.$$

Evaluating $B''(0)$ is the hardest one.

First note that

$$J_2(0) = \frac{2}{\pi}\int_0^{\pi} \cos(0\sin\vartheta - 2\vartheta)\,d\vartheta = \frac{2}{\pi}\int_0^{\pi}\cos 2\vartheta\, d\vartheta = \frac{2}{\pi}\,\frac{\sin 2\pi - \sin 0}{2} = 0$$

Then:

$$B''(\rho) = \frac{d^2}{d\rho^2}\int_0^{+\infty} b(k)\, J_0(k\rho)\,dk = \int_0^{+\infty} b(k)\left[\frac{d^2}{d\rho^2} J_0(k\rho)\right] dk =$$

$$= \int_0^{+\infty} b(k)\,\frac{d}{d\rho}\left[-kJ_1(k\rho)\right] dk = -\int_0^{+\infty} k\,b(k)\,\frac{d}{d\rho}\left[J_1(k\rho)\right] dk =$$

$$= -\int_0^{+\infty} k^2 b(k)\,\frac{1}{2}\left[J_0(k\rho) - J_2(k\rho)\right] dk \Rightarrow$$

$$\Rightarrow B''(0) = -\frac{1}{2}\int_0^{+\infty} k^2 b(k)\left[J_0(0) - J_2(0)\right] dk = -\frac{1}{2}\int_0^{+\infty} k^2 b(k)\left[1-0\right] dk =$$

$$= -\frac{1}{2}\int_0^{+\infty} k^2 b(k)\,dk. \qquad \square$$

In a simulation it is possible to "evaluate" these integrals directly from the Fourier coefficients $\hat{\psi}_p$, $\vec{p} \in A_L$ without estimating $b(k)$. The following theorem can be used on all three integrals:

Thm: $$\boxed{\int_0^{+\infty} k^{\alpha} b(k)\,dk = \lim_{L\to+\infty} \frac{4\pi^2}{L^2} \sum_{\vec{p}\in A_L^*} |\hat{\psi}_p|^2 \|\vec{p}\|^{\alpha}}$$, where $A_L^* = \frac{2\pi}{L}\mathbb{Z}^2 - \{(0,0)\}$.

Proof

Recall that $b(k) \cong \int_{C_k} |\psi(\vec{p})|^2\, d\vec{p}$. where C_k is a circle of radius k.

Then:

$$I_\alpha = \int_0^{+\infty} k^{\alpha} b(k)\,dk = \int_0^{+\infty} k^{\alpha}\left[\int_{C_k} |\psi(\vec{p})|^2 d\vec{p}\right] dk = \int_0^{+\infty}\int_{C_k} k^{\alpha} |\psi(\vec{p})|^2\, d\vec{p}\, dk =$$

$$= \int_0^{+\infty}\int_{C_k} \|\vec{p}\|^{\alpha} |\psi(\vec{p})|^2\, d\vec{p}\, dk = \int_{\mathbb{R}^2 - \{0\}} \|\vec{p}\|^{\alpha} |\psi(\vec{p})|^2\, d\vec{p}$$

$$= \lim_{L\to+\infty} \sum_{\vec{p}\in A_L^*} \|\vec{p}\|^a \, |\Psi(\vec{p})|^2 \, \Delta p_L = \lim_{L\to+\infty} \frac{4\pi^2}{L^2} \sum_{\vec{p}\in A_L^*} \|\vec{p}\|^a \, |\Psi(\vec{p})|^2 =$$

$$= \lim_{L\to+\infty} \frac{4\pi^2}{L^2} \sum_{\vec{p}\in A_L^*} \|\vec{p}\|^a \, |\hat{\psi}_p|^2 \qquad \square$$

▼ Energy and enstrophy.

Def: The energy field $E(t,\vec{r})$ and the enstrophy field $G(t,\vec{r})$ of 2d turbulence are defined as the following probabilistic averages:

$$\boxed{\begin{aligned} E(t,\vec{r}) &= \frac{1}{2}\int_{\Omega}[u^2(t,\vec{r},\omega)+v^2(t,\vec{r},\omega)]\,d\mu(\omega). \\ G(t,\vec{r}) &= \frac{1}{2}\int_{\Omega} J^2(t,\vec{r},\omega)\,d\mu(\omega). \end{aligned}}$$

In the following discussion we will relate the energy and enstrophy with the correlation function and define the energy spectrum and the enstrophy spectrum. Then we will derive various results about these spectra, including a result about how they evolve as a function of time.

Def: The total energy $E(t)$ and total enstrophy $G(t)$ are defined in terms of $E(t,\vec{r})$ and $G(t,\vec{r})$ by:

$$\boxed{\begin{aligned} E(t) &= \langle E(t,\vec{r})\rangle = \lim_{L\to+\infty}\frac{1}{L^2}\int_{S_L} E(t,\vec{r})\,d\vec{r} \\ G(t) &= \langle G(t,\vec{r})\rangle = \lim_{L\to+\infty}\frac{1}{L^2}\int_{S_L} G(t,\vec{r})\,d\vec{r} \end{aligned}}$$

▶ Notation: We will use the bracket notation to denote spatial averaging as in the definition above and reserve the overbar notation for probabilistic averaging.

Using this notation, we can abbreviate our definitions as follows:

$$E(t) = \left\langle \frac{1}{2}\overline{(u^2+v^2)}\right\rangle \quad \text{and} \quad G(t) = \left\langle \frac{1}{2}\overline{J^2}\right\rangle$$

$$E(t,\vec{r}) = \frac{1}{2}\overline{(u^2+v^2)} \quad \text{and} \quad G(t) = \frac{1}{2}\overline{J^2}$$ ◀

▼ Directional derivatives.

Let $F(\vec{r}_1,\vec{r}_2)$ be a function of two points $\vec{r}_1$ and $\vec{r}_2$. It is possible to define two kinds of directional derivatives: one with respect to $\vec{r}_1$ and one with respect to $\vec{r}_2$. It is necessary then to carefully define operators for differentiation that take this distinction into account. We given then the following definitions:

<u>Def</u>: We define: $\boxed{\begin{aligned}\nabla_1 F(\vec{r}_1,\vec{r}_2) &= \text{the gradient of } F \text{ wrt } \vec{r}_1\\ \nabla_2 F(\vec{r}_1,\vec{r}_2) &= \text{the gradient of } F \text{ wrt } \vec{r}_2\end{aligned}}$

<u>Def</u>: Let $\vec{a}\in\mathbb{R}^2$ with $\|\vec{a}\|$ be a vector that indicates direction. Then we define the directional derivatives of F as follows:

$$\boxed{\begin{aligned} D_1(\vec{a})F_{\theta}(\vec{r}_1,\vec{r}_2) &= \nabla_1 F(\vec{r}_1,\vec{r}_2)\cdot\vec{a}\\ D_2(\vec{a})F(\vec{r}_1,\vec{r}_2) &= \nabla_2 F(\vec{r}_1,\vec{r}_2)\cdot\vec{a}\end{aligned}}$$

► We use the notation $\vec{i}=(1,0)$ and $\vec{j}=(0,1)$ to indicate the horizontal and vertical directions. ◄

Now consider the function $\rho(\vec{r}_1,\vec{r}_2)=\|\vec{r}_1-\vec{r}_2\|$. We would like to know the gradients and the directional derivatives of $\rho(\vec{r}_1,\vec{r}_2)$. First we show a technical statement:

<u>Lemma</u>: Let $\vec{r}=(x,y)$ and $\vec{a}=(a_1,a_2)$. Then: if ∇ is gradient with respect to $\vec{r}$, then: $\nabla\|\vec{r}-\vec{a}\| = \frac{1}{\|\vec{r}-\vec{a}\|}(\vec{r}-\vec{a})$.

<u>Proof</u>

$\|\vec{r}-\vec{a}\| = \sqrt{(x-a_1)^2+(y-a_2)^2}$, therefore:

$$\frac{\partial}{\partial x}\|\vec{r}-\vec{a}\| = \frac{\partial}{\partial x}\sqrt{(x-a_1)^2+(y-a_2)^2} =$$

$$= \frac{1}{2\sqrt{(x-a_1)^2+(y-a_2)^2}}\frac{\partial}{\partial x}\left[(x-a_1)^2+(y-a_2)^2\right] =$$

$$= \frac{2(x-a_1)}{2\|\vec{r}-\vec{a}\|} = \frac{x-a_1}{\|\vec{r}-\vec{a}\|}$$

Similarly: $\frac{\partial}{\partial y}\|\vec{r}-\vec{a}\| = \frac{y-a_2}{\|\vec{r}-\vec{a}\|}$, therefore $\nabla\|\vec{r}-\vec{a}\| = \frac{1}{\|\vec{r}-\vec{a}\|}(\vec{r}-\vec{a})$. □

A direct consequence of this result is that:

Corollary: $\boxed{\nabla_1 \rho(\vec{r}_1, \vec{r}_2) = \frac{\vec{r}_1 - \vec{r}_2}{\|\vec{r}_1 - \vec{r}_2\|}}$ $\boxed{\nabla_2 \rho(\vec{r}_1, \vec{r}_2) = \frac{\vec{r}_2 - \vec{r}_1}{\|\vec{r}_1 - \vec{r}_2\|}}$

Corollary: Let $F(\vec{r}_1, \vec{r}_2) = f(\rho(\vec{r}_1, \vec{r}_2))$ be a homogeneous and isotropic function and let $\vec{r}_1, \vec{r}_2, \rho$ be fixed. Then:

$$\boxed{\begin{aligned} \nabla_1 F(\vec{r}_1, \vec{r}_2) &= \frac{f'(\rho)}{\rho}(\vec{r}_1 - \vec{r}_2) \\ \nabla_2 F(\vec{r}_1, \vec{r}_2) &= \frac{f'(\rho)}{\rho}(\vec{r}_2 - \vec{r}_1). \end{aligned}}$$

▼ The energy spectrum

In a homogeneous and isotropic flow, the correlation of the streamfunction ψ at two distinct points $\vec{r}_1$ and $\vec{r}_2$ depends only on the distance $\rho = \|\vec{r}_1 - \vec{r}_2\|$ between the two points:

$$\overline{\psi(\vec{r}_1)\,\psi(\vec{r}_2)} = B(\rho).$$

This result does not extend to the velocity components u and v. However $\overline{u(r_1)u(r_2)} + \overline{v(r_1)v(r_2)}$ can be shown to be dependent only on ρ:

Thm: In homogeneous and isotropic flow:

$$\boxed{\overline{u(\vec{r}_1)u(\vec{r}_2)} + \overline{v(\vec{r}_1)v(\vec{r}_2)}. = -B''(\rho) - \frac{B'(\rho)}{\rho}}$$

Proof

Recall that $u = -\partial\psi/\partial y = D(-\vec{j})\psi$ and
$v = \partial\psi/\partial x = D(\vec{i})\psi$.

First we evaluate $\overline{u(\vec{r}_1)u(\vec{r}_2)}$:

$$\overline{u(\vec{r}_1)u(\vec{r}_2)} = \int_{\Omega} u(\vec{r}_1, \vartheta)\, u(\vec{r}_2, \vartheta)\, d\mu(\vartheta) =$$

$$= \int_{\Omega} [D_1(-\vec{j})\,\psi(\vec{r}_1, \vartheta)][D_2(-\vec{j})\,\psi(\vec{r}_2, \vartheta)]\, d\mu(\vartheta). =$$

$$= \int_{\Omega} D_1(-\vec{j}) D_2(-\vec{j}) \left[\psi(\vec{r}_1,\vartheta)\, \psi(\vec{r}_2,\vartheta) \right] d\mu(\vartheta) =$$

$$= D_1(\vec{j}) D_2(\vec{j}) \int_{\Omega} \psi(\vec{r}_1,\vartheta)\psi(\vec{r}_2,\vartheta)\, d\mu(\vartheta). = D_1(\vec{j}) D_2(\vec{j})\, B(\rho(\vec{r}_1,\vec{r}_2))$$

$$= D_1(\vec{j}) \left[\nabla_2 B(\rho(\vec{r}_1,\vec{r}_2)) \cdot \vec{j} \right] = D_1(\vec{j}) \left[\frac{B'(\rho)}{\rho} (\vec{r}_2 - \vec{r}_1) \cdot \vec{j} \right] =$$

$$= D_1(\vec{j}) \left[\frac{B'(\rho)}{\rho} (y_2 - y_1) \right] =$$

$$= \frac{B'(\rho)}{\rho} D_1(\vec{j})(y_2 - y_1) + (y_2 - y_1) D_1(\vec{j}) \frac{B'(\rho)}{\rho} =$$

$$= \frac{B'(\rho)}{\rho} \frac{\partial}{\partial y_1} (y_2 - y_1) + (y_2 - y_1) \frac{\partial}{\partial \rho} \left(\frac{B'(\rho)}{\rho} \right) \frac{1}{\rho} (\vec{r}_1 - \vec{r}_2) \cdot \vec{j} =$$

$$= \frac{B'(\rho)}{\rho} (-1) + (y_2 - y_1) \frac{\rho B''(\rho) - B'(\rho)}{\rho^2} \frac{1}{\rho} (y_1 - y_2) =$$

$$= -\frac{B'(\rho)}{\rho} - \frac{(y_1 - y_2)^2}{\rho^3} \left[\rho B''(\rho) - B'(\rho) \right].$$

Similarly we obtain:

$$\overline{v(\vec{r}_1) v(\vec{r}_2)} = D_1(\vec{i}) D_2(\vec{i})\, B(\rho(\vec{r}_1,\vec{r}_2)) =$$

$$= -\frac{B'(\rho)}{\rho} - \frac{(x_1 - x_2)^2}{\rho^3} \left[\rho B''(\rho) - B'(\rho) \right]$$

Adding both equations we get:

$$\overline{u(\vec{r}_1) u(\vec{r}_2)} + \overline{v(\vec{r}_1) v(\vec{r}_2)} = -\frac{2B'(\rho)}{\rho} - \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{\rho^3} \left[\rho B''(\rho) - B'(\rho) \right] =$$

$$= -\frac{2B'(\rho)}{\rho} - \frac{\rho B''(\rho) - B'(\rho)}{\rho} =$$

$$= -B''(\rho) - \frac{B'(\rho)}{\rho}. \qquad \square$$

Since this quantity is nice we will call it the <u>velocity correlation function</u> and write it:

$$\boxed{\Gamma(\vec{r}_1,\vec{r}_2) \equiv \overline{u(\vec{r}_1) u(\vec{r}_2)} + \overline{v(\vec{r}_1) v(\vec{r}_2)}.}$$

Under homogeneity and isotropy we get $\Gamma(\rho) = -B''(\rho) - \frac{B'(\rho)}{\rho}.$
Under the limit $\rho \to 0$ we get our main result about energy:

▸ <u>notation</u>: When we wish to indicate the dependence of $B(\rho)$ and $\Gamma(\rho)$ on time t, we will write it as $B(t,\rho)$ and $\Gamma(t,\rho)$. Nevertheless, primes will still indicate differentiation with respect to ρ. So $B'(t,\rho) = \frac{\partial}{\partial \rho} B(t,\rho)$ and so on .... ◂

<u>Thm</u>: In homogeneous and isotropic flow the energy field and the total energy are both equal to $B''(t,0)$:

$$\boxed{E(t) = E(t,\vec{r}) = -B''(t,0)} \quad , \forall \vec{r} \in \mathbb{R}^2$$

<u>Proof</u>

$$E(t,\vec{r}) = \frac{1}{2}\left[\overline{u(t,\vec{r})u(t,\vec{r})} + \overline{v(t,\vec{r})v(t,\vec{r})}\right] =$$

$$= \frac{1}{2}\lim_{\rho\to 0}\Gamma(t,\rho) = \frac{1}{2}\lim_{\rho\to 0}\left[-B''(\rho) - \frac{B'(\rho)}{\rho}\right]$$

$$= \frac{1}{2}\left[-B''(0) - \lim_{\rho\to 0}\frac{B'(\rho)}{\rho}\right]$$

Since $B'(0) = 0 \Rightarrow$ the De L'Hospital theorem applies $\Rightarrow$

$$\Rightarrow \lim_{\rho\to 0}\frac{B'(\rho)}{\rho} = \lim_{\rho\to 0} B''(\rho) = B''(0).$$

therefore: $E(t,\vec{r}) = \frac{1}{2}\left[-B''(t,0) - B''(t,0)\right] = -B''(t,0)$. $, \forall \vec{r} \in \mathbb{R}^2$.

Since $E(t,\vec{r})$ does not depend on $\vec{r}$, it follows that it equals its spatial average:

$$E(t) = \langle E(t,\vec{r})\rangle = -\langle B''(t,0)\rangle = -B''(t,0). \qquad \square$$

This result motivates the definition of the <u>energy spectrum</u>.

<u>Def</u>: The <u>energy spectrum</u> $\mathcal{E}(k)$ of turbulent homogeneous and isotropic flow is defined in terms of the streamfunction spectrum by:

$$\boxed{\mathcal{E}(k) = \frac{k^2 \mathcal{B}(k)}{\blacksquare}}$$

The reason why we call it the energy spectrum is because it is a multiresolution analysis of the total energy by wavenumber.

Prop : In homogeneous and isotropic flow

$$\boxed{E(t) = \frac{1}{2}\int_0^{+\infty} \mathcal{E}(t,k)\,dk}$$

Proof

Recall from page 20 that $B''(0) = -\frac{1}{2}\int_0^{+\infty} k^2 \mathcal{E}(k)\,dk$
It follows that:

$$E(t) = -B''(t,0) = -\frac{1}{2}\int_0^{+\infty} k^2 \mathcal{E}(k,t)\,dk = \frac{1}{2}\int_0^{+\infty} \mathcal{E}(t,k)\,dk. \qquad \square$$

Another characterization of the energy spectrum is its relation with the velocity correlation function $\Gamma(\rho)$. First we show a property of Bessel functions that we need:

Lemma : $\boxed{J_0''(x) + \frac{J_0'(x)}{x} = -J_0(x)}$

Proof

Our argument assumes the following identities about Bessel functions:

$$J_0'(x) = -J_1(x) \quad , \quad J_1'(x) = \frac{J_0(x) - J_2(x)}{2} \quad , \quad J_2(x) = \frac{2}{x} J_1(x) - J_0(x).$$

Note that:

$$J_0''(x) = [-J_1(x)]' = \frac{1}{2}[J_2(x) - J_0(x)]$$

$$\frac{J_0'(x)}{x} = -\frac{J_1(x)}{x} = -\frac{1}{2}\frac{2}{x}J_1(x) = -\frac{1}{2}[J_2(x) + J_0(x)]$$

therefore:

$$J_0''(x) + \frac{J_0'(x)}{x} = \frac{1}{2}[\cancel{J_2(x)} - J_0(x)] - \frac{1}{2}[\cancel{J_2(x)} + J_0(x)] =$$

$$= -\frac{J_0(x)}{2} - \frac{J_0(x)}{2} = -J_0(x) \qquad \square$$

Now we can show that $\mathcal{E}(t,k)$ gives the Bessel expansion of the velocity correlation function:

<u>Thm</u>: The velocity correlation function $\Gamma(\rho)$ has the following Bessel expansion:

$$\boxed{\Gamma(\rho) = \int_0^{+\infty} \mathcal{E}(k)\, J_0(k\rho)\, dk}$$

<u>Proof</u>

Recall that $B(\rho) = \int_0^{+\infty} b(k)\, J_0(k\rho)\, dk$, therefore:

$$\Gamma(\rho) = -B''(\rho) - \frac{B'(\rho)}{\rho} =$$

$$= -\frac{\partial^2}{\partial \rho^2}\int_0^{+\infty} b(k)\, J_0(k\rho)\, dk - \frac{1}{\rho}\frac{\partial}{\partial \rho}\int_0^{+\infty} b(k)\, J_0(k\rho)\, dk =$$

$$= \int_0^{+\infty} b(k)\left[-\frac{\partial^2}{\partial \rho^2} J_0(k\rho) - \frac{1}{\rho}\frac{\partial}{\partial \rho} J_0(k\rho)\right] dk =$$

$$= \int_0^{+\infty} b(k)\left[-k^2 J_0''(k\rho) - \frac{k J_0'(k\rho)}{\rho}\right] dk =$$

$$= \int_0^{+\infty} k^2 b(k)\left[-J_0''(k\rho) - \frac{J_0'(k\rho)}{k\rho}\right] dk = \int_0^{+\infty} k^2 b(k)\, J_0(k\rho)\, dk$$

$$= \int_0^{+\infty} \mathcal{E}(k)\, J_0(k\rho)\, dk. \qquad \square$$

The enstrophy spectrum

In a homogeneous and isotropic flow we will show that, unlike the velocity components u and v, the vorticity field ζ is also homogeneous and isotropic. This fact will then motivate the definition of the enstrophy spectrum. We begin by defining the vorticity correlation function:

Def : The vorticity correlation function $Z(\vec{r}_1,\vec{r}_2)$ is defined as the correlation between the vorticity at $\vec{r}_1$ and the vorticity at $\vec{r}_2$:

$$\boxed{Z(\vec{r}_1\vartheta,\vec{r}_2) = \int_{\Omega} \zeta(\vec{r}_1,\vartheta)\,\zeta(\vec{r}_2,\vartheta)\,d\mu(\vartheta)}$$

Prop : The vorticity correlation function is related with the streamfunction correlation function by:

$$\boxed{Z(\vec{r}_1,\vec{r}_2) = \nabla_1^2 \nabla_2^2 B(\rho(\vec{r}_1,\vec{r}_2)).}$$

Proof

Recall that $\zeta(\vec{r}_1,\vartheta) = \nabla_1^2\psi(\vec{r}_1,\vartheta)$ and
$\zeta(\vec{r}_2,\vartheta) = \nabla_2^2\psi(\vec{r}_2,\vartheta)$.

Then:

$$\begin{aligned} Z(\vec{r}_1,\vec{r}_2) &= \int_{\Omega} \zeta(\vec{r}_1,\vartheta)\,\zeta(\vec{r}_2,\vartheta)\,d\mu(\vartheta) = \\ &= \int_{\Omega} [\nabla_1^2\psi(\vec{r}_1,\vartheta)][\nabla_2^2\psi(\vec{r}_2,\vartheta)]\,d\mu(\vartheta) = \\ &= \int_{\Omega} \nabla_1^2\nabla_2^2[\psi(\vec{r}_1,\vartheta)\,\psi(\vec{r}_2,\vartheta)]\,d\mu(\vartheta) = \\ &= \nabla_1^2\nabla_2^2 \int_{\Omega} \psi(\vec{r}_1,\vartheta)\,\psi(\vec{r}_2,\vartheta)\,d\mu(\vartheta) = \nabla_1^2\nabla_2^2 B(\rho(\vec{r}_1,\vec{r}_2)) \qquad \square \end{aligned}$$

Now we show that the Laplacian operators ∇_1^2 and ∇_2^2 preserve homogeneity and isotropy, to conclude that ζ is also homogeneous and isotropic.

Thm : Let $F(\vec{r}_1,\vec{r}_2) = f(\rho(\vec{r}_1,\vec{r}_2))$ where $\rho(\vec{r}_1,\vec{r}_2) = \|\vec{r}_1 - \vec{r}_2\|$.
Then:

$$\boxed{\nabla_1^2 F(\vec{r}_1,\vec{r}_2) = \nabla_2^2 F(\vec{r}_1,\vec{r}_2) = f''(\rho) + \frac{f'(\rho)}{\rho}}$$

Proof

We begin by evaluating $D_1^2(\vec{i}) F(\vec{r}_1,\vec{r}_2)$ and $D_1^2(\vec{j}) F(\vec{r}_1,\vec{r}_2)$.

$$D_1^2(\vec{i}) F(\vec{r}_1,\vec{r}_2) = D_1(\vec{i})\left[D_1(\vec{i}) f(\rho(\vec{r}_1,\vec{r}_2))\right] = D_1(\vec{i})\left[\nabla_1 f(\rho(\vec{r}_1,\vec{r}_2)) \cdot \vec{i}\right] =$$

$$= D_1(\vec{i})\left[\frac{f'(\rho)}{\rho}(x_1 - x_2)\right] =$$

$$= \frac{f'(\rho)}{\rho} D_1(\vec{i})(x_1 - x_2) + (x_1 - x_2) D_1(\vec{i})\left[\frac{f'(\rho)}{\rho}\right] =$$

$$= \frac{f'(\rho)}{\rho} + (x_1 - x_2)\left[\frac{1}{\rho}\frac{f''(\rho)\rho - f'(\rho)}{\rho^2}(x_1 - x_2)\right] =$$

$$= \frac{f'(\rho)}{\rho} + \frac{(x_1 - x_2)^2}{\rho^2} \cdot \frac{\rho f''(\rho) - f'(\rho)}{\rho}$$

By a similar argument we have:

$$D_1^2(\vec{j}) F(\vec{r}_1,\vec{r}_2) = \frac{f'(\rho)}{\rho} + \frac{(y_1 - y_2)^2}{\rho^2} \cdot \frac{\rho f''(\rho) - f'(\rho)}{\rho}$$

Combining these two equations we have:

$$\nabla_1^2 F(\vec{r}_1,\vec{r}_2) = D_1^2(\vec{i}) F(\vec{r}_1,\vec{r}_2) + D_1^2(\vec{j}) F(\vec{r}_1,\vec{r}_2) =$$

$$= \frac{2f'(\rho)}{\rho} + \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{\rho^2} \cdot \frac{\rho f''(\rho) - f'(\rho)}{\rho} =$$

$$= \frac{2f'(\rho)}{\rho} + \frac{\rho f''(\rho) - f'(\rho)}{\rho} = \frac{\rho f''(\rho) + f'(\rho)}{\rho} = f''(\rho) + \frac{f'(\rho)}{\rho}$$

Because $F(\vec{r}_1,\vec{r}_2) = f(\rho(\vec{r}_1,\vec{r}_2)) = f(\rho(\vec{r}_2,\vec{r}_1)) = F(\vec{r}_2,\vec{r}_1)$. it follows that

$$\nabla_2^2 F(\vec{r}_1,\vec{r}_2) = \nabla_1^2 F(\vec{r}_1,\vec{r}_2) = f''(\rho) + \frac{f'(\rho)}{\rho} \qquad \square$$

A direct consequence of this theorem is that the functions $J_0(k\rho(\vec{r}_1,\vec{r}_2))$ that we use to define the spectra of correlation functions are eigenfunctions of ∇_1^2 and ∇_2^2:

<u>Lemma</u>: $$\boxed{\begin{array}{l}\nabla_1^2 J_0(k\rho(\vec{r}_1,\vec{r}_2)) = \nabla_2^2 J_0(k\rho(\vec{r}_1,\vec{r}_2)) = -k^2 J_0(k\rho(\vec{r}_1,\vec{r}_2)) \\ \nabla_1^2\nabla_2^2 J_0(k\rho(\vec{r}_1,\vec{r}_2)) = k^4 J_0(k\rho(\vec{r}_1,\vec{r}_2)).\end{array}}$$

<u>Proof</u>

Let $k>0$ be given. Define $f(\rho) = J_0(k\rho)$. Then:

$$\nabla_1^2 J_0(k\rho(\vec{r}_1,\vec{r}_2)) = f''(\rho) + \frac{f'(\rho)}{\rho} = k^2 J_0''(k\rho) + \frac{k J_0'(k\rho)}{\rho} =$$
$$= k^2\left[J_0''(k\rho) + \frac{J_0'(k\rho)}{k\rho}\right] = k^2\left[-J_0(k\rho)\right] =$$
$$= -k^2 J_0(k\rho).$$

Similarly $\nabla_2^2 J_0(k\rho(\vec{r}_1,\vec{r}_2)) = -k^2 J_0(k\rho)$.

Finally:

$$\nabla_1^2\nabla_2^2 J_0(k\rho(\vec{r}_1,\vec{r}_2)) = \nabla_1^2\left[-k^2 J_0(k\rho(\vec{r}_1,\vec{r}_2))\right] =$$
$$= -k^2\nabla_1^2 J_0(k\rho(\vec{r}_1,\vec{r}_2)) = -k^2(-k^2) J_0(k\rho(\vec{r}_1,\vec{r}_2)) =$$
$$= k^4 J_0(k\rho(\vec{r}_1,\vec{r}_2)). \qquad \square$$

<u>Corollary:</u> In homogeneous and isotropic flow, the vorticity correlation function depends only on ρ:

$$\boxed{Z(\vec{r}_1,\vec{r}_2) = Z(\rho(\vec{r}_1,\vec{r}_2))}$$

Using this result, we define the <u>enstrophy spectrum</u>:

<u>Def</u>: The enstrophy spectrum $G(k)$ is the Bessel transform of $Z(\rho)$:

$$\boxed{\begin{array}{l} Z(\rho) = \displaystyle\int_0^{+\infty} G(k) J_0(k\rho)\,dk. \\ \hline G(k) = \displaystyle\int_0^{+\infty} k\rho Z(\rho) J_0(k\rho)\,d\rho \end{array}}$$

Although the relation between $Z(\rho)$ and $B(\rho)$ is complicated, the relation between the spectra is very simple:

Thm: $\boxed{G(k) = k^4 \mathcal{B}(k).}$

Proof

$$Z(\rho) = \nabla_1^2 \nabla_2^2 B(\rho(\vec{r}_1, \vec{r}_2)) = \nabla_1^2 \nabla_2^2 \int_0^{+\infty} \mathcal{B}(k) J_0(k\rho(\vec{r}_1,\vec{r}_2))\, dk =$$

$$= \int_0^{+\infty} \mathcal{B}(k) \left[\nabla_1^2 \nabla_2^2 J_0(k\rho(\vec{r}_1,\vec{r}_2)) \right] dk =$$

$$= \int_0^{+\infty} \mathcal{B}(k)\, k^4\, J_0(k\rho(\vec{r}_1,\vec{r}_2))\, dk = \int_0^{+\infty} G(k) J_0(k\rho(\vec{r}_1,\vec{r}_2))\, dk \Rightarrow$$

$$\Rightarrow G(k) = k^4 \mathcal{B}(k). \quad \square$$

Finally, we get a result about the total enstrophy just like the one we have about total energy.

Thm: $\boxed{G(t) = G(t,\vec{r}) = \int_0^{+\infty} \frac{G(t,k)}{2}\, dk.}$, $\forall \vec{r} \in \mathbb{R}^2$

Proof

$$G(t,\vec{r}) = \frac{1}{2} \overline{J(t,\vec{r}) J(t,\vec{r})} = \frac{1}{2} Z(0) =$$

$$= \frac{1}{2} \int_0^{+\infty} G(k) J_0(k \cdot 0)\, dk = \frac{1}{2} \int_0^{+\infty} G(t,k)\, dk. \quad , \forall \vec{r} \in \mathbb{R}^2$$

because $J_0(0) = 1$. Since $G(t,\vec{r})$ does not depend on $\vec{r}$, it follows that

$$G(t) = \langle G(t,\vec{r}) \rangle = G(t,\vec{r}), \ \forall \vec{r} \in \mathbb{R}^2 \quad \square$$

Summary of facts about spectra

The fundamental property of homogeneous and isotropic turbulence is that the correlation of the streamfunction fluctuations at two distinct points $\vec{r}_1$ and $\vec{r}_2$ depends only on the distance $\rho(\vec{r}_1,\vec{r}_2) = \|\vec{r}_1 - \vec{r}_2\|$ between the points. This defines the correlation function $B(\rho)$ such that

$$\overline{\psi(\vec{r}_1)\psi(\vec{r}_2)} = B(\rho(\vec{r}_1,\vec{r}_2))$$

The streamfunction spectrum $b(k)$ is given as the Bessel function transform of $B(\rho)$:

$$b(k) = \int_0^{+\infty} k\rho\, B(\rho)\, J_0(k\rho)\, d\rho \longleftrightarrow B(\rho) = \int_0^{+\infty} b(k)\, J_0(k\rho)\, dk$$

The meaning of k as wavenumber is established by the following fundamental result about the 2d Fourier transform $\Psi(\vec{k})$ of the streamfunction:

$$\overline{\Psi(\vec{k})\Psi(\vec{p})^*} = \frac{1}{2\pi}\frac{b(\|\vec{k}\|)}{\|\vec{k}\|}\,\delta(\vec{k}-\vec{p})$$

When $\vec{k} \neq \vec{p}$, it follows that $\Psi(\vec{k})$ and $\Psi(\vec{p})$ are not correlated. The most interesting result however is obtained when $\vec{k} = \vec{p}$:

$$\boxed{b(k) = 2\pi k\,\overline{|\Psi(\vec{k})|^2}}\,, \quad \forall \vec{k} \in \mathbb{R}^2 : \|\vec{k}\| = k.$$

which establishes the meaning of $b(k)$ as "itensity of turbulence for wavenumbers with length k". Using an ergodic hypothesis we can estimate $b(k)$ from a particular instance of the flow with the following line integral:

$$b(k) \cong \int_{C_k} |\Psi(\vec{p},\omega)|^2\, d\vec{p}$$

where $C_k = \{\vec{p} \in \mathbb{R}^2 \mid \|\vec{p}\| = k\}$. In a simulation where we idealize homogeneous and isotropic flow with bound flow in a cube with side L and periodic boundary conditions, $\Psi(\vec{p})$ is not known to us. Instead we only know the fourier coefficients $\hat{\psi}_p$ defined on a lattice of wavenumbers:

$$A_L = \frac{2\pi}{L}\mathbb{Z}^2$$

As $L \to +\infty$, $\lim_{L\to+\infty} A_L = \mathbb{R}^2$ because the lattice becomes increasingly more dense. Also, the Fourier coefficients converge to $\Psi(\vec{p})$ for their corresponding wavenumber. From this observation we get the following estimate of $b(k)$:

$$b(k) \cong \lim_{\varepsilon\to 0} \lim_{L\to+\infty} \left[\frac{4\pi^2}{\varepsilon L^2} \sum_{\vec{p}\in A_L \cap D(k,\varepsilon)} |\hat{\psi}_p|^2 \right]$$

where $D(k,\varepsilon) = \{\vec{p}\in\mathbb{R}^2 \mid k \leq \|\vec{p}\| \leq k+\varepsilon\}$

To justify the idealization in these simulations the integral length scale $L(\psi)$ must be $L(\psi) \ll L$. This way the periodic boundary condition is, plausibly, harmless since a region within the cube is large enough to contain the large-scale coherence of the turbulent flow. Also, we must retain enough wavenumbers to resolve the associated differential length scale $\lambda(\psi)$. So the wavenumber cut-off must be such that:

$$K \gg \frac{2\pi}{\lambda(\psi)}.$$

These length scales are given by:

$$L(\psi) = \frac{1}{B(0)} \int_0^{+\infty} B(\rho)\,d\rho \quad \text{and} \quad \lambda(\psi) = \left[-\frac{2B(0)}{B''(0)} \right]^{1/2}$$

the terms of which can all be written in terms of integrals of the streamfunction spectrum:

$$B(0) = \int_0^{+\infty} b(k)\,dk \quad ; \quad B''(0) = \int_0^{+\infty} -\frac{k^2 b(k)}{2}\,dk \quad ; \quad \int_0^{+\infty} B(\rho)\,d\rho = \int_0^{+\infty} \frac{b(k)}{k}\,dk.$$

In a simulation all of these three integrals are estimated using the following result:

$$\int_0^{+\infty} k^a b(k)\,dk = \lim_{L\to+\infty} \frac{4\pi^2}{L^2} \sum_{\vec{p}\in A_L} |\hat{\psi}_p|^2 \|\vec{p}\|^a$$

which avoids the errors and hassle of tuning ε.

Now we turn to the energy and enstrophy spectra: These are defined by the velocity correlation function and the vorticity correlation function:

$$\overline{u(\vec{r}_1)u(\vec{r}_2)} + \overline{v(\vec{r}_1)v(\vec{r}_2)} = \Gamma(\rho(\vec{r}_1,\vec{r}_2)) = \int_0^{+\infty} \mathcal{E}(k) J_0(k\rho(\vec{r}_1,\vec{r}_2))\,dk$$

$$\overline{\zeta(\vec{r}_1)\zeta(\vec{r}_2)} = Z(\rho(\vec{r}_1,\vec{r}_2)) = \int_0^{+\infty} \mathcal{G}(k) J_0(k\rho(\vec{r}_1,\vec{r}_2))\,dk$$

where $\Gamma(\rho)$ = velocity correlation function
$Z(\rho)$ = vorticity correlation function.
$\mathcal{E}(k)$ = energy spectrum
$\mathcal{G}(k)$ = enstrophy spectrum.

After some non-trivial development we showed that

$$\mathcal{E}(k) = k^2 \mathcal{B}(k) \quad \text{and} \quad \mathcal{G}(k) = k^4 \mathcal{B}(k).$$

This also implies that $\mathcal{G}(k) = k^2 \mathcal{E}(k)$.

A characterization of the energy and enstrophy spectra is given ~~from~~ in terms of the energy and enstrophy of the turbulence. We define the energy field as:

$$E(t,\vec{r}) = \frac{1}{2}\left[\overline{u^2(\vec{r}_1)} + \overline{v^2(\vec{r})}\right]$$

and the enstrophy field as

$$G(t,\vec{r}) = \frac{1}{2}\overline{\zeta^2(\vec{r})}.$$

The "total" energy and enstrophy are defined as:

$$E(t) = \langle E(t,\vec{r})\rangle \quad \text{and} \quad G(t) = \langle G(t,\vec{r})\rangle$$

The spectra $\mathcal{E}(k)$ and $\mathcal{G}(k)$ are multiresolution analysis of the energy and enstrophy:

$$\boxed{\begin{aligned} E(t) &= E(t,\vec{r}) = \frac{1}{2}\int_0^{+\infty} \mathcal{E}(t,k)\,dk. \\ G(t) &= G(t,\vec{r}) = \frac{1}{2}\int_0^{+\infty} \mathcal{G}(t,k)\,dk \end{aligned}}$$

These integrals take the form $\int_0^{+\infty} k^a \mathcal{B}(k)\,dk$ as well, so in a simulation they can be estimated using the same method as with the length scales $L(\psi)$ and $\lambda(\psi)$.

Now that we have established the three spectra, $\mathcal{B}(k)$, $\mathcal{E}(k)$, and $\mathcal{G}(k)$, we will proceed to derive their conservation laws. Before we do that, however, we would like to make one additional remark about the physical interpretation of the energy and enstrophy spectra.

Notes on spectra

Every spectrum is associated with a correlation function:

Hom + Is. $\longrightarrow$ $\overline{\psi(\vec r_1)\,\psi(\vec r_2)} = B(\rho(\vec r_1,\vec r_2))$.

corresponding spectrum:

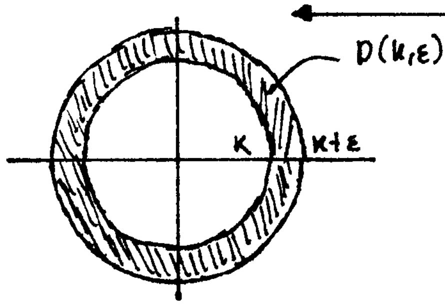
$$B(\rho) = \int_0^{+\infty} b(k)\,J_0(k\rho)\,dk \longleftrightarrow b(k) = \int_0^{+\infty} k\rho\, B(\rho)\, J_0(k\rho)\,d\rho$$

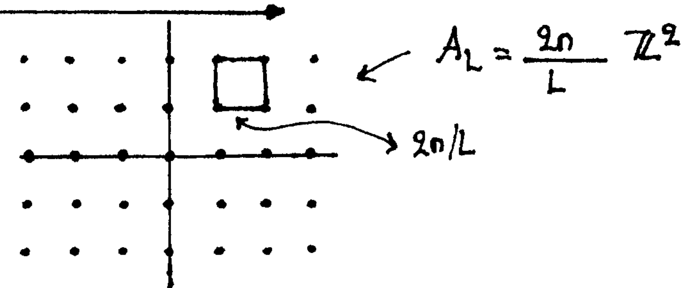
▸ Addition theorem on 2d Fourier

$$\int_{\mathbb{R}^2} J_0(k_0\rho(\vec r_1,\vec r_2))\exp(-i\vec k\cdot\vec r_1)\,d\vec r_1 = 2\pi\,\frac{\delta(\|\vec k\|-k_0)}{k_0}\,\exp(-i\vec k\cdot\vec r_2). \blacktriangleleft$$

$$\hookrightarrow\ \Psi(\vec k) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\psi(\vec r)\exp(-i\vec k\cdot\vec r)\,d\vec r$$

$$\vec k \neq \vec p \Rightarrow \overline{\Psi(\vec k)\,\Psi(\vec p)^*} = 0 \quad\text{and}\quad \underline{\underline{b(k) = 2\pi k\,\overline{|\Psi(\vec k)|^2}}}\ ,\ \|\vec k\| = k$$





$$\psi(\vec r) = \sum_{\vec p\in A_L}\hat\psi_p\exp(i\vec p\cdot\vec r) \longleftrightarrow b(k) \approx \frac{4\pi^2}{\varepsilon L^2}\sum_{\vec p\in A_L\cap D(k,\varepsilon)}|\hat\psi_p|^2$$

$$\int_0^{+\infty} k^a b(k)\,dk \approx \frac{4\pi^2}{L^2}\sum_{\vec p\in A_L}|\hat\psi_p|^2\cdot\|\vec p\|^a$$

$$L(\psi) = \frac{1}{B(0)}\int_0^{+\infty} B(\rho)\,d\rho \qquad B(0) = \int_0^{+\infty} b(k)\,dk\ ,\quad B''(0) = \int_0^{+\infty} k^2 b(k)\,dk$$

$$\lambda(\psi) = \left[-\frac{2B(0)}{B''(0)}\right]^{1/2} \qquad \int_0^{+\infty} B(\rho)\,d\rho = \int_0^{+\infty}\frac{b(k)}{k}\,dk.$$

$$\overline{u_1u_2} + \overline{v_1v_2} = -B''(\rho) - \frac{B'(\rho)}{\rho} \equiv \Gamma(\rho) \longleftarrow \text{velocity correlation}$$

$$\overline{\zeta_1\zeta_2} = \nabla_1^2\nabla_2^2 B(\rho(\vec r_1,\vec r_2)) \equiv Z(\rho) \longleftarrow \text{vorticity correlation.}$$

$$\Gamma(\rho) \equiv \int_0^{+\infty} \mathcal{E}(k)\, J_0(k\rho)\, dk \longleftrightarrow \text{energy spectrum} \quad \mathcal{E}(k) = k^2 \mathcal{B}(k)$$

$$Z(\rho) = \int_0^{+\infty} \mathcal{G}(k)\, J_0(k\rho)\, dk \longleftrightarrow \text{enstrophy spectrum} \quad \mathcal{G}(k) = k^4 \mathcal{B}(k)$$

$\hookrightarrow$ (addition theorem).

$$\boxed{\begin{aligned} \mathcal{E}(k) &= 2\pi k \left[\overline{|\hat{u}(\vec{k})|^2} + \overline{|\hat{v}(\vec{k})|^2} \right] \\ \mathcal{G}(k) &= 2\pi k \left[\overline{|\hat{\zeta}(\vec{k})|^2} \right] \end{aligned}}$$

Total energy + enstrophy:

$$E = \frac{1}{2} \langle \overline{u^2 + v^2} \rangle = \frac{1}{2} \int_0^{+\infty} \mathcal{E}(k)\, dk$$

$$G = \frac{1}{2} \langle \overline{\zeta^2} \rangle = \frac{1}{2} \int_0^{+\infty} \mathcal{G}(k)\, dk.$$

$$\longleftrightarrow$$

Conservation laws:

$$\underset{\displaystyle Z(\rho)}{\underset{\updownarrow}{\frac{\partial}{\partial t} \overline{\zeta_1 \zeta_2}}} + \underset{\displaystyle T_Z(\rho)}{\underset{\updownarrow}{\overline{\zeta_1 J(\psi_2, \zeta_2)} + \overline{\zeta_2 J(\psi_1, \zeta_1)}}} = \underset{\displaystyle Z(\rho).}{\underset{\updownarrow}{\nu (\nabla_1^2 + \nabla_2^2) \overline{\zeta_1 \zeta_2}}}$$

$$\tau_Z(\rho) \equiv \overline{\zeta_1 J(\psi_2, \zeta_2)} + \overline{\zeta_2 J(\psi_1, \zeta_1)} = \int_0^{+\infty} T_Z(k)\, J_0(k\rho)\, dk.$$