Turbulence as a dynamical system

▼ Preliminary concepts.

<u>Def</u>: A set E is a <u>σ-algebra</u> on a set Ω if and only if

a) $\emptyset \in E$ and $\Omega \in E$

b) If $A \subseteq \mathbb{N}$ and $\forall i \in A : E_i \in E$ then:

$$\bigcup_{i\in A} E_i \in E \quad \text{and} \quad \bigcap_{i \in A} E_i \in E$$

c) $E_1 \in E$ and $E_2 \in E \Rightarrow E_1 - E_2 \in E$.

<u>examples</u>: the powerset $\mathcal{P}(\Omega)$ is one σ-algebra on Ω.
the set $\{\emptyset, \Omega\}$ is another trivial σ-algebra on Ω.
the set of countable unions of intervals is a σ-algebra on $\mathbb{R}$.

● Measure theory.

▶ The objective of measure theory is to associate subsets of a set Ω with a real number that encodes a notion of "size". The concepts "length", "area" and "volume" are examples of this. ◀

To define a measure, first we construct an <u>outer measure</u> on the entire powerset $\mathcal{P}(\Omega)$ of Ω, that must have the following properties.

<u>Def</u>: An <u>outer measure</u> on Ω is a function $\mu : \mathcal{P}(\Omega) \to \mathbb{R}$ such that:

a) $\mu(\emptyset) = 0$

b) If $A \subset B \Rightarrow \mu(A) \leq \mu(B)$. (monotonicity)

c) $\mu\left(\bigcup_{i=1}^{+\infty} A_i\right) \leq \sum_{i=1}^{+\infty} \mu(A_i)$ (subadditivity). ◀

Once we have such a definition, then we define a subset of $\mathcal{P}(\Omega)$ that contains subsets of Ω on which μ has stronger properties that capture our intuition of measure better:

<u>Def</u>: A set $E \subset \Omega$ is called <u>measurable</u> if and only if:

$$\forall T \in \mathcal{P}(\Omega) : \quad \mu(E \cap T) + \mu((\Omega - E) \cap T) = \mu(T). \quad ◀$$

Let $\mathfrak{M}$ be the set of all measurable subsets of Ω.

<u>Thm</u>: $\boxed{\mathcal{M} \text{ is a } \sigma\text{-algebra on } \Omega.}$

<u>Thm</u>: If μ is an outer measure on Ω, then:

α) $\forall A, B \in \mathcal{M} : A \subset B \Rightarrow \mu(A) < \mu(B)$. (strict monotonicity)
β) $\forall A_i \in \mathcal{M} : \mu\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} \mu(A_i)$. (additivity)

A function $\mu : \mathcal{M} \longrightarrow \mathbb{R}$ that has these stricter properties is called a <u>measure</u> on $\mathcal{M}$.
If $\mu(\Omega) = 1$, ~~then~~ and μ is a measure, then we call it a <u>probability measure</u>

- <u>Dynamical systems</u>

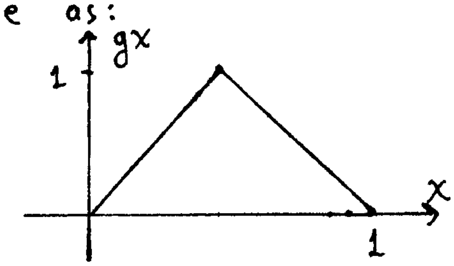
<u>Def</u>: A quadruplet $(\Omega, \mathcal{M}, \mu, G)$ is a <u>dynamical system</u> on Ω if and only if
α) μ is a probability measure on Ω.
β) $\mathcal{M}$ is the set of measurable subsets of Ω under μ.
c) G is an <u>abelian group</u> of transformations $g : \mathcal{M} \longrightarrow \mathcal{M}$ such that

$$\mu(gA) = \mu(A) \quad , \forall A \in \mathcal{M}, \forall g \in G.$$

<u>example</u>: A real dynamical system is the <u>tent-map</u>.
$\Omega = [0,1]$.
μ = <u>Lebesgue measure</u> on $\mathbb{R}$ (it is a probability measure in Ω)
$\mathcal{M}$ = defined by μ, Ω.
G represents <u>iterations</u> which we define as:
$g : \Omega \longrightarrow \Omega$ with

$$gx = \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2 - 2x & \text{if otherwise} \end{cases}$$

$G = \{g^n \mid n \in \mathbb{Z}\}$



- G can be interpreted as a group of <u>time-shifts</u> and adds the notion of evolution, the <u>dynamics</u>, provided that it conserves the measure μ.

An unproven conjecture is that we can treat the NV equations as a dynamical system:

$\underset{\sim}{\Omega}$ = the set of all initial conditions ϖ for the fluid state that satisfy the boundary conditions.
$\mu, \mathfrak{M}$ = are assumed definable by conjecture.
G = a group of <u>time-shifts</u> defined as follows:

▶ Let $\vec{v}(t, \vec{r}, \varpi)$ = the fluid velocity at time t, location $\vec{r}$ if the initial condition is ϖ.
The existence and uniqueness of a solution like that to the NV equations is a conjecture but we assume that it is true. We define G to be a group of transformations $g_t : \underset{\sim}{\Omega} \longrightarrow \underset{\sim}{\Omega}$ that is isomorphic to $\mathbb{R}$ ($G \cong \mathbb{R}$), $t \in \mathbb{R}$ such that

$$\forall g_t \in G : \; g_t \varpi = \vec{v}_o(\vec{r}) \equiv \vec{v}(t, \vec{r}, \varpi).$$

So g_t maps the initial state ϖ of the fluid to ~~the~~ a snapshot $\vec{v}_o(\vec{r})$ of its state after time t has elapsed ◀

Note that by construction,

$$\boxed{\vec{v}(t, \vec{r}, g_{t_o} \varpi) = \vec{v}(t + t_o, \vec{r}, \varpi)}$$

Because $\vec{v}$ has this property, we say that $\vec{v}$ is <u>stationary</u>, by definition.

● <u>Lebesgue integral and ergodic theorem</u>

<u>Def</u>: A <u>directed set</u> D is a pair $(D, <)$ where $<$ is a relation on D such that
a) $\alpha < \alpha$, $\forall \alpha \in D$
b) $\alpha < b$ and $b < c \Rightarrow \alpha < c$, $\forall \alpha, b, c \in D$
c) $\forall \alpha, b \in D, \exists c \in D : \alpha < c$ and $b < c$

<u>Def</u>: A <u>net</u> is a function $\varphi : \alpha \in D \longrightarrow x_\alpha \in \mathbb{R}$ where D is a directed set.

<u>Def</u>: A net x_a <u>converges</u> to ℓ $\left(\lim_{\alpha\in D} x_a = \ell\right)$ if:

$$\lim_{a\in D} x_a = \ell \Leftrightarrow \forall \varepsilon>0, \exists a_0\in D : \forall a\in D : (\alpha>\alpha_0 \to |x_a - x_{a_0}|<\varepsilon)$$

▶ Prove the usual theorems about limits ◀
Specialize limits of sequences, limits of functions <u>and</u> limits of partitions.

Let Ω be a set, μ a measure and $\mathfrak{M}$ the measurable subsets of Ω. Let $E\in\mathfrak{M}$ be a given measurable subset of Ω.

<u>Def</u>: A finite set $P=\{E_1,E_2,\dots,E_n\}$ is a <u>partition</u> of E if and only if

$$\bigcup_{i=1}^{n} E_i = E$$

and $E_i\cap E_j=\emptyset$, $\forall i,j$

▶ Let $\pi(E)$ the set of all partitions of E.

<u>Def</u>: Let $P,Q\in\pi(E)$ be two partitions. Define:
$P<Q \Leftrightarrow \forall a\in Q, \exists b\in P: a\subseteq b$.
(Q refinement of P)

<u>Def</u>: Let f be a function $f:\Omega\to\mathbb{R}$, $E\in\mathfrak{M}$, $P\in\pi(E)$.
The left + right Riemann sums are defined as:

$$L(f,P) = \sum_{e\in P}\mu(e)\inf\{f(\omega)\mid\omega\in e\}$$

$$U(f,P) = \sum_{e\in P}\mu(e)\sup\{f(\omega)\mid\omega\in e\}.$$

<u>Definition of Lebesgue integral</u>

A fundction $f:\Omega\to\mathbb{R}$ is integrable on $E\in\mathfrak{M}$ over measure μ if and only if:

$$\lim_{P\in\pi(E)} L(f,P) = \lim_{P\in\pi(E)} U(f,P) = \ell\in\mathbb{R}.$$

We then denote:

$$\ell = \int_E f\,d\mu$$

Def: A <u>random variable</u> in a dynamical system is a function $\varphi : \Omega \to \mathbb{R}$ which maps an initial condition ϑ to a value in $\mathbb{R}$.

Def: Given a random variable φ, we define the function
$f_\varphi(t,\vartheta) = \varphi(g_t \vartheta)$. , $\forall t \in \mathbb{R}$.
We call this the time evolution of φ.

Def: The <u>average</u> of φ is defined to be the integral:

$$\langle \varphi \rangle = \int_\Omega \varphi(\vartheta)\, d\mu(\vartheta)$$

We also call this the <u>probabilistic average</u> of $f(t,\vartheta)$:

$$\boxed{\overline{f(t,\vartheta)} \equiv F(t) = \int_\Omega f(t,\vartheta)\, d\mu(\vartheta).}$$

→ Probability density functions

With any function $f(t,\vartheta)$ we can associate a probability density function $p(t,x)$ defined by:

$$\boxed{p(t,x) = \frac{d}{dx} \mu(\{\vartheta \in \Omega \mid f(t,\vartheta) \leqslant x\})}$$

It is easy to show that

$$\int_{-\infty}^{+\infty} p(t,x)\,dx = \int_{-\infty}^{+\infty} \frac{d}{dx} \mu(\{\vartheta \in \Omega \mid f(t,\vartheta) \leq x\})\, dx =$$

$$= \mu(\{\vartheta \in \Omega \mid f(t,x) \leqslant +\infty\}) - \mu(\{\vartheta \in \Omega \mid f(t,x) \leqslant -\infty\}) =$$

$$= \mu(\Omega) - \mu(\varnothing) = 1 - 0 = 1.$$

A more difficult result is that:

$$\boxed{\overline{f(t)} = \int_\Omega f(t,\vartheta)\, d\mu(\vartheta) = \int_{-\infty}^{+\infty} x\, p(x)\, dx}$$

The pdf function provides with a way for evaluating the probabilistic average. Another approach is by using the ergodic theorem:

Birkhoff's ergodic theorem

<u>Def</u>: Let $(\Omega, \mathfrak{M}, \mu, G)$ be a dynamical system. We say that the system has a <u>transitive metric</u> if

$$\forall A \in \mathfrak{M} : (g_t A = A,\ \forall g_t \in G) \Leftrightarrow \mu(A) = 0 \vee \mu(A) = 1$$

(the only measurable subsets of Ω that are in $\mathfrak{M}$ which are invariant under all the time shifts in G are the ones that have either measure one or measure zero).

<u>Thm</u>: (<u>Ergodic theorem</u>)

Let $(\Omega, \mathfrak{M}, \mu, G)$ be a dynamical system with a transitive metric. Let $\varphi : \Omega \longrightarrow \mathbb{R}$ be a random variable. Then the probabilistic average $\langle \varphi \rangle$ is given by:

$$\langle \varphi \rangle = \int_\Omega \varphi(\vartheta)\, d\mu(\vartheta) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \varphi(g_t \vartheta)\, dt \ , \ \forall \vartheta \in \Omega - W$$

where $\mu(W) = 0$. ◀

For <u>almost</u> all initial conditions $\vartheta \in \Omega$ except for a set W of conditions of measure 0, $\langle \varphi \rangle$ can be obtained by time-averaging the evolution of φ under the time-shifting prescribed by the group G.

For the evolution function $f(t, \vartheta) \equiv \varphi(g_t \vartheta)$, this result yields:

$$\overline{f(t)} = \int_\Omega f(t, \vartheta)\, d\mu(\vartheta) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \cancel{\varphi(g_t \vartheta)\, dt}\ f(t, g$$

$$= \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(t, g_\tau \vartheta)\, d\cancel{\mu}\tau = \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(t+\tau, \vartheta)\, d\tau$$

The integral time-scale

A practical question raised by the ergodic theorem is: as we take the limit $T \to +\infty$ at what order of magnitude T_I does the limit converge such that the error becomes negligible? In other words we want:

$$T \gg T_I \Rightarrow \frac{1}{T}\int_0^T \varphi(g_\tau \omega)\, d\tau \approx \langle \varphi \rangle .$$

To answer this, we introduce the idea of correlation.

The correlation function.

Def: Let $\varphi_1 : \Omega \to \mathbb{R}$ and $\varphi_2 : \Omega \to \mathbb{R}$ be two random variables. Their correlation Γ is the number defined as:

$$\boxed{\Gamma = \langle (\varphi_1 - \langle \varphi_1 \rangle)(\varphi_2 - \langle \varphi_2 \rangle) \rangle}$$

Prop: If Γ is the correlation of φ_1 and φ_2, then:

$$\langle \varphi_1 \varphi_2 \rangle = \langle \varphi_1 \rangle \langle \varphi_2 \rangle + \Gamma$$

Proof

Define $\tilde{\varphi}_1(\omega) = \varphi_1(\omega) - \langle \varphi_1 \rangle \Rightarrow \varphi_1 = \langle \varphi_1 \rangle + \tilde{\varphi}_1$

and $\tilde{\varphi}_2(\omega) = \varphi_2(\omega) - \langle \varphi_2 \rangle . \Rightarrow \varphi_2 = \langle \varphi_2 \rangle + \tilde{\varphi}_2$

Then $\Gamma = \langle \tilde{\varphi}_1 \tilde{\varphi}_2 \rangle$. and $\langle \tilde{\varphi}_1 \rangle = \langle \tilde{\varphi}_2 \rangle = 0$.

We have then:

$$\begin{aligned}
\langle \varphi_1 \varphi_2 \rangle &= \langle (\tilde{\varphi}_1 + \langle \varphi_1 \rangle)(\tilde{\varphi}_2 + \langle \varphi_2 \rangle) \rangle = \\
&= \langle \tilde{\varphi}_1 \tilde{\varphi}_2 + \tilde{\varphi}_1 \langle \varphi_2 \rangle + \tilde{\varphi}_2 \langle \varphi_1 \rangle + \langle \varphi_1 \rangle \langle \varphi_2 \rangle \rangle = \\
&= \langle \tilde{\varphi}_1 \tilde{\varphi}_2 \rangle + \langle \tilde{\varphi}_1 \langle \varphi_2 \rangle \rangle + \langle \tilde{\varphi}_2 \langle \varphi_1 \rangle \rangle + \langle \langle \varphi_1 \rangle \langle \varphi_2 \rangle \rangle = \\
&= \Gamma + \langle \varphi_2 \rangle \langle \tilde{\varphi}_1 \rangle + \langle \varphi_1 \rangle \langle \tilde{\varphi}_2 \rangle + \langle \varphi_1 \rangle \langle \varphi_2 \rangle = \\
&= \Gamma + \langle \varphi_2 \rangle \cdot 0 + \langle \varphi_1 \rangle \cdot 0 + \langle \varphi_1 \rangle \langle \varphi_2 \rangle = \\
&= \langle \varphi_1 \rangle \langle \varphi_2 \rangle + \Gamma \qquad \square
\end{aligned}$$

This proposition gives a physical interpretation to the notion of Γ.

Now we extend this definition to the correlation function.

<u>Def</u> : Let $f(t,\vartheta)$ be an arbitrary function $f: A \times \Omega \to \mathbb{R}$
Let $\varphi_1(\vartheta) = f(t_1,\vartheta)$ and $\varphi_2(\vartheta) = f(t_2,\vartheta)$. be two random variables defined by f at given points $t_1, t_2 \in A$. Then we define:

$$\boxed{\Gamma(t_1,t_2) = \int_\Omega (\varphi_1(\vartheta) - \langle\varphi_1\rangle)(\varphi_2(\vartheta) - \langle\varphi_2\rangle)\, d\mu(\vartheta).}$$

the <u>correlation function</u> $\Gamma: A \times A \to \mathbb{R}$. ◀

A is usually a time domain, a space domain or a space + time domain.

<u>Correlation of a stationary fluctuation.</u>

Let $(\Omega, \mathfrak{M}, \mu, G)$ be a dynamical system and $f(t,\vartheta): A \times \Omega \to \mathbb{R}$ a stationary function that tracks the evolution of the system.

<u>Prop:</u> If the system has a transitive metric and $f(t,\vartheta)$ is stationary, and A is a vector space on $\mathbb{R}$, then: the correlation function depends only on the difference $\Delta t = t_2 - t_1$: $\Gamma(t_1,t_2) = \gamma(t_2 - t_1)$.

<u>Proof</u>

Claim $\Gamma(t_1+\tau, t_2+\tau) = \Gamma(t_1,t_2)$. Let $\tilde{f}$ be the fluctuation of f.

$$\Gamma(t_1+\tau, t_2+\tau) = \int_\Omega \tilde{f}(t_1+\tau,\vartheta)\tilde{f}(t_2+\tau,\vartheta)\, d\mu(\vartheta) =$$

$$= \int_\Omega \tilde{f}(t_1, g_\tau\vartheta)\tilde{f}(t_2, g_\tau\vartheta)\, d\mu(\vartheta) =$$

$$= \lim_{P \in \pi(\Omega)} \sum_{e \in P} \mu(e) \sup_{\vartheta \in e} \left[\tilde{f}(t_1, g_\tau\vartheta)\tilde{f}(t_2, g_\tau\vartheta)\right] =$$

$$= \lim_{P \in \pi(\underline{o})} \sum_{e \in P} \mu(g_\tau e) \sup_{\vartheta \in e} \left[\tilde{f}(t_1, g_\tau \vartheta) \tilde{f}(t_2, g_\tau \vartheta) \right]$$

(because $\mu(g_\tau e) = \mu(e)$, bc G preserves the measure μ)

$$= \int_{g_\tau \underline{o}} \tilde{f}(t_1, \vartheta) \tilde{f}(t_2, \vartheta)\, d\mu(\vartheta) = \int_{\underline{o}} \tilde{f}(t_1, \vartheta) \tilde{f}(t_2, \vartheta)\, d\mu(\vartheta) = \Gamma(t_1, t_2).$$

because $\left.\begin{array}{l}\mu \text{ transitive} \\ \mu(\underline{o}) = 1\end{array}\right\} \Rightarrow g_\tau \underline{o} = \underline{o}, \ \forall g_\tau \in G.$

Since $\Gamma(t_1+\tau, t_2+\tau) = \Gamma(t_1, t_2)$, $\forall \tau > 0$, then with some abuse of notation we write $\Gamma(t_1, t_2) = \Gamma(t_2 - t_1)$ □

<u>Thm</u> : Let $f(t, \vartheta)$ be a stationary function with $\overline{f(t)} = 0$ and $t \in \mathbb{R}$. Define

$$\varphi_T(\vartheta) = \frac{1}{T} \int_0^T f(t, \vartheta)\, dt$$

be an approximation of $\overline{f(t)}$ (which should be zero).
Then:

$$\langle \varphi_T \varphi_T \rangle \leqslant \frac{2}{T} \int_0^{+\infty} d\tau\, |\Gamma(\tau)|$$

is an error estimate of the variance of $\varphi_T(\vartheta)$.

<u>Proof</u>

$$\langle \varphi_T \varphi_T \rangle = \int_{\underline{o}} \varphi_T(\vartheta) \varphi_T(\vartheta)\, d\mu(\underline{o}) = \int_{\underline{o}} \left[\frac{1}{T} \int_0^T f(t, \vartheta)\, dt \right]^2 d\mu(\underline{o}) =$$

$$= \frac{1}{T^2} \int_{\underline{o}} d\mu(\vartheta) \int_0^T dt_1 \int_0^T dt_2\, f(t_1, \vartheta) f(t_2, \vartheta) =$$

$$= \frac{1}{T^2} \iint_{[0,T]^2} dt_1 dt_2 \left[\int_{\underline{o}} d\mu(\vartheta)\, f(t_1, \vartheta) f(t_2, \vartheta) \right] =$$

$$= \frac{1}{T^2} \iint_{[0,T]^2} dt_1 dt_2\, \Gamma(t_2 - t_1) = \cdots =$$

$$= \frac{2}{T^2} \int_0^T dt_1 \int_0^{t_1} dt_2\, \Gamma(t_2) \leq \frac{2}{T} \int_0^{+\infty} d\tau\, |\Gamma(\tau)| \qquad \square$$

Def : The integral time scale is defined as:

$$\boxed{T_I = \frac{1}{\Gamma(0)} \int_0^{+\infty} |\Gamma(\tau)| \, d\tau .}$$

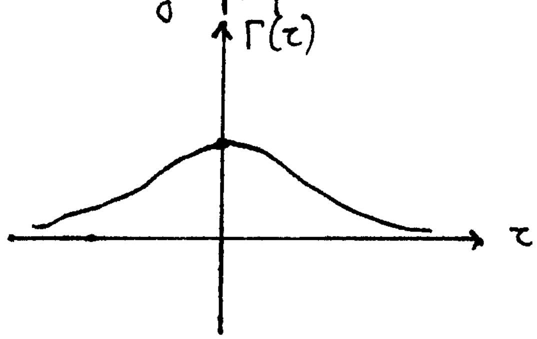
- Note that $\Gamma(0) = \overline{f(t) f(t)}$ is the variance of $f(t, \vartheta)$. We divide the integral with $\Gamma(0)$ to obtain a T_I whose dimensions are indeed time. If we substitute this definition we have:

$$\langle \varphi_T \varphi_T \rangle \leq \frac{2}{T} \int_0^{+\infty} d\tau \, |\Gamma(\tau)| = 2\Gamma(0) \frac{T_I}{T}$$

So the time-average variance becomes negligible when $T \gg T_I$. When $T \approx T_I$, then it is comparable to the variance $\Gamma(0)$ of the fluctuation itself.

Properties of the correlation function.

The correlation function $\Gamma(\tau)$ looks like this and has the following properties:



1) $\Gamma(-\tau) = \Gamma(\tau)$
2) $|\Gamma(\tau)| \leq \Gamma(0) \quad , \forall \tau \in \mathbb{R}$.
3) $\Gamma'(0) = 0$

So it is symmetric around $\tau = 0$, it has a max at $\tau = 0$ and it is bounded by $\Gamma(0)$.

Now we prove these properties:

Prop: $\boxed{\Gamma(-\tau) = \Gamma(\tau).}$

Proof

Recall that $\Gamma(t_1 + \tau, t_2 + \tau) = \Gamma(t_1, t_2)$
and $\Gamma(t_1, t_2) = \Gamma(t_2, t_1)$. (from the definition)

It follows that

$$\Gamma(-\tau) = \Gamma(t, t-\tau) = \Gamma(t+\tau, t-\tau+\tau) = \Gamma(t+\tau, t) = $$
$$= \Gamma(t, t+\tau) = \Gamma(\tau) \quad \square$$

<u>Thm</u> : $\boxed{|\Gamma(\tau)| \leqslant \Gamma(0) \quad , \forall \tau \in \mathbb{R}}$

<u>Proof</u>

Let $f(t,\vartheta)$ be <u>a centered</u> function, such that $\overline{f(t)} = 0$, and let $\Gamma(\tau) = \overline{f(t) f(t+\tau)}$. with $t \in \mathbb{R}$ given.
Define I as follows:

$$I = \int_{\Omega} [f(t,\vartheta) \pm f(t+\tau,\vartheta)]^2 d\mu(\vartheta) =$$
$$= \int_{\Omega} [f^2(t,\vartheta) + f^2(t+\tau,\vartheta) \pm 2 f(t,\vartheta) f(t+\tau,\vartheta)] \, d\mu(\vartheta) =$$
$$= \int_{\Omega} f^2(t,\vartheta) \, d\mu(\vartheta) + \int_{\Omega} f^2(t+\tau,\vartheta) \, d\mu(\vartheta) \pm 2 \int_{\Omega} f(t,\vartheta) f(t+\tau,\vartheta) \, d\mu(\vartheta)$$
$$= \Gamma(0) + \Gamma(0) \pm 2\Gamma(\tau) = 2\Gamma(0) \pm 2\Gamma(\tau).$$

Note that $[f(t,\vartheta) \pm f(t+\tau,\vartheta)]^2 \geqslant 0 \,, \forall \vartheta \in \Omega \Rightarrow$

$$\Rightarrow I \geqslant 0 \Rightarrow \begin{cases} \Gamma(0) + \Gamma(\tau) \geqslant 0 \\ \Gamma(0) - \Gamma(\tau) \geqslant 0 \end{cases} \Rightarrow -\Gamma(0) \leqslant \Gamma(\tau) \leqslant \Gamma(0)$$
$$\Rightarrow |\Gamma(\tau)| \leqslant \Gamma(0) \quad \square$$

► If the function $f(t,\vartheta)$ is differentiable wrt t, it also follows that $\Gamma(\tau)$ is differentiable. wrt τ.
Let $f'(t,\vartheta) = \frac{\partial}{\partial t} f(t,\vartheta)$. Then:

$$\Gamma'(\tau) = \frac{\partial}{\partial \tau} \int_{\Omega} f(t,\vartheta) f(t+\tau,\vartheta) \, d\mu(\vartheta) =$$
$$= \int_{\Omega} \frac{\partial}{\partial \tau} \left[f(t,\vartheta) f(t+\tau,\vartheta) \right] d\mu(\vartheta) =$$
$$= \int_{\Omega} f(t,\vartheta) f'(t+\tau,\vartheta) \, d\mu(\vartheta). \quad , \forall \tau \in \mathbb{R}. \quad ◄$$

The main reason why these observations are important, is because we can use them to show the following result:

Thm: If $f(t,\vartheta)$ is a stationary function, then it is uncorrelated with its time-derivative: $\overline{f(t)f'(t)} = 0$

Proof

$$\overline{f(t)f'(t)} = \int_{\Omega} f(t,\vartheta)\, f'(t,\vartheta)\, d\mu(\vartheta) = \Gamma'(0). \qquad (1)$$

However note that
$|\Gamma(\tau)| \leq \Gamma(0)$, $\forall \tau \in \mathbb{R} \Rightarrow \Gamma$ has a local max at $\tau = 0$; Γ is differentiable $\Rightarrow$

$\Rightarrow \Gamma'(0) = 0 \Rightarrow \overline{f(t)f'(t)} = 0$ $\square$

This does not generalize with higher-order derivatives. In fact, we get the following result for the 2nd-order derivative:

Prop: If $f(t,\vartheta)$ is a stationary function, then: $\overline{f(t)f''(t)} = -\overline{f'(t)f'(t)}$

Proof

Note that $\dfrac{\partial}{\partial t}\left(f\dfrac{\partial f}{\partial t}\right) = \dfrac{\partial f}{\partial t}\dfrac{\partial f}{\partial t} + f\dfrac{\partial^2 f}{\partial t^2} \Rightarrow$

$$\Rightarrow f\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(f\frac{\partial f}{\partial t}\right) - \frac{\partial f}{\partial t}\frac{\partial f}{\partial t} \qquad (1).$$

From the previous result:

$$\overline{\frac{\partial}{\partial t}\left(f\frac{\partial f}{\partial t}\right)} = \int_{\Omega} \frac{\partial}{\partial t}\left(f\frac{\partial f}{\partial t}\right) d\mu(\vartheta) = \frac{\partial}{\partial t}\int_{\Omega} f\frac{\partial f}{\partial t}\, d\mu(\vartheta) =$$

$$= \frac{\partial}{\partial t}\left(\overline{f(t)f'(t)}\right) = 0$$

therefore: $\overline{f\dfrac{\partial^2 f}{\partial t^2}} = -\overline{\dfrac{\partial f}{\partial t}\dfrac{\partial f}{\partial t}}$ $\square$

▼ Associated differential time-scale

The integral time-scale T_I provides a measure of the time needed to obtain a time-average that approximates well the probabilistic average.

The associated differential time scale is formally defined in terms of the Taylor expansion of $\Gamma(\tau)$:

$$\boxed{\Gamma(\tau) = \Gamma(0)\left(1 - \frac{\tau^2}{T_s^2} + o(\tau^4)\right)}$$

The lack of ~~low~~ odd-order terms is justified by $\Gamma(\tau) = \Gamma(-\tau), \forall \tau \in \mathbb{R}$. The T_s measures the ~~so~~ scale short-term fluctuations, whereas T_I measures the scale in which long-term patterns in the statistics of fluctuations emerge.

Thm : The associated differential time-scale is given by:

$$T_s = \left[-\frac{2\Gamma(0)}{\Gamma''(0)}\right]^{1/2}$$

Proof

First we derive the assumed form of the Taylor expansion:

$$\Gamma(\tau) = \Gamma(0) + \tau\Gamma'(0) + \frac{\tau^2}{2}\Gamma''(0) + o(\tau^3) =$$

$$= \Gamma(0) + \frac{\tau^2}{2}\Gamma''(0) + o(\tau^3) \quad , \text{because } \Gamma'(0) = 0 .$$

Next we show that $\Gamma''(0)$ is indeed a negative number.
Let $f(t,\omega)$ be the stationary function that corresponds to $\Gamma(\tau)$.
Then,

$$\Gamma''(0) = \overline{f(t)f''(t)} = -\overline{f'(t)f'(t)} = -\int_\Omega [f'(t,\omega)]^2 \, d\mu(\omega).$$

Since $[f'(t,\omega)]^2 \geq 0, \forall \omega \in \Omega \Rightarrow \Gamma''(0) \leq 0$.
therefore our definition of T_s is consistent. We have:

$$-\frac{\Gamma(0)}{T_s^2} = \frac{\Gamma''(0)}{2} \Rightarrow T_s^2 = -\frac{2\Gamma(0)}{\Gamma''(0)} \Rightarrow T_s = \left[-\frac{2\Gamma(0)}{\Gamma''(0)}\right]^{1/2} \quad \square$$

Symmetry and the correlation function.

In this discussion we examine symmetry in general. We specialize later to symmetry under translation and rotation which corresponds to homogeneity, isotropy and their consequences.

The context is a dynamical system $(\Omega, \mathcal{M}, \mu, G)$ where Ω is the set of all possible initial states ϑ and G contains time-shift operators $g: \Omega \to \Omega$ that govern the evolution of the system in time. We assume that the state of the system is functions $\varphi_0(x)$ such that $\varphi_0 : V \to \mathbb{R}$ where V is a vector space on $\mathbb{R}$; so $\varphi_0 \in \Omega$. The space V is $\mathbb{R}^2$ for 2d flow, and $\mathbb{R}^3$ for 3d flow.

In this context we define:

$$\tilde{\varphi}(t, x, \vartheta) = (g_t \vartheta)(x)$$

which by construction is a stationary function:

$$\tilde{\varphi}(t+\tau, x, \vartheta) = (g_{t+\tau}\vartheta)(x) = g_t(g_\tau \vartheta)(x) = \tilde{\varphi}(t, x, g_\tau \vartheta).$$

therefore the time correlation function has all the properties that we have described:

$$\Gamma(\tau) = \overline{\varphi(t,x)\varphi(t+\tau,x)}.$$

We also assume that μ is a transitive measure, therefore the ergodic theorem applies.

Def: We will call an abelian group H a transformation group if it contains mappings $h : V \to V$.

notation: If $x \in V$, the image of x under a transformation $h \in H$ will be written as hx. There is also an associated transformation $\vartheta \in \Omega \to \mathring{h}\vartheta \in \Omega$ which we define by:

$$\vartheta' = \mathring{h}\vartheta \Leftrightarrow \vartheta'(x) = (\mathring{h}\vartheta)(x), \ \forall x \in V.$$

A transformation group is interesting if it is also a symmetry group. Note that for $h \in H$ and $g_t \in G$, we can define $\mathring{h} g_t$ and $g_t \mathring{h}$ as compositions of mappings $\Omega \to \Omega$. The union $G \cup H$ is then a group, also, but the elements of G do not have to commute with the elements of H.

<u>Def</u>: A transformation group H is a <u>symmetry group</u> if and only if:

$$\forall g_t \in G, \forall h \in H : g_t h^\circ = h^\circ g_t .$$

↪ A transformation group will be a symmetry group if the governing equations that yield $\varphi(t, x, \vartheta)$ are invariant under the transformation. Since we abstract these equations into the notion of the time-shift group G, our definition of the symmetry group is expressed in relation to the group G. The following shows that the definition means the right thing.

<u>Prop</u>: If H is a symmetry group, then:

$$\boxed{\tilde{\varphi}(t, hx, \vartheta) = \tilde{\varphi}(t, x, h^\circ \vartheta)} , \forall h \in H$$

<u>Proof</u>

Let $h \in H$ be given.

$$\tilde{\varphi}(t, hx, \vartheta) = (g_t \vartheta)(hx) = (h^\circ g_t \vartheta)(x) \overset{H \text{ symmetry group}}{=}$$

$$= (g_t h^\circ \vartheta)(x) = \tilde{\varphi}(t, x, h^\circ \vartheta) \qquad \square$$

↪ This means that $\tilde{\varphi}(t, hx, \vartheta)$ is the solution that we obtain if we use $h^\circ \vartheta$ as the initial condition.

<u>Def</u>: Let $(\Omega, \mathcal{M}, \mu, G)$ be a dynamical system. We say that ~~a sym~~ the dynamical system is <u>symmetric</u> under a symmetry group H if it preserves measure as well:

$$\mu(h^\circ A) = \mu(A), \forall A \in \mathcal{M}, \forall h \in H$$

<u>Def</u>: We also say that a dynamical system $(\Omega, \mathcal{M}, \mu, G)$ is <u>ergodic</u> under a symmetry group H if:

$$\forall A \in \mathcal{M} : (h^\circ A = A, \forall h \in H) \Leftrightarrow \mu(A) = 0 \vee \mu(A) = 1 .$$

- A direct consequence, if the system is indeed ergodic is that $h\Omega = \Omega$, $\forall h \in H$.

Consequences on the correlation function

The definition for the correlation function can be generalized. First we define the fluctuation of $\varphi(t,x,\vartheta)$:

Def: The fluctuation of $\tilde{\varphi}(t,x,\vartheta)$ is:

$$\boxed{\varphi(t,x,\vartheta) = \tilde{\varphi}(t,x,\vartheta) - \int_{\Omega} \tilde{\varphi}(t,x,\vartheta_0)\, d\mu(\vartheta_0).}$$

- A direct consequence is that $\overline{\varphi(t,x)} = 0$ and: $\tilde{\varphi}(t,x,\vartheta) = \overline{\tilde{\varphi}(t,x)} + \varphi(t,x,\vartheta)$.

Def: The generalized correlation function $\Gamma(t_1,t_2,x_1,x_2)$ is defined as:

$$\boxed{\Gamma(t_1,t_2,x_1,x_2) = \int_{\Omega} \varphi(t_1,x_1,\vartheta)\varphi(t_2,x_2,\vartheta)\, d\mu(\vartheta)}$$

Now note that stationarity as well as symmetries are also true with the fluctuations $\varphi(t,x,\vartheta)$. First, we prove the following lemma about dynamical systems:

Lemma: Let $f:\Omega \to \mathbb{R}$ be any random variable and $g:\Omega\to\Omega$ a transformation such that $g\Omega = \Omega$ and $\forall \vartheta\in\Omega: \mu(\vartheta)=\mu(g\vartheta)$.
Then $\int_{\Omega} f(g\vartheta)\, d\mu(\vartheta) = \int_{\Omega} f(\vartheta)\, d\mu(\vartheta)$.

Proof

$$\begin{aligned}
\int_{\Omega} f(g\vartheta)\, d\mu(\vartheta) &= \lim_{P\in\Pi(\Omega)} \sum_{e\in P} \mu(e) \sup_{\vartheta\in e}\left[f(g\vartheta)\right] = \quad \Big) \ \mu(g\vartheta)=\mu(\vartheta) \\
&= \lim_{P\in\Pi(\Omega)} \sum_{e\in P} \mu(ge) \sup_{\vartheta\in e}\left[f(g\vartheta)\right] = \\
&= \lim_{P\in\Pi(g\Omega)} \sum_{e\in P} \mu(e) \sup_{\vartheta\in e}\left[f(\vartheta)\right] = \\
&= \int_{g\Omega} f(\vartheta)\, d\mu(\vartheta) = \int_{\Omega} f(\vartheta)\, d\mu(\vartheta). \qquad \square
\end{aligned}$$

<u>Prop</u> : $\boxed{\varphi(t+\tau, x, \vartheta) = \varphi(t, x, g_\tau \vartheta)}$, $\forall g_\tau \in G$.

<u>Proof</u>
Assume that metric transitivity applies. Then
$\mu(\underline{\Omega}) = 1 \Rightarrow g_\tau \underline{\Omega} = \underline{\Omega}$, $\forall g_\tau \in G \Rightarrow$ the Lemma applies.
therefore,

$$\begin{aligned}
\varphi(t+\tau, x, \vartheta) &= \tilde{\varphi}(t+\tau, x, \vartheta) - \int_{\underline{\Omega}} \tilde{\varphi}(t+\tau, x, \vartheta_0)\, d\mu(\vartheta_0) = \\
&= \tilde{\varphi}(t, x, g_\tau \vartheta) - \int_{\underline{\Omega}} \tilde{\varphi}(t, x, g_\tau \vartheta_0)\, d\mu(\vartheta_0) = \\
&= \tilde{\varphi}(t, x, g_\tau \vartheta) - \int_{\underline{\Omega}} \tilde{\varphi}(t, x, \vartheta_0)\, d\mu(\vartheta_0) = \\
&= \varphi(t, x, g_\tau \vartheta). \qquad \square
\end{aligned}$$

<u>Prop</u>: If $(\underline{\Omega}, \mathfrak{M}, \mu, G)$ is symmetric and ergodic under a symmetry group H, then:

$$\boxed{\varphi(t, hx, \vartheta) = \varphi(t, x, h^\circ \vartheta)} \ , \ \forall h \in H$$

<u>Proof</u>
Let $h \in H$ be given.
$(\underline{\Omega}, \mathfrak{M}, \mu, G)$ symmetric under $H \Rightarrow \mu(h^\circ \vartheta) = \mu(\vartheta)$, $\forall \vartheta \in \underline{\Omega}$, $\forall h \in H$ $\Big\} \Rightarrow$
$(\underline{\Omega}, \mathfrak{M}, \mu, G)$ ergodic under $H \Big\} \Rightarrow h^\circ \underline{\Omega} = \underline{\Omega}$
$\mu(\underline{\Omega}) = 1$
$\Rightarrow$ the Lemma applies for $h^\circ : \underline{\Omega} \to \underline{\Omega}$
Also because
~~$(\underline{\Omega}, \mathfrak{M}, \mu, G)$ symmetric under H~~
H symmetry group $\Rightarrow \tilde{\varphi}(t, hx, \vartheta) = \tilde{\varphi}(t, x, h^\circ \vartheta)$
It follows from all this that:

$$\begin{aligned}
\varphi(t, hx, \vartheta) &= \tilde{\varphi}(t, hx, \vartheta) - \int_{\underline{\Omega}} \tilde{\varphi}(t, hx, \vartheta_0)\, d\mu(\vartheta_0) = \\
&= \tilde{\varphi}(t, x, h^\circ \vartheta) - \int_{\underline{\Omega}} \tilde{\varphi}(t, x, h^\circ \vartheta_0)\, d\mu(\vartheta_0) = \\
&= \tilde{\varphi}(t, x, h^\circ \vartheta) - \int_{\underline{\Omega}} \tilde{\varphi}(t, x, \vartheta_0)\, d\mu(\vartheta_0) = \\
&= \varphi(t, x, h^\circ \vartheta) \qquad \square
\end{aligned}$$

Also note that because φ is stationary, we can show, using the same argument as before, that

$$\Gamma(t_1+\tau, t_2+\tau, x_1, x_2) = \Gamma(t_1, t_2, x_1, x_2)$$

and therefore:

$$\Gamma(t_1, t_2, x_1, x_2) \equiv \Gamma(\Delta t, x_1, x_2).$$

with $\Delta t = t_2 - t_1$. The following theorem extends this result to x.

<u>Thm</u>: If $(\underline{\Omega}, m, \mu, G)$ is a dynamical system that is symmetric and ergodic wrt a symmetry group H, then

$$\boxed{\forall h \in H : \Gamma(\tau, hx_1, hx_2) = \Gamma(\tau, x_1, x_2).}$$

<u>Proof</u>

$$\begin{aligned}
\Gamma(\tau, hx_1, hx_2) &= \int_{\underline{\Omega}} \varphi(t, hx_1, \vartheta)\varphi(t+\tau, hx_2, \vartheta)\, d\mu(\vartheta) = \quad \text{propositions} \\
&= \int_{\underline{\Omega}} \varphi(t, x_1, h^\circ\vartheta)\varphi(t+\tau, x_2, h^\circ\vartheta)\, d\mu(\vartheta) = \quad \text{Lemma} \\
&= \int_{\underline{\Omega}} \varphi(t, x_1, \vartheta)\varphi(t+\tau, x_2, \vartheta)\, d\mu(\vartheta) = \\
&= \Gamma(\tau, x_1, x_2). \qquad \square
\end{aligned}$$

A consequence is that we can reduce $\Gamma(\tau, x_1, x_2)$ to $\Gamma(\tau, a)$. But what is a? That depends on the symmetry group H. Define the following relation on $V \times V$:

$$(x_1, x_2) \overset{H}{\sim} (y_1, y_2) \Leftrightarrow \exists h \in H : (y_1, y_2) = (hx_1, hx_2).$$

The relation $eq(H)$ is an <u>equivalence relation</u> because it satisfies:

$$X \overset{H}{\sim} Y \Leftrightarrow Y \overset{H}{\sim} X$$
$$X \overset{H}{\sim} X.$$
$$X \overset{H}{\sim} Y \wedge Y \overset{H}{\sim} Z \Rightarrow X \overset{H}{\sim} Z.$$

which is easy to prove. (it follows from the properties of H as a group). Consider the set of equivalence classes:

$$A = V \times V / eq(H).$$

For a given $a \in A : \forall (x_1, x_2), (y_1, y_2) \in a : \Gamma(\tau, x_1, x_2) = \Gamma(\tau, y_1, y_2)$. so there is a unique value, which we can associate with a, and denote as $\Gamma(\tau, a)$, such that

$$\forall (x_1, x_2) \in a : \Gamma(\tau, x_1, x_2) = \Gamma(\tau, a).$$

In all the special cases we will ever consider, the equivalence classes A are isomorphic to $[0, +\infty)$.