Spectrum of random fields

Let $\varphi : \mathbb{R}^n \times \underline{\Omega} \times O(n) \longrightarrow \mathbb{R}^m$ be a random field. For this entire section we assume that

$$\langle \varphi(x) \rangle = 0 \quad , \quad \forall x \in \mathbb{R}^n$$

where $\langle\ \rangle$ indicates ensemble average (i.e integrating out the dependence on $\vartheta \in \underline{\Omega}$). In other words, we will be solely concerned with the fluctuation of the actual field. In fact, it is to the fluctuation that we attribute homogeneity, etc.

Recall that $\mathbb{R}^n$ and $[0,+\infty) \times SO(n)$ (the spherical representation of $\mathbb{R}^n$) are isomorphic. In both representations, let

$$S(\rho) = \{ \vec{r} \in \mathbb{R}^n \mid \|\vec{r}\| \leqslant \rho \}$$

be a ball of radius ρ. Equivalently:

$$S(\rho) = \{ (r, A) \in [0,+\infty) \times SO(n) \mid r \leqslant \rho \}$$

We denote $d\vec{r}$ the volume measure for $\vec{r}$, and $d\Omega(\vec{r})$ the solid angle measure for $\vec{r}$, when that is not obvious. Finally we define dr such that $d\vec{r} = r^{n-1}\, dr\, d\Omega(\vec{r})$.

We start, following the 1d development (p.49), by generalizing filtering:

Definition : Let φ be a random field with Fourier transform

$$\boxed{\hat{\varphi}(\vec{k}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi(\vec{r}) \exp[-i\vec{k}\cdot\vec{r}]\, d\vec{r}}$$

Then the low-pass-filtered $\varphi^{<}_{k_0}$ and the high-pass-filtered $\varphi^{>}_{k_0}$ random fields are defined by

$\varphi^{<}_{k_0}(\vec{r}) = \int_{S(\rho)} \hat{\varphi}(\vec{k}) \exp[i\vec{k}\cdot\vec{r}]\, d\vec{k}$	$\varphi^{>}_{k_0}(\vec{r}) = \int_{\mathbb{R}^n - S(\rho)} \hat{\varphi}(\vec{k}) \exp[i\vec{k}\cdot\vec{r}]\, d\vec{k}$

In general dimensions, the spectrum and the cumulative spectrum are defined as <u>tensors</u>:

<u>Definition</u>: Let φ be a random field. We define:
a) The <u>cumulative spectrum tensor</u> by

$$\boxed{\boldsymbol{\mathcal{E}}(\vec{r},k) = \frac{1}{2}\int_{\Omega} \varphi_k^<(\vec{r},\varpi)\otimes\varphi_k^<(\vec{r},\varpi)\, d\mu(\varpi)}$$

b) The <u>spectrum tensor</u> by

$$\boxed{\mathbf{E}(\vec{r},k) = \frac{d}{dk}\boldsymbol{\mathcal{E}}(\vec{r},k)}$$

<u>Definition</u>: Let φ be a random field. The <u>cumulative spectrum</u> and the <u>spectrum</u> of φ are defined as the traces of the corresponding tensors:

$$\boxed{\begin{aligned} \mathcal{E}(\vec{r},k) &= \operatorname{tr}\boldsymbol{\mathcal{E}}(\vec{r},k) \\ E(\vec{r},k) &= \operatorname{tr}\mathbf{E}(\vec{r},k) \end{aligned}}$$

From the trace identity
$$\operatorname{tr}(a\otimes b) = a\cdot b$$
It follows immediately that

$$\begin{aligned} \mathcal{E}(\vec{r},k) &= \frac{1}{2}\int_{\Omega} \varphi_k^<(\vec{r},\varpi)\cdot\varphi_k^<(\vec{r},\varpi)\, d\mu(\varpi) \\ &= \frac{1}{2}\int_{\Omega} \|\varphi_k^<(\vec{r},\varpi)\|^2\, d\mu(\varpi) \end{aligned}$$

where $\|\varphi_k^<\|$ is the norm of $\varphi_k^<$ which is either a scalar, in which case it is trivial, or a 2d or 3d vector.

The correlation tensor of a random field

Next, we introduce the correlation tensor $\mathbf{B}$ which we define as follows:

<u>Definition</u>: Let φ be a random field. The <u>correlation tensor</u> $\mathbf{B}$ is then given by:

$$\boxed{\mathbf{B}(\vec{r}_1,\vec{r}_2)=\int_{\Omega}\varphi(\vec{r}_1,\omega)\otimes\varphi(\vec{r}_2,\omega)\,d\mu(\omega)}$$

It is easy to see (using theorem on p. 33) that if

$$\varphi \text{ homogeneous } \Rightarrow \mathbf{B}(\vec{r}_1+\vec{r},\vec{r}_2+\vec{r})=\mathbf{B}(\vec{r}_1,\vec{r}_2)$$
$$\Rightarrow \mathbf{B}(\vec{r}_1,\vec{r}_2)=\mathbf{B}_0(\vec{r}_2-\vec{r}_1).$$

so the correlation tensor then depends on only one variable. If φ is also isotropic, then we can show that $\mathbf{B}_0$ is an isotropic tensor (in the sense of p. 82–83).

<u>Proposition</u>: If φ is a homogeneous and isotropic field, then the correlation tensor $\mathbf{B}(\vec{r}_1,\vec{r}_2)$ can be written as

$$\mathbf{B}(\vec{r}_1,\vec{r}_2)=\mathbf{B}_0(\vec{r}) \quad , \quad \vec{r}=\vec{r}_2-\vec{r}_1$$

with $\mathbf{B}_0$ an isotropic tensor.

<u>Proof</u>

Let $A\in O(n)$ be given. Then

$$\mathbf{B}_0(A\vec{r})=\int_{\Omega}\varphi(A\vec{r}_1,\omega)\otimes\varphi(A\vec{r}_2,\omega)\,d\mu(\omega)=$$
$$=\int_{\Omega}\left[m_\varphi(A)\varphi(\vec{r}_1,g(A)\omega)\right]\otimes\left[m_\varphi(A)\varphi(\vec{r}_2,g(A)\omega)\right]d\mu(\omega)=$$
$$=\int_{\Omega}\left[\oplus^2 m_\varphi(A)\right]\left[\varphi(\vec{r}_1,g(A)\omega)\otimes\varphi(\vec{r}_2,g(A)\omega)\right]d\mu(\omega)=$$

$$= \oplus^2 m_\varphi(A) \int_{\underline{O}} \varphi(\vec{r}_1, g(A)\vartheta) \otimes \varphi(\vec{r}_2, g(A)\vartheta)\, d\mu(\vartheta) =$$

$$= \oplus^2 m_\varphi(A) \int_{\underline{O}} \varphi(\vec{r}_1,\vartheta) \otimes \varphi(\vec{r}_2,\vartheta)\, d\mu(\vartheta) =$$

$$= \left(\oplus^2 m_\varphi(A)\right) \mathbf{B}_0(\vec{r}) \quad , \quad \forall A \in O(n).$$

It follows that $\mathbf{B}_0$ is isotropic because $\oplus^2 m_\varphi = m_{\mathbf{B}_0}$ is the transformation low of $\mathbf{B}_0$ itself. □

▼ Energy spectrum for homogeneous flow

Now we briefly review the consequences of homogeneity on the energy spectrum. Ultimately we want to analyze local homogeneity and local isotropy. The results of such analysis can be carried over to this more special situation using the main result that we want to prove here:

$$\varphi \text{ homogeneous} \Rightarrow \mathbf{E}(\vec{r}, k) = \mathbf{E}(k).$$

We follow our development of the simpler case on p. 51:

Proposition: Let φ and ψ be two random fields such that $\langle\varphi\rangle = \langle\psi\rangle = 0$ of the form: $\varphi, \psi : \mathbb{R}^n \times \underline{O} \times O(n) \longrightarrow \mathbb{R}^m$

Then:

$$\boxed{\varphi \simeq \psi \Rightarrow (\forall k_0 > 0)(\varphi^<_{k_0} \simeq \psi^<_{k_0})}$$

Proof

Let $\hat{\varphi}, \hat{\psi}$ be the Fourier transforms of φ, ψ respectively. Then because $\varphi(\vec{r},\vartheta) = \psi(\vec{r}, g\vartheta)$ for some g:

$$\hat{\varphi}(\vec{k},\vartheta) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi(\vec{r},\vartheta) \exp[-i\vec{k}\cdot\vec{r}]\, d\vec{r} =$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \psi(\vec{r}, g\vartheta) \exp[-i\vec{k}\cdot\vec{r}]\, d\vec{r} = \hat{\psi}(\vec{k}, g\vartheta).$$

It follows that:

$$\varphi_{k_0}^{<}(\vec{r},\vartheta) = \int_{S(k_0)} \hat{\varphi}(\vec{k},\vartheta) \exp[i\vec{k}\cdot\vec{r}]\, d\vec{k} =$$
$$= \int_{S(k_0)} \hat{\psi}(\vec{k},g\vartheta) \exp[i\vec{k}\cdot\vec{r}]\, d\vec{k} = \psi_{k_0}^{<}(\vec{r},g\vartheta), \ \forall \vec{r} \in \mathbb{R}^n \Rightarrow$$
$$\Rightarrow \varphi_{k_0}^{<} \simeq \psi_{k_0}^{<}. \quad \square$$

<u>Proposition</u>: Let φ be a random field with $\langle\varphi\rangle = 0$.
Then:

$$\boxed{\varphi \text{ homogeneous} \Rightarrow \nabla\langle\varphi(\vec{r})\otimes\varphi(\vec{r})\rangle = 0}$$

where ∇ = gradient wrt $\vec{r}$.

<u>Proof</u>

$$\langle\varphi(\vec{r})\otimes\varphi(\vec{r})\rangle = \mathbf{B}(\vec{r},\vec{r}) = \mathbf{B}_0(\vec{r}-\vec{r}) = \mathbf{B}_0(0) = \text{constant} \Rightarrow$$
$$\Rightarrow \nabla\langle\varphi(\vec{r})\otimes\varphi(\vec{r})\rangle = 0 \quad \square$$

Combining these two results we obtain:

<u>Theorem</u>: Let φ be a random field with $\langle\varphi\rangle = 0$
Then:

$$\boxed{\varphi \text{ homogeneous} \Rightarrow \mathbf{E}(\vec{r},k) = \mathbf{E}(k)}$$

<u>Proof</u>

$$\varphi \text{ homogeneous} \Rightarrow \varphi(\vec{r}+\vec{r}_0) \simeq \varphi(\vec{r}) \Rightarrow$$
$$\Rightarrow \varphi_k^{<}(\vec{r}+\vec{r}_0) \simeq \varphi_k^{<}(\vec{r}) \Rightarrow$$
$$\Rightarrow \varphi_k^{<} \text{ homogeneous} \Rightarrow \nabla\langle\varphi_k^{<}(\vec{r})\otimes\varphi_k^{<}(\vec{r})\rangle = 0$$

It follows that:

$$\nabla \mathbf{E}(\vec{r},k) = \nabla\left[\frac{1}{2}\frac{\partial}{\partial k}\langle \varphi_k^<(\vec{r}) \otimes \varphi_k^<(\vec{r})\rangle\right] =$$

$$= \frac{1}{2}\frac{\partial}{\partial k}\left[\nabla \langle \varphi_k^<(\vec{r}) \otimes \varphi_k^<(\vec{r})\rangle\right] = 0 \Rightarrow$$

$$\Rightarrow \mathbf{E}(\vec{r},k) = \mathbf{E}(k) \qquad \square$$

Note that the assumption $\mathbf{E}(\vec{r},k) = \mathbf{E}(k)$ with $\vec{r} \in \mathbb{R}^n$, in an infinite domain, implies that the total energy of φ is infinite! It is still possible to work with this assumption however with the help of the theory of generalized functions. For an example of such analysis see p. 55-61. If such analysis appears eccentric it is because it brings out the paradox that homogeneous φ in an infinite domain has infinite energy, and yet we need the domain to be infinite in order for isotropy to make sense. The reason why we can hide this infinity under the rug is because the energy is homogeneously distributed and its value at any point offers a unique way to talk about an energy. However if we drop this assumption, then we can no longer do that. Then we have to assume that the total energy is finite when integrated over $\vec{r} \in \mathbb{R}^n$, and work with that. In the following, we will analyze both assumptions and follow them to their logical conclusions.

First, we introduce some generalizations to n-dimensional Fourier transform theory.

Spherical Fourier transforms

The following development is motivated by the problem of evaluating the following integral:

$$I = \int_{\mathbb{R}^n} \varphi(\|\vec{r}\|) \exp(-i\vec{k}\cdot\vec{r})\, d\vec{r}$$

Recall that the n-dimensional space has a spherical isomorphism $\mathbb{R}^n \cong [0,+\infty) \times SO(n)$ where the volume measure is given by

$$dV = r^{n-1}\, dr\, d\underline{\Omega}(n)$$

where $d\underline{\Omega}(n)$ is the solid-angle measure on $SO(n)$ given by:

$$d\Omega(n) = \prod_{k=1}^{n} (\sin\varphi_k)^{n-1-k}\, d\varphi_k \quad , \quad \vec{\varphi} \in [0,\pi]^{n-2} \times [0,2\pi)$$

See p.76. As a point of departure we introduce the generalized spherical Bessel function:

Definition: The generalized spherical Bessel function $\Phi_n(k)$, $k>0$ is defined as the ratio of the following integrals:

$$\boxed{\Phi_n(x) = \frac{\int_{SO(n)} \exp[ix\vec{e}\cdot A\vec{e}]\, d\Omega(n)}{\int_{SO(n)} d\Omega(n)}}$$

where $\vec{e} = (1,0,\dots,0)$ is the unit vector

It is easy to see that this definition has been normalized such that $\Phi_n(0) = 1$. We will denote the normalization constant as:

$$\boxed{\gamma_n \equiv \int_{SO(n)} d\Omega(n)}$$

Evaluating $\Phi_n(x)$ and γ_n

We begin by evaluating γ_n. To do that we make use of the following integration formulas:

$$\int_0^\pi \sin^{2m}\varphi \, d\varphi = \int_0^\pi \cos^{2m}\varphi \, d\varphi = \frac{(2m-1)!!}{(2m)!!}\pi$$

$$\int_0^\pi \sin^{2m+1}\varphi \, d\varphi = \int_0^\pi \cos^{2m+1}\varphi \, d\varphi = \frac{(2m)!!}{(2m+1)!!}2$$

where the double factorial is defined by

$$0!! = 1 \qquad 1!! = 1$$
$$n!! = n \cdot (n-2)!!$$

We also rely on the following properties of the gamma function:

$$\Gamma(n) = (n-1)! \qquad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \quad , \quad n \in \mathbb{N}^*$$

Proposition : $\boxed{\gamma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}}$

Proof

Define $R_n \equiv [0,\pi]^{n-2} \times [0,2\pi)$ and also let $I(k) \equiv \int_0^\pi (\sin\varphi)^k \, d\varphi$ abbreviate the integrals that we mentioned earlier.

$$\gamma_n = \int_{SO(n)} d\Omega(n) = \int_{R_n} \prod_{k=1}^{n-1} (\sin\varphi_k)^{n-1-k} \, d\varphi_k =$$

$$= \left[\prod_{k=1}^{n-2} \int_0^\pi (\sin\varphi_k)^{n-1-k} \, d\varphi_k\right] \int_0^{2\pi} d\varphi_{n-1} =$$

$$= 2\pi \prod_{k=1}^{n-2} I(n-1-k) = 2\pi \prod_{k=1}^{n-2} I(k)$$

To evaluate the product we distinguish two cases:

a) If $n = 2m$ = even then

$$\gamma_n = 2\pi \prod_{k=1}^{n-2} I(k) = 2\pi \prod_{k=1}^{m-1} I(2k) I(2k-1) =$$

$$= 2\pi \prod_{k=1}^{m-1} \left\{ \frac{(2k-1)!!}{(2k)!!} \frac{(2k-2)!!}{(2k-1)!!} 2\pi \right\} = (2\pi)^m \prod_{k=1}^{m-1} \frac{(2k-2)!!}{(2k)!!} =$$

$$= (2\pi)^m \prod_{k=1}^{m-1} \frac{1}{2k} = \frac{(2\pi)^m}{2^{m-1}(m-1)!} = \frac{2\pi^m}{(m-1)!} = \frac{2\pi^m}{\Gamma(m)} =$$

$$= \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

b) If $n = 2m+1$ = odd then

$$\gamma_n = 2\pi \prod_{k=1}^{n-2} I(k) = 2\pi \, I(n-2) \prod_{k=1}^{m-1} I(2k) I(2k-1) =$$

$$= \frac{2\pi^m}{(m-1)!} I(n-2) = \frac{2\pi^m}{(m-1)!} I(2m-1) = \frac{2\pi^m}{(m-1)!} \frac{(2m-2)!!}{(2m-1)!!} 2$$

$$= \frac{2\pi^m}{(m-1)!} \frac{2^{m-1}(m-1)!}{\Gamma(m+\frac{1}{2}) 2^m / \sqrt{\pi}} \cdot 2 =$$

$$= 2\pi^{m+1/2} \frac{1}{\Gamma(m+1/2)} = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \qquad \square$$

The interpretation of γ_n is that the "surface area" of an n-dimensional sphere is $\gamma_n r^{n-1}$. Thus:

for $n=2$: $\gamma_2 = \dfrac{2\pi^{2/2}}{\Gamma(2/2)} = 2\pi \longrightarrow 2\pi r$ = circumference of circle

$n=3$: $\gamma_3 = \dfrac{2\pi^{3/2}}{\Gamma(3/2)} = \dfrac{2\pi^{3/2}}{(1/2)\Gamma(1/2)} = \dfrac{4\pi^{3/2}}{\pi^{1/2}} = 4\pi$

$\longrightarrow 4\pi r^2$ = area of 3d sphere.

To evaluate the $\Phi_n(x)$ itself, we will also need the following formulas:

$$\int_0^\pi \cos(x\cos\varphi)\sin^{2m}\varphi\, d\varphi = \sqrt{\pi}\ \Gamma\left(m+\frac{1}{2}\right)\left(\frac{2}{x}\right)^m J_m(x)$$

$$\int_0^\pi \sin(x\cos\varphi)\sin^{2m}\varphi\, d\varphi = 0$$

<u>Proposition</u>: $\boxed{\Phi_n(x) = \frac{1}{\gamma_n}(2\pi)^{n/2}\frac{J_{(n-2)/2}(x)}{x^{(n-2)/2}}}$

<u>Proof</u>

Define $R_n = [0,\pi]^{n-2}\times[0,2\pi)$ and also let $I(k) = \int_0^\pi (\sin\varphi)^k d\varphi$

Let $e = (1,0,0,\dots,0)$ a unit vector and let $A\in SO(n)$ with corresponding angle vector $\vec{\varphi}$. Then we find that:

$\vec{e}\cdot A\vec{e} = (A\vec{e})_1 = \cos\varphi_1$, using the relations from p. 76.

It follows that

$$\Phi_n(x) = \frac{1}{\gamma_n}\int_{SO(n)} \exp[ix\vec{e}\cdot A\vec{e}]\, d\Omega(n) =$$

$$= \frac{1}{\gamma_n}\int_{R_n} \exp(ix\cos\varphi_1)\prod_{k=1}^{n-1}(\sin\varphi_k)^{n-1-k}\, d\varphi_k =$$

$$= \frac{1}{\gamma_n}\left[\int_0^\pi \exp(ix\cos\varphi_1)\sin^{n-2}\varphi_1\, d\varphi_1\right]\left[\prod_{k=2}^{n-2}\int_0^\pi (\sin\varphi_k)^{n-1-k}d\varphi_k\right]\left[\int_0^{2\pi} d\varphi_{n-1}\right]$$

~~Consider these factors separately~~:

$$= \frac{1}{\gamma_n}\left[2\pi\prod_{k=2}^{n-2} I(n-1-k)\right]\int_0^\pi \exp(ix\cos\varphi_1)\sin^{n-2}\varphi_1\, d\varphi_1$$

$$= \frac{1}{\gamma_n}\left[2\pi \prod_{k=1}^{n-3} I(k)\right] \int_0^{\pi} [\cos(x\cos\varphi_1) + i\sin(x\cos\varphi_1)] \sin^{n-2}\varphi_1 \, d\varphi_1 =$$

$$= \frac{\gamma_{n-1}}{\gamma_n} \int_0^{\pi} \cos(x\cos\varphi_1) \sin^{n-2}\varphi_1 \, d\varphi_1 =$$

$$= \frac{\gamma_{n-1}}{\gamma_n} \sqrt{\pi}\, \Gamma\left(\frac{n-2}{2} + \frac{1}{2}\right)\left(\frac{2}{x}\right)^{(n-2)/2} J_{(n-2)/2}(x) =$$

$$= \frac{1}{\gamma_n}\left[\frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)}\right] \sqrt{\pi}\, \Gamma\left(\frac{n-1}{2}\right)\left(\frac{2}{x}\right)^{(n-2)/2} J_{(n-2)/2}(x)$$

$$= \frac{1}{\gamma_n} (2\pi)^{n/2} \frac{J_{(n-2)/2}(x)}{x^{(n-2)/2}} \qquad \square$$

Since we will be working with $n=2$ and $n=3$, we will evaluate $\Phi_n(x)$ for those two cases:

Corollary: $\boxed{\Phi_2(x) = J_0(x)} \quad \boxed{\Phi_3(x) = \frac{\sin x}{x}}$

Proof

Recall that $\gamma_2 = 2\pi$ and $\gamma_3 = 4\pi^2$. It follows that

$$\Phi_2(x) = \frac{1}{\gamma_2}(2\pi)^{2/2}\frac{J_{(2-2)/2}(x)}{x^{(2-2)/2}} = \frac{1}{2\pi}(2\pi) J_0(x) = J_0(x)$$

$$\Phi_3(x) = \frac{1}{\gamma_3}(2\pi)^{3/2}\frac{J_{(3-2)/2}(x)}{x^{(3-2)/2}} = \frac{1}{4\pi^2}(2\pi)^{3/2}\frac{J_{1/2}(x)}{x^{1/2}} =$$

$$= \frac{1}{4\pi^2}(2\pi)^{3/2}\frac{1}{\sqrt{x}}\sqrt{\frac{2}{\pi x}}\,\sin x =$$

$$= \frac{\sin x}{x} \qquad \square$$

Another handy observation is that $\Phi_n(-x) = \Phi_n(x)$. It is easy to see that this holds for $n=2$ and $n=3$. To bring this out in general, note that $\boxed{\Phi_n(x) = \frac{\gamma_{n-1}}{\gamma_n} \int_0^{\pi} \cos(x\cos\varphi)\sin^{n-2}\varphi \, d\varphi}$

Since $\cos(-x\cos\varphi) = \cos(x\cos\varphi)$, it follows immediately.

Finally there is two more results that we would like to mention:

Proposition: $\boxed{|\Phi_n(x)| \leq 1 \ , \ \forall x \in \mathbb{R}}$ $\boxed{\lim_{x\to 0} \frac{1-\Phi_n(x)}{x^2} = 1/(2n)}$

Proof

$$|\Phi_n(x)| = \frac{1}{\gamma_n} \left| \int_{SO(n)} \exp[ix\vec{e}\cdot A\vec{e}] \, d\underline{\rho}(n) \right| \leq$$
$$\leq \frac{1}{\gamma_n} \int_{SO(n)} |\exp[ix\vec{e}\cdot A\vec{e}]| \, d\underline{\rho}(n) =$$
$$= \frac{1}{\gamma_n} \int_{SO(n)} 1 \cdot d\underline{\rho}(n) = \frac{1}{\gamma_n} \gamma_n = 1$$

$$\lim_{x\to 0} \frac{1-\Phi_n(x)}{x^2} = \lim_{x\to 0} \left\{ \frac{1}{x^2} \frac{1}{\gamma_n} \left[\gamma_n - \int_{SO(n)} \exp[ix\vec{e}\cdot A\vec{e}] \, d\underline{\rho}(n) \right] \right\}$$

$$= \lim_{x\to 0} \frac{1}{\gamma_n x^2} \left[\gamma_n - \int_{SO(n)} \cos(x\vec{e}\cdot A\vec{e}) \, d\underline{\rho}(n) \right] =$$
$$= \lim_{x\to 0} \frac{1}{\gamma_n} \int_{SO(n)} \frac{1-\cos(x\vec{e}\cdot A\vec{e})}{x^2} \, d\underline{\rho}(n) =$$
$$= \frac{1}{\gamma_n} \int_{SO(n)} \lim_{x\to 0} \left(\frac{1-\cos(x\vec{e}\cdot A\vec{e})}{x^2} \right) d\underline{\rho}(n).$$
$$= \frac{1}{\gamma_n} \int_{SO(n)} (\vec{e}\cdot A\vec{e}) \lim_{x\to 0} \frac{\sin(x\vec{e}\cdot A\vec{e})}{2x} \, d\underline{\rho}(n)$$
$$= \frac{1}{\gamma_n} \int_{SO(n)} \frac{(\vec{e}\cdot A\vec{e})^2}{2} \, d\underline{\rho}(n).$$

$$= \frac{1}{\gamma_n} \int_{SO(n)} \frac{\cos^2\varphi_1}{2} \, d\Omega(n) =$$ (see p. 97 for similar result)

$$= \frac{1}{\gamma_n} \left[2\pi \prod_{k=2}^{n-2} I(n-1-k) \right] \int_0^\pi \frac{\cos^2\varphi_1}{2} \sin^{n-2}\varphi_1 \, d\varphi_1$$

$$= \frac{1}{\gamma_n} \left[2\pi \prod_{k=1}^{n-3} I(k) \right] \int_0^\pi \frac{1-\sin^2\varphi}{2} \sin^{n-2}\varphi \, d\varphi =$$

$$= \frac{\gamma_{n-1}}{2\gamma_n} \int_0^\pi \sin^{n-2}\varphi \, d\varphi - \frac{\gamma_{n-1}}{\gamma_n} \int_0^\pi \sin^n \varphi \, d\varphi =$$

$$= \frac{\gamma_{n-1}}{2\gamma_n} \left[I(n-2) - I(n) \right] = \frac{\gamma_{n-1}}{\gamma_n} \left[I(n-2) - \frac{n-1}{n} I(n-2) \right] =$$

$$= \frac{\gamma_{n-1} I(n-2)}{2\gamma_n} \left[1 - \frac{n-1}{n} \right] = \frac{\gamma_n}{2\gamma_n} \frac{n-n+1}{n} = \frac{1}{2n} \quad \square$$

These properties are analogous to the properties of $\cos x$ that we used to prove the relation between the structure function power law and the energy spectrum power law. We will use these generalized properties to prove that this relation persists independent of the number of dimensions of the problem.

Defining the spherical transform

We now return to the problem that we posed on p. 94:

Theorem: $$\boxed{\int_{\mathbb{R}^n} \varphi(\|\vec{r}\|)\exp(i\vec{k}\cdot\vec{r})\,d\vec{r} = \int_0^{+\infty} \gamma_n \rho^{n-1} \varphi(\rho)\,\Phi_n(\|\vec{k}\|\rho)\,d\rho}$$

Proof

Let $\vec{k} \in \mathbb{R}^n$ be given and choose a unit vector $\vec{e}$ such that $\vec{k} = \|\vec{k}\|\vec{e}$. Similarly let $\vec{r} = \rho A\vec{e}$ with $A \in SO(n)$. Then $\vec{k}\cdot\vec{r} = (\|\vec{k}\|\vec{e})\cdot(\rho A\vec{e}) = \|\vec{k}\|\rho\,\vec{e}\cdot A\vec{e}$.
It follows that:

$$I = \int_{\mathbb{R}^n} \varphi(\|\vec{r}\|)\exp(i\vec{k}\cdot\vec{r})\,d\vec{r} = \int_0^{+\infty} \rho^{n-1}d\rho \int_{SO(n)} d\Omega \left[\varphi(\rho)\exp(i\|\vec{k}\|\rho\,\vec{e}\cdot A\vec{e})\right] =$$

$$= \int_0^{+\infty} \rho^{n-1}\varphi(\rho)\left\{\int_{SO(n)} \exp[i\|\vec{k}\|\rho\,\vec{e}\cdot A\vec{e}]\,d\Omega\right\} d\rho =$$

$$= \int_0^{+\infty} \rho^{n-1}\varphi(\rho)\left[\gamma_n \Phi_n(\|\vec{k}\|\rho)\right] d\rho =$$

$$= \int_0^{+\infty} \gamma_n \rho^{n-1}\varphi(\rho)\,\Phi_n(\|\vec{k}\|\rho)\,d\rho. \qquad \square$$

For $n=2$ and $n=3$ we obtain:

Corollary:
$$\boxed{\begin{aligned} \int_{\mathbb{R}^2} \varphi(\|\vec{r}\|)\exp(i\vec{k}\cdot\vec{r})\,d\vec{r} &= \int_0^{+\infty} 2\pi\rho\,\varphi(\rho)\,J_0(\|\vec{k}\|\rho)\,d\rho \\ \int_{\mathbb{R}^3} \varphi(\|\vec{r}\|)\exp(i\vec{k}\cdot\vec{r})\,d\vec{r} &= \int_0^{+\infty} 4\pi\rho^2\,\varphi(\rho)\,\frac{\sin(\|\vec{k}\|\rho)}{\|\vec{k}\|\rho}\,d\rho \end{aligned}}$$

Theorem: (Spherical transform theorem)

The following equivalence is always true:

$$\boxed{\varphi(\rho) = \int_0^{+\infty} \hat{\varphi}(k)\,\Phi_n(k\rho)\,dk \iff \hat{\varphi}(k) = \frac{\gamma_n^2}{(2\pi)^n}\int_0^{+\infty} (k\rho)^{n-1}\varphi(\rho)\,\Phi_n(k\rho)\,d\rho}$$

Proof

$$\varphi(\rho) = \int_0^{+\infty} \hat{\varphi}(k)\,\Phi_n(k\rho)\,dk = \int_0^{+\infty} \gamma_n k^{n-1}\frac{\hat{\varphi}(k)}{\gamma_n k^{n-1}}\,\Phi_n(k\rho)\,dk$$

$$= \int_{\mathbb{R}^n} \frac{\hat{\varphi}(\|\vec{k}\|)}{\gamma_n\|\vec{k}\|^{n-1}} \exp(i\vec{k}\cdot(\rho\vec{e}))\,d\vec{k} =$$

$$= \int_{\mathbb{R}^n} \frac{\hat{\varphi}(\|\vec{k}\|)}{\gamma_n\|\vec{k}\|^{n-1}} \exp(i\vec{k}\cdot\vec{r})\,d\vec{k} \quad , \quad \forall \vec{r}\in\mathbb{R}^n : \|\vec{r}\| = \rho \iff$$

$$\iff \hat{\varphi}(k) = \gamma_n\|\vec{k}\|^{n-1}\frac{\hat{\varphi}(\|\vec{k}\|)}{\gamma_n\|\vec{k}\|} = \gamma_n\|\vec{k}\|^{n-1}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\varphi(\|\vec{r}\|)\exp(i\vec{k}\cdot\vec{r})\,d\vec{r}$$

$$= \gamma_n\|\vec{k}\|^{n-1}\frac{1}{(2\pi)^n}\int_0^{+\infty}\gamma_n\rho^{n-1}\varphi(\rho)\,\Phi_n(\|\vec{k}\|\rho)\,d\rho =$$

$$= \frac{1}{(2\pi)^n}\int_0^{+\infty}\gamma_n^2(k\rho)^{n-1}\varphi(\rho)\,\Phi_n(k\rho)\,d\rho \qquad \square$$

Thus we may define a transform pair which is a generalization of the cosine transform for isotropic functions in n-dimensions:

$$\boxed{\begin{aligned} \varphi(\rho) &= \int_0^{+\infty}\hat{\varphi}(k)\,\Phi_n(k\rho)\,dk \\ \hat{\varphi}(k) &= \frac{\gamma_n^2}{(2\pi)^n}\int_0^{+\infty}(k\rho)^{n-1}\varphi(\rho)\,\Phi_n(k\rho)\,d\rho \end{aligned}}$$

For $n=2$ we obtain the Bessel transform and for $n=3$ the sinc transform:

$$\varphi(\rho) = \int_0^{+\infty} \hat{\varphi}(k) J_0(k\rho)\, dk \qquad \hat{\varphi}(\rho) = \int_0^{+\infty} \hat{\varphi}(k) \frac{\sin(k\rho)}{k\rho}\, dk$$

$$\underset{n=2}{\varphi(\rho) = \int_0^{+\infty} k\rho\, \varphi(\rho) J_0(k\rho)\, dk} \qquad \underset{n=3}{\hat{\varphi}(k) = \frac{2}{\pi} \int_0^{+\infty} \varphi(\rho)\, k\rho \sin(k\rho)\, d\rho}$$

For reference we will call $\hat{\varphi}(k)$ the <u>spherical transform</u> of $\varphi(\rho)$, and it is related with the Fourier transform as follows:

<u>Proposition</u>: Suppose that $\varphi(\rho)$, $\rho \in [0, +\infty)$ can be expanded as:

$$\varphi(\rho) = \int_0^{+\infty} \hat{\varphi}(k)\, \Phi_n(k\rho)\, dk$$

$$\varphi(\|\vec{r}\|) = \int_{\mathbb{R}^n} \tilde{\varphi}(\vec{k}) \exp(i\vec{k}\cdot\vec{r})\, d\vec{k}$$

Then

$$\boxed{\tilde{\varphi}(\vec{k}) = \frac{\hat{\varphi}(\|\vec{k}\|)}{\gamma_n k^{n-1}}}$$

<u>Proof</u>

$$\begin{aligned}
\hat{\varphi}(\vec{k}) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi(\|\vec{r}\|) \exp(-i\vec{k}\cdot\vec{r})\, d\vec{r} = \\
&= \frac{1}{(2\pi)^n} \int_0^{+\infty} \gamma_n \rho^{n-1} \varphi(\rho) \Phi_n(\|\vec{k}\|\rho)\, d\rho = \\
&= \frac{1}{\gamma_n k^{n-1}} \frac{\gamma_n^2}{(2\pi)^n} \int_0^{+\infty} (k\rho)^{n-1} \varphi(\rho)\, \Phi_n(\|\vec{k}\|\rho)\, d\rho = \\
&= \frac{\hat{\varphi}(\|\vec{k}\|)}{\gamma_n k^{n-1}} \qquad \square
\end{aligned}$$

The spherical transform theorem implies the following orthogonality relation:

$$\boxed{\int_0^{+\infty} (k\rho)^{n-1} \Phi_n(k_0\rho)\, \Phi_n(k\rho)\, d\rho = \frac{(2\pi)^n}{\gamma_n^2} \delta(k-k_0).}$$

The addition theorem for n-dimensional Fourier transform

Theorem: Let $k_0 > 0$ be given. Then:

$$\boxed{\int_{\mathbb{R}^n} \phi_n(k_0\|\vec{r}_1-\vec{r}_2\|)\exp(-i\vec{k}\cdot\vec{r}_1)\,d\vec{r}_1 = (2\pi)^n \frac{\delta(k_0-|k|)}{\gamma_n k_0^{n-1}} \exp(-i\vec{k}\cdot\vec{r}_2).}$$

Proof

$$\int_0^{+\infty} \delta(k-k_0)\phi_n(k\rho)\,dk = \phi_n(k_0\rho) \Rightarrow \delta(k-k_0) = \frac{\gamma_n^2}{(2\pi)^n}\int_0^{+\infty}(k\rho)^n\phi_n(k_0\rho)\phi_n(k\rho)\,d\rho$$

Our expression above holds only for $k>0$. To generalize it for all k we must place it under absolute value. Then we have:

$$\begin{aligned}
\delta(k_0-|k|) &= \frac{\gamma_n^2}{(2\pi)^n}\int_0^{+\infty}(k\rho)^{n-1}\phi_n(k\rho)\phi_n(k_0\rho)\,d\rho = \qquad \text{(by orthogonality)}\\
&= \frac{\gamma_n^2}{(2\pi)^n}\int_0^{+\infty}(k_0\rho)^{n-1}\phi_n(k\rho)\phi_n(k_0\rho)\,d\rho =\\
&= \frac{\gamma_n k_0^{n-1}}{(2\pi)^n}\int_0^{+\infty}\gamma_n\rho^{n-1}\phi_n(k_0\rho)\phi_n(k\rho)\,d\rho = \qquad \text{(theorem, p. 101)}\\
&= \frac{\gamma_n k_0^{n-1}}{(2\pi)^n}\int_{\mathbb{R}^n}\phi_n(k_0\|\vec{r}\|)\exp(-i\vec{k}\cdot\vec{r})\,d\vec{r} = \quad, \forall \vec{k} : \|\vec{k}\| = k.\\
&\qquad\qquad \text{(substitution } \vec{r}_1 = \vec{r}+\vec{r}_2\text{).}\\
&= \frac{\gamma_n k_0^{n-1}}{(2\pi)^n}\int_{\mathbb{R}^n}\phi_n(k_0\|\vec{r}_1-\vec{r}_2\|)\exp(-i\vec{k}\cdot(\vec{r}_1-\vec{r}_2))\,d\vec{r}_1 =\\
&= \frac{\gamma_n k_0^{n-1}}{(2\pi)^n}\exp(i\vec{k}\cdot\vec{r}_2)\int_{\mathbb{R}^n}\phi_n(k_0\|\vec{r}_1-\vec{r}_2\|)\exp(-i\vec{k}\cdot\vec{r}_1)\,d\vec{r}_1
\end{aligned}$$

therefore:

$$\int_{\mathbb{R}^n}\phi_n(k_0\|\vec{r}_1-\vec{r}_2\|)\exp(-i\vec{k}\cdot\vec{r}_1)\,d\vec{r}_1 = \frac{(2\pi)^n}{\gamma_n k_0^{n-1}}\delta(k_0-|k|)\exp(-i\vec{k}\cdot\vec{r}_2) \quad \square$$

Spherical Fourier transforms summary

Definitions

$$\gamma_n = \int_{SO(n)} d\Omega = 2\pi \prod_{k=1}^{n-2} I(k) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad \text{where} \quad I(k) = \int_0^{\pi} (\sin\varphi)^k d\varphi$$

$$\phi_n(x) = \frac{1}{\gamma_n} \int_{SO(n)} \exp[ix\vec{e}\cdot(A\vec{e})]\, d\Omega = \frac{\gamma_{n-1}}{\gamma_n} \int_0^{\pi} \cos(x\cos\varphi) \sin^{n-2}\varphi\, d\varphi =$$

$$= \frac{(2\pi)^{n/2}}{\gamma_n} \frac{J_{(n-2)/2}(x)}{x^{(n-2)/2}} \longrightarrow \phi_2(x) = J_0(x) \quad \phi_3(x) = \frac{\sin x}{x}$$

$$\gamma_2 = 2\pi\ , \ \gamma_3 = 4\pi$$

Properties

$$\phi_n(0) = 1\ ,\quad \phi_n(-x) = \phi_n(x)\ ,\quad |\phi_n(x)| \leqslant 1\ ,\quad \lim_{x\to 0} \frac{1-\phi_n(x)}{x^2} = \frac{1}{2n}$$

$$\gamma_n = \gamma_{n-1} I(n-2)\ ,\quad I(n) = \frac{n-1}{n} I(n-2)\ ,\quad I(0) = \pi\ ,\ I(1) = 2$$

$$\int_{\mathbb{R}^n} \varphi(\|\vec{r}\|) \exp(i\vec{k}\cdot\vec{r})\, d\vec{r} = \int_0^{+\infty} \gamma_n \rho^{n-1} \varphi(\rho)\, \phi_n(\|\vec{k}\|\rho)\, d\rho$$

$$\int_0^{+\infty} (k\rho)^{n-1} \phi_n(k\rho)\, \phi_n(k_0\rho)\, d\rho = \frac{(2\pi)^n}{\gamma_n^2} \delta(k-k_0)\ ,\quad \forall k>0,\ \forall k_0>0$$

$$\int_{\mathbb{R}^n} \phi_n(k_0\|\vec{r}_1 - \vec{r}_2\|) \exp(-i\vec{k}\cdot\vec{r}_1)\, d\vec{r}_1 = (2\pi)^n \frac{\delta(k_0 - |k|)}{\gamma_n k_0^{n-1}} \exp(-i\vec{k}\cdot\vec{r}_2)\ ,\quad \forall k_0>0$$

$$\hookrightarrow \int_{\mathbb{R}^n} \exp(-i\vec{k}\cdot\vec{r}_1)\, d\vec{r}_1 = (2\pi)^n \delta(\vec{k}) \qquad \text{for} \quad k_0 = 0$$

Transforms

$\varphi(\rho) = \int_0^{+\infty} \hat{\varphi}(k)\, \phi_n(k\rho)\, dk$	$\hat{\varphi}(k) = \frac{\gamma_n^2}{(2\pi)^n} \int_0^{+\infty} (k\rho)^{n-1} \varphi(\rho)\, \phi_n(k\rho)\, d\rho$
$\varphi(\rho) = \int_0^{+\infty} \hat{\varphi}(k)\, J_0(k\rho)\, dk$	$\hat{\varphi}(k) = \int_0^{+\infty} k\rho\, \varphi(\rho)\, J_0(k\rho)\, dk$ $\quad (n=2)$
$\varphi(\rho) = \int_0^{+\infty} \hat{\varphi}(k) \frac{\sin(k\rho)}{k\rho}\, dk$	$\hat{\varphi}(k) = \frac{2}{\pi} \int_0^{+\infty} \varphi(\rho)\, k\rho \sin(k\rho)\, d\rho$ $\quad (n=3)$

Energy spectrum and correlation spectrum

This is the most general relationship that we can write down about the energy spectrum:

Theorem: Let $\varphi : \mathbb{R}^n \times \Omega \times O(n) \longrightarrow \mathbb{R}^m$ be a random field with $\langle\varphi\rangle = 0$ and let $\tilde{\varphi}$ be the Fourier transform of φ. Then the energy spectrum is given by:

$$\boxed{\mathbf{E}(\vec{r}, k) = \int_{\mathbb{R}^n} d\vec{k}_0 \int_{SO(n)} d\Omega \; k^{n-1} \langle \tilde{\varphi}(\vec{k}_0) \otimes \tilde{\varphi}^*(kA\vec{e}) \rangle H(k - \|\vec{k}_0\|) \exp[i(\vec{k}_0 - kA\vec{e}) \cdot \vec{r}]}$$

Proof

First note that

$$\varphi_k^<(\vec{r}) = \int_{S(k)} \tilde{\varphi}(\vec{k}) \exp(i\vec{k}\cdot\vec{r})\, d\vec{k} = \int_{S(k)} \tilde{\varphi}^*(\vec{k}) \exp(-i\vec{k}\cdot\vec{r})\, d\vec{k}$$

Differentiating wrt k we have

$$\frac{d}{dk} \varphi_k^<(\vec{r}) = \frac{d}{dk} \int_{S(k)} \tilde{\varphi}^*(\vec{k}) \exp(-i\vec{k}\cdot\vec{r})\, d\vec{k} =$$

$$= \frac{d}{dk} \int_0^k dk_0 \int_{SO(n)} d\Omega \left[k_0^{n-1} \tilde{\varphi}^*(k_0 A\vec{e}) \exp(-i(k_0 A\vec{e})\cdot\vec{r}) \right] =$$

$$= \int_{SO(n)} d\Omega \; k^{n-1} \tilde{\varphi}^*(kA\vec{e}) \exp(-i(kA\vec{e})\cdot\vec{r}).$$

By definition we have:

$$\mathbf{E}(\vec{r}, k) = \frac{d}{dk} \left\langle \frac{1}{2} \varphi_k^<(\vec{r}) \otimes \varphi_k^<(\vec{r}) \right\rangle = \frac{1}{2} \left\langle \frac{d}{dk} [\varphi_k^<(\vec{r}) \otimes \varphi_k^<(\vec{r})] \right\rangle$$

$$= \left\langle \varphi_k^<(\vec{r}) \otimes \left[\frac{d}{dk} \varphi_k^<(\vec{r}) \right] \right\rangle =$$

$$= \left\langle \varphi_K^{<}(\vec{r}) \otimes \int_{SO(n)} d\underline{\Omega}\; k^{n-1}\, \tilde{\varphi}^*(kA\vec{e}) \exp(-i(kA\vec{e})\cdot\vec{r}) \right\rangle =$$

$$= \int_{SO(n)} d\underline{\Omega}\; k^{n-1} \langle \varphi_K^{<}(\vec{r}) \otimes \tilde{\varphi}^*(kA\vec{e}) \rangle \exp(-i(kA\vec{e})\cdot\vec{r})$$

$$= \int_{S(K)} d\vec{k}_0 \int_{SO(n)} d\underline{\Omega}\; k^{n-1} \langle \tilde{\varphi}(\vec{k}_0) \otimes \tilde{\varphi}^*(kA\vec{e}) \rangle \exp[i(\vec{k}_0 - kA\vec{e})\cdot\vec{r}]$$

$$= \int_{\mathbb{R}^n} d\vec{k}_0 \int_{SO(n)} d\underline{\Omega}\; k^{n-1} \langle \tilde{\varphi}(\vec{k}_0) \otimes \tilde{\varphi}(kA\vec{e}) \rangle H(K - \|\vec{k}_0\|) \exp[i(\vec{k}_0 - kA\vec{e})\cdot\vec{r}]$$

□

In the above, $H(x) = \begin{cases} 0, & x<0 \\ 1/2, & x=0 \\ 1, & x>0 \end{cases}$ is the Heavyside function.

There are two ways to continue our development.

a) We may assume $E(\vec{r},K) = E(K)$ and study $E(k)$. This corresponds to a system whose energy is homogenously distributed over $\mathbb{R}^n$ and it is mathematically equivalent to the traditional development of spectrum dynamics by Batchelor and others. Note that this is in a way unphysical because the total energy is infinite.

b) We assume that the total energy is <u>finite</u> and reinterpret $E(k)$ as that total energy:

$$\boxed{\mathbf{E}(k) = \int_{\mathbb{R}^n} d\vec{r}\; \mathbf{E}(\vec{r},k).}$$

Although these two cases are independent from one another (i.e. one can not be reduced to the other, and $E(k)$ has no common generalization) most results that we obtain have the same mathematical form <u>if</u> we carefully reinterpret all the other quantities, such as $B(\rho)$, $b(k)$, etc.

The correlation spectrum tensor

The theorem, p. 105 that we have derived for the energy spectrum tensor $\mathbf{E}(\vec{r},k)$ involves the quantity

$$\langle \tilde{\varphi}(\vec{p}) \otimes \overset{*}{\tilde{\varphi}}(\vec{q}) \rangle$$

We now turn our attention to this average and study it in more detail.

<u>Definition</u>: Let φ be a random field with $\langle\varphi\rangle = 0$. Then:

a) The <u>correlation spectrum tensor</u> $\tilde{\mathbf{B}}(\vec{k},\vec{p})$ of φ is given by the following relation:

$$\boxed{\tilde{\mathbf{B}}(\vec{p},\vec{q}) = \langle \tilde{\varphi}(\vec{p}) \otimes \overset{*}{\tilde{\varphi}}(\vec{q}) \rangle \ , \ \forall \vec{p},\vec{q} \in \mathbb{R}^n}$$

b) The <u>correlation spectrum</u> $B(\vec{k},\vec{p})$ of φ is the trace of the corresponding correlation tensor:

$$\boxed{\tilde{B}\,\cancel{\mathbf{B}}(\vec{p},\vec{q}) = \mathrm{tr}\,\mathbf{B}(\vec{p},\vec{q}) = \langle \tilde{\varphi}(\vec{p}) \cdot \overset{*}{\tilde{\varphi}}(\vec{q}) \rangle}$$

In actual physical situations the spectrum tensor $\mathbf{E}$ can be recovered from its trace $E = \mathrm{tr}\,\mathbf{E}$ so from this point we will focus exclusively on evaluating this trace. We will return to the tensor itself in the context of the actual physics of turbulence. We distinguish mainly two cases:

Homogeneous and isotropic correlation $\mathbf{B}(\vec{r}_1,\vec{r}_2)$

The integral for $\mathbf{E}(\vec{r},t)$ can be evaluated in closed form if the correlation $\mathbf{B}(\vec{r}_1,\vec{r}_2)$ is homogeneous and isotropic. because the correlation spectrum takes on a particularly simple form. Physically this holds iff φ is homogeneous and isotropic.

Recall from p. 90 that when that is the case $\mathbf{B}(\vec{r}_1,\vec{r}_2)$ can be written in terms of an isotropic tensor $\mathbf{B}_0(\vec{r})$.
Let $\underline{b(k)}$ be the spherical Bessel function expansion of the trace of that tensor:

$$\boxed{B_0(\rho) = \operatorname{tr} \mathbf{B}_0(\vec{r}) = \int_0^{+\infty} b(k)\,\Phi_n(k\rho)\,dk} \quad , \quad \rho = \|\vec{r}\|$$

Let us also state what we mean by homogeneous and isotropic when we refer to a field of $\underline{\text{two}}$ variables (recall that we have only given a definition of isotropy for fields of one variable on p. 82). We specialize our definition to tensors, for clarity.

<u>Definition</u>: Let $\mathbf{B}(\vec{p},\vec{q})$ be a tensor field of two variables $\vec{p},\vec{q} \in \mathbb{R}^n$. We will say that

a)	$\mathbf{B}$ homogeneous	$\Leftrightarrow$	$\mathbf{B}(\vec{r}_1+\vec{r},\vec{r}_2+\vec{r}) = \mathbf{B}(\vec{r}_1,\vec{r}_2)$	$,\forall \vec{r}\in\mathbb{R}^n$
b)	$\mathbf{B}$ isotropic	$\Leftrightarrow$	$\mathbf{B}(A\vec{r}_1,A\vec{r}_2) = (\oplus^2 A)\,\mathbf{B}(\vec{r}_1,\vec{r}_2)$	$,\forall A\in O(n)$

It is easy to see that if φ is homogeneous and isotropic then the same holds for $\mathbf{B}$ under this definition.
Finally note carefully the relation between $\mathbf{B}$ and $\tilde{\mathbf{B}}$:

$$\boxed{\mathbf{B}(\vec{r}_1,\vec{r}_2) = \int_{\mathbb{R}^n} d\vec{p} \int_{\mathbb{R}^n} d\vec{q}\; \tilde{\mathbf{B}}(\vec{p},\vec{q}) \exp\left[i\vec{p}\cdot\vec{r}_1 - i\vec{q}\cdot\vec{r}_2\right]}$$

$$\boxed{\tilde{\mathbf{B}}(\vec{p},\vec{q}) = \frac{1}{(4\pi^2)^n} \int_{\mathbb{R}^n} d\vec{r}_1 \int_{\mathbb{R}^n} d\vec{r}_2\; \mathbf{B}(\vec{r}_1,\vec{r}_2) \exp\left[i\vec{q}\cdot\vec{r}_2 - i\vec{p}\cdot\vec{r}_1\right]}$$

The minus sign appears because of the complex conjugate that appears in the definition of $\tilde{\mathbf{B}}$:

$$\mathbf{B}(\vec{p},\vec{q}) = \langle \tilde{\varphi}(\vec{p}) \otimes \tilde{\varphi}^*(\vec{q}) \rangle .$$

and it simplifies some aspects of our computations.
Our main result is the following theorem:

<u>Theorem</u>: If $\mathbf{B}(\vec r_1, \vec r_2)$ is a homogeneous and isotropic tensor, then the trace $\operatorname{tr}\tilde{\mathbf{B}}(\vec p, \vec q)$ is given by:

$$\boxed{\operatorname{tr}\tilde{\mathbf{B}}(\vec p, \vec q) = \frac{b[\|\vec p\|]}{\gamma_n \|\vec p\|^{n-1}}\, \delta(\vec q - \vec p)}$$

<u>Proof</u>

$$\operatorname{tr}\tilde{\mathbf{B}}(\vec p, \vec q) = \frac{1}{(4\pi^2)^n}\int_{\mathbb{R}^{2n}} d\vec r_1\, d\vec r_2\; \operatorname{tr}\mathbf{B}(\vec r_1, \vec r_2)\exp[i\vec q\cdot\vec r_2 - i\vec p\cdot\vec r_1] =$$

$$= \frac{1}{(4\pi^2)^n}\int_{\mathbb{R}^{2n}} d\vec r_1\, d\vec r_2\; B_0(\|\vec r_2 - \vec r_1\|)\exp[i\vec q\cdot\vec r_2 - i\vec p\cdot\vec r_1] =$$

$$= \frac{1}{(4\pi^2)^n}\int_{\mathbb{R}^{2n}} d\vec r_1\, d\vec r_2 \left\{\int_0^{+\infty} b(k)\,\Phi_n(k\|\vec r_1 - \vec r_2\|)\,dk\right\}\exp[i\vec q\cdot\vec r_2 - i\vec p\cdot\vec r_1] =$$

$$= \frac{1}{(4\pi^2)^n}\int_0^{+\infty} b(k)\, f(k, \vec p, \vec q)$$

where $f(k)$ is the following integral:

$$f^{\vec p, \vec q}(k) = \int_{\mathbb{R}^{2n}} d\vec r_1\, d\vec r_2\; \Phi_n(k\|\vec r_1 - \vec r_2\|)\exp[i\vec q\cdot\vec r_2 - i\vec p\cdot\vec r_1] =$$

$$= \int_{\mathbb{R}^n} d\vec r_2\, \exp(i\vec q\cdot\vec r_2)\left\{\int_{\mathbb{R}^n}\Phi_n(k\|\vec r_1 - \vec r_2\|)\exp[-i\vec p\cdot\vec r_1]\,d\vec r_1\right\} = \text{(addition thm)}$$

$$= \int_{\mathbb{R}^n} d\vec r_2\, \exp(i\vec q\cdot\vec r_2)\left\{(2\pi)^n\frac{\delta(\|\vec p\| - k)}{\gamma_n k^{n-1}}\exp[-i\vec p\cdot\vec r_2]\right\}$$

$$= (2\pi)^n\frac{\delta(\|\vec p\| - k)}{\gamma_n k^{n-1}}\int_{\mathbb{R}^n} d\vec r_2\, \exp[i(\vec q - \vec p)\cdot\vec r_2]$$

$$= (2\pi)^n\frac{\delta(\|\vec p\| - k)}{\gamma_n k^{n-1}}\,(2\pi)^n\,\delta(\vec q - \vec p) =$$

$$= \frac{(4\pi^2)^n}{\gamma_n k^{n-1}}\,\delta(\|\vec p\| - k)\,\delta(\vec q - \vec p).$$

Backsubstituting to the original integral we obtain:

$$\operatorname{tr} \tilde{\mathbf{B}}(\vec{p},\vec{q}) = \frac{1}{(4\pi^2)^n}\int_0^{+\infty} b(k)\,\frac{(4\pi^2)^n}{\gamma_n k^{n-1}}\,\delta(k-\|\vec{p}\|)\,\delta(\vec{q}-\vec{p})\,dk =$$

$$= \frac{1}{(4\pi^2)^n}\, b(\|\vec{p}\|)\,\frac{(4\pi^2)^n}{\gamma_n \|\vec{p}\|^{n-1}}\,\delta(\vec{q}-\vec{p}) =$$

$$= \frac{b(\|\vec{p}\|)}{\gamma_n \|\vec{p}\|^{n-1}}\,\delta(\vec{q}-\vec{p}) \qquad \square$$

Now we can show that $E(k)$ and $b(k)$ are related:

Theorem: If $\mathbf{B}(\vec{r}_1,\vec{r}_2)$ is a homogeneous and isotropic tensor, then the energy spectrum is given by:

$$\boxed{E(\vec{r},k) = \frac{b(k)}{2}}$$

Proof

$$E(\vec{r},k) = \int_{\mathbb{R}^n} d\vec{k}_0 \int_{SO(n)} d\underline{o}\; k^{n-1}\, \tilde{B}(\vec{k}_0, kA\vec{e})\, H(k-\|\vec{k}_0\|)\exp[i(\vec{k}_0 - kA\vec{e})\cdot\vec{r}]$$

$$= \int_{\mathbb{R}^n} d\vec{k}_0 \int_{SO(n)} d\underline{o}\; k^{n-1}\left[\frac{b(\|kA\vec{e}\|)}{\gamma_n \|kA\vec{e}\|^{n-1}}\,\delta(\vec{k}_0 - kA\vec{e})\right] H(k-\|\vec{k}_0\|)\exp[i(\vec{k}_0 - kA\vec{e})\cdot\vec{r}]$$

$$= \int_{SO(n)} d\underline{o}\; k^{n-1}\,\frac{b(\|kA\vec{e}\|)}{\gamma_n \|kA\vec{e}\|^{n-1}}\, H(k-\|kA\vec{e}\|)\exp[i(kA\vec{e}-kA\vec{e})\cdot\vec{r}] =$$

$$= \int_{SO(n)} d\underline{o}\; k^{n-1}\,\frac{b(k)}{\gamma_n k^{n-1}}\, H(0)\exp(0) =$$

$$= \frac{b(k)}{2\gamma_n}\int_{SO(n)} d\underline{o} = \frac{b(k)}{2} \qquad \square$$

Since $b(k)$ is related with $B_0(\rho)$ (p. 108) we get the Wienier – Khinchin relations between the energy spectrum and correlation:

$$\boxed{\begin{aligned} B_0(\rho) &= \int_0^{+\infty} 2E(\vec{r},k)\,\Phi_n(k\rho)\,dk \\ E(\vec{r},k) &= \frac{1}{2}\,\frac{\gamma_n^2}{(2\pi)^n}\int_0^{+\infty} (k\rho)^{n-1}\, B_0(\rho)\,\Phi_n(k\rho)\,d\rho \end{aligned}}$$

Finite energy and averaged correlation

Suppose that the total energy is finite and that the energy spectrum $\mathbf{E}(k)$ is being reinterpreted by:

$$\boxed{\mathbf{E}(k) = \int_{\mathbb{R}^n} d\vec{r}\ \mathbf{E}(\vec{r}, k).}$$

We propose the following reinterpretations:

<u>Definition:</u> Let φ be a random field with correlation tensor $\mathbf{B}(\vec{r}_1, \vec{r}_2)$. Then we define:

a) The <u>averaged correlation</u> tensor by:

$$\boxed{\mathbf{B}(\vec{r}) = \int_{\mathbb{R}^n} d\vec{r}_0\ \mathbf{B}(\vec{r}_0, \vec{r}_0 + \vec{r})}$$

b) The <u>correlation spectrum</u> $b(k)$ by the following expansion:

$$\boxed{\frac{1}{\gamma_n} \int_{SO(n)} d\Omega\ \ B(\rho A \vec{e}) = \int_0^{+\infty} b(k)\, \Phi_n(k\rho)\, dk}$$

We will show that the relation $E(k) = b(k)/2$ continues to hold, for <u>any</u> random field! Our reinterpretations relinquish enough information about φ that we do not have to pin it down with any assumptions of statistical symmetry. This allows us to work with asymptotic statistical symmetries which, physically, are the most realistic.
We begin with relations between $B(\vec{r})$ and $b(k)$ and the $\tilde{B}(\vec{p}, \vec{q})$:

<u>Proposition</u> :
$$\boxed{\mathbf{B}(\vec{r}) = (2\pi)^n \int_{\mathbb{R}^n} d\vec{k}\, \widetilde{\mathbf{B}}(\vec{k},\vec{k}) \exp(-i\vec{k}\cdot\vec{r}).}$$

<u>Proof</u>

$$\mathbf{B}(\vec{r}) = \int_{\mathbb{R}^n} \mathbf{B}(\vec{\rho},\vec{\rho}+\vec{r})\, d\vec{\rho} =$$
$$= \int_{\mathbb{R}^n} d\vec{\rho} \int_{\mathbb{R}^{2n}} d\vec{p}\, d\vec{q}\, \widetilde{\mathbf{B}}(\vec{p},\vec{q}) \exp[i\vec{p}\cdot\vec{\rho} - i\vec{q}\cdot(\vec{\rho}+\vec{r})]$$
$$= \int_{\mathbb{R}^{2n}} d\vec{p}\, d\vec{q}\, \widetilde{\mathbf{B}}(\vec{p},\vec{q}) \exp(-i\vec{q}\cdot\vec{r}) \left\{ \int_{\mathbb{R}^n} d\vec{\rho} \exp[i(\vec{p}-\vec{q})\cdot\vec{\rho}] \right\} =$$
$$= \int_{\mathbb{R}^{2n}} d\vec{p}\, d\vec{q}\, \widetilde{\mathbf{B}}(\vec{p},\vec{q}) \exp(-i\vec{q}\cdot\vec{r})\, (2\pi)^n \delta(\vec{p}-\vec{q})$$
$$= (2\pi)^n \int_{\mathbb{R}^n} d\vec{k}\, \widetilde{\mathbf{B}}(\vec{k},\vec{k}) \exp(-i\vec{k}\cdot\vec{\rho}). \qquad \square$$

<u>Proposition</u> :
$$\boxed{b(k) = (2\pi)^n k^{n-1} \int_{SO(n)} \widetilde{B}(kA\vec{e}, kA\vec{e})\, d\underline{O}}$$

<u>Proof</u>

$$\frac{1}{\gamma_n} \int_{SO(n)} d\underline{O}\, B(\rho A\vec{e}) = \frac{1}{\gamma_n} \int_{SO(n)} d\underline{O} \left\{ (2\pi)^n \int_{\mathbb{R}^n} d\vec{k}\, \widetilde{\mathbf{B}}(\vec{k},\vec{k}) \exp(-i\vec{k}\cdot(\rho A\vec{e})) \right\} =$$
$$= \frac{(2\pi)^n}{\gamma_n} \int_{\mathbb{R}^n} d\vec{k}\, \widetilde{\mathbf{B}}(\vec{k},\vec{k}) \left\{ \int_{SO(n)} d\underline{O} \exp[-i\vec{k}\cdot(\rho A\vec{e})] \right\}$$
$$= \frac{(2\pi)^n}{\gamma_n} \int_{\mathbb{R}^n} d\vec{k}\, \widetilde{B}(\vec{k},\vec{k})\, \gamma_n \Phi_n(\|\vec{k}\|\rho)$$
$$= (2\pi)^n \int_0^{+\infty} dk \int_{SO(n)} d\underline{O}\, \widetilde{B}(kA\vec{e}, kA\vec{e})\, \Phi_n(k\rho)\, k^{n-1} =$$
$$= (2\pi)^n \int_0^{+\infty} dk \left\{ k^{n-1} \int_{SO(n)} \widetilde{B}(kA\vec{e}, kA\vec{e})\, d\underline{O} \right\} \Phi_n(k\rho) \Rightarrow$$

$$\Rightarrow b(k) = (2\pi)^n k^{n-1} \int_{SO(n)} \widetilde{B}(kA\vec{e}, kA\vec{e})\, d\underline{O} \qquad \square$$

Proposition : $\boxed{b(k) = (2\pi)^n \gamma_n k^{n-1} \int_{\mathbb{R}^n} d\vec{r}\ B(\vec{r})\, \phi_n(k\|\vec{r}\|)}$

Proof

Taking the inverse transform of the first proposition we get:

$$\tilde{B}(\vec{k},\vec{k}) = \int_{\mathbb{R}^n} d\vec{r}\ B(\vec{r}) \exp(i\vec{k}\cdot\vec{r}).$$

Substitute into the previous proposition.

$$\begin{aligned} b(k) &= (2\pi)^n k^{n-1} \int_{SO(n)} \tilde{B}(kA\vec{e}, kA\vec{e})\, d\Omega = \\ &= (2\pi)^n k^{n-1} \int_{SO(n)} \left\{ \int_{\mathbb{R}^n} d\vec{r}\ B(\vec{r}) \exp(i(kA\vec{e})\cdot\vec{r}) \right\} d\Omega \\ &= (2\pi)^n k^{n-1} \int_{\mathbb{R}^n} d\vec{r}\ B(\vec{r}) \left\{ \int_{SO(n)} \exp[i(kA\vec{e})\cdot\vec{r}]\, d\Omega \right\} \\ &= (2\pi)^n \gamma_n k^{n-1} \int_{\mathbb{R}^n} d\vec{r}\ B(\vec{r})\, \phi_n(k\|\vec{r}\|) \qquad \square \end{aligned}$$

If $B(\vec{r})$ is isotropic, i.e. $B(\vec{r}) = B_0(\|\vec{r}\|)$, then by definition:

$$\frac{1}{\gamma_n} \int_{SO(n)} d\Omega\ B(\rho A\vec{e}) = B_0(\rho) = \int_0^{+\infty} b(k)\, \phi_n(k\rho)\, dk$$

which yields a spherical Fourier transform pair:

$$B_0(\rho) = \int_0^{+\infty} b(k)\, \phi_n(k\rho)\, dk$$

$$b(k) = \frac{\gamma_n^2}{(2\pi)^n} \int_0^{+\infty} (k\rho)^{n-1} B_0(\rho)\, \phi_n(k\rho)\, d\rho$$

Note from all of the above that the only information that is retained after all the averaging is the modes $\tilde{B}(\vec{k},\vec{k})$. We now show that this information is sufficient for evaluating the energy spectrum.

Theorem : $\boxed{E(k) = \frac{b(k)}{2}}$

Proof

$$E(k) = \int_{\mathbb{R}^n} d\vec{r}\; E(\vec{r}, k) =$$

$$= \int_{\mathbb{R}^n} d\vec{r} \int_{\mathbb{R}^n} d\vec{k}_0 \int_{SO(n)} d\underline{o}\; k^{n-1} \tilde{B}(\vec{k}_0, kA\vec{e}) H(k - \|\vec{k}_0\|) \exp[i(\vec{k}_0 - kA\vec{e})\cdot\vec{r}]$$

$$= \int_{\mathbb{R}^n} d\vec{k}_0 \int_{SO(n)} d\underline{o}\; k^{n-1} \tilde{B}(\vec{k}_0, kA\vec{e}) H(k - \|\vec{k}_0\|) \left\{ \int_{\mathbb{R}^n} d\vec{r}\; \exp[i(\vec{k}_0 - kA\vec{e})\cdot\vec{r}] \right\}$$

$$= \int_{\mathbb{R}^n} d\vec{k}_0 \int_{SO(n)} d\underline{o}\; k^{n-1} \tilde{B}(\vec{k}_0, kA\vec{e}) H(k - \|\vec{k}_0\|) (2\pi)^n \delta(\vec{k}_0 - kA\vec{e})$$

$$= \int_{SO(n)} d\underline{o}\; k^{n-1} \tilde{B}(kA\vec{e}, kA\vec{e}) H(k - \|kA\vec{e}\|) (2\pi)^n$$

$$= (2\pi)^n k^{n-1} \int_{SO(n)} \tilde{B}(kA\vec{e}, kA\vec{e}) H(0)\, d\underline{o} =$$

$$= \frac{1}{2} \left\{ (2\pi)^n k^{n-1} \int_{SO(n)} \tilde{B}(kA\vec{e}, kA\vec{e})\, d\underline{o} \right\} = \frac{b(k)}{2} \qquad \square$$

In terms of the averaged correlation $B(\vec{r})$ this relationship becomes:

$$\boxed{E(k) = \frac{(2\pi)^n}{2} \gamma_n k^{n-1} \int_{\mathbb{R}^n} d\vec{r}\; B(\vec{r})\, \Phi_n(k\|\vec{r}\|)}$$

Note that $E(k)$ has less information than $B(\vec{r})$, so we can not invert this relationship and solve for $B(\vec{r})$. However, it is possible to obtain the spherically averaged $B(\vec{r})$ over a shell of radius ρ:

$$\boxed{B_0(\rho) \equiv \frac{1}{\gamma_n} \int_{SO(n)} B(\rho A\vec{e})\, d\underline{o} = 2 \int_0^{+\infty} E(k)\, \Phi_n(k\rho)\, dk}$$

▼ The 2nd-order structure function

We introduce 2nd-order structure functions and show their relationship with the energy spectrum:
Let φ be a random field, and let $\delta\varphi$ be given by

$$\delta\varphi(\vec{r},\vec{\ell}) \equiv \varphi(\vec{r}+\vec{\ell}) - \varphi(\vec{r})$$

Then we define:

Definition:
a) The local 2nd-order structure tensor is given by:

$$\boxed{\mathbf{D}(\vec{r},\vec{\ell}) = \int_{\Omega} \delta\varphi(\vec{r},\vec{\ell},\varpi) \otimes \delta\varphi(\vec{r},\vec{\ell},\varpi)\, d\mu(\varpi)}$$

b) The local 2nd-order structure function is the trace of the corresponding tensor:

$$\boxed{D(\vec{r},\vec{\ell}) = \operatorname{tr} \mathbf{D}(\vec{r},\vec{\ell}) = \langle |\delta\varphi(\vec{r},\vec{\ell})|^2 \rangle}$$

There are two ways to remove the dependence on $\vec{r}$.

●1 Assume local homogeneity and isotropy.

Proposition: $\boxed{\varphi \text{ locally homogeneous} \Longrightarrow \mathbf{D}(\vec{r},\vec{\ell}) = \mathbf{D}_0(\vec{\ell})}$

Proof

Let $\vec{r},\vec{r}_0 \in \mathbb{R}^n$ be given. Since φ is locally homogeneous, there is a $g:\Omega \to \Omega$ such that $g^{-1}\Omega = \Omega$ modμ and $\mu(g^{-1}A) = \mu(A)$, $A \in \mathcal{B}$ and

$$\delta\varphi(\vec{r}+\vec{r}_0,\vec{\ell},\varpi) = \delta\varphi(\vec{r},\vec{\ell},g\varpi)$$

It follows that:

$$\mathbf{D}(\vec{r}+\vec{r}_0,\vec{\ell}) = \int_{\Omega} \left[\otimes^2 \delta\varphi(\vec{r}+\vec{r}_0,\vec{\ell},\vartheta)\right] d\mu(\vartheta) =$$
$$= \int_{\Omega} \left[\otimes^2 \delta\varphi(\vec{r},\vec{\ell},g\vartheta)\right] d\mu(\vartheta) =$$
$$= \int_{\Omega} \left[\otimes^2 \delta\varphi(\vec{r},\vec{\ell},\vartheta)\right] d\mu(\vartheta) = \mathbf{D}(\vec{r},\vec{\ell}) \ , \ \forall \vec{r}_0 \in \mathbb{R}^n$$

therefore there is an $\mathbf{D}_0$ such that $\mathbf{D}(\vec{r},\vec{\ell}) = \mathbf{D}_0(\vec{\ell})$. $\square$

<u>Proposition</u>: $\boxed{\left.\begin{array}{l}\varphi \text{ locally homogeneous} \\ \varphi \text{ locally isotropic}\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\mathbf{D}(\vec{r},\vec{\ell}) = \mathbf{D}_0(\vec{\ell}) \\ \mathbf{D}_0 \text{ isotropic tensor.}\end{array}\right.}$

<u>Proof</u>

From the previous proposition we get $\mathbf{D}(\vec{r},\vec{\ell}) = \mathbf{D}_0(\vec{\ell})$.
Let $A \in O(n)$ be given. Since $\delta\varphi$ is locally isotropic, there is a transformation $g: \Omega \to \Omega$ such that

$$\delta\varphi(\vec{r}, A\vec{\ell}, \vartheta) = m_\varphi(A)\,\delta\varphi(\vec{r},\vec{\ell},g\vartheta)$$

where m_φ is the transformation law of φ. It follows that:

~~$\mathbf{D}_0(\vec{\ell}) =$~~

$$\mathbf{D}_0(A\vec{\ell}) = \int_{\Omega} \otimes^2 \delta\varphi(\vec{r}, A\vec{\ell},\vartheta)\, d\mu(\vartheta) =$$
$$= \int_{\Omega} \otimes^2\left[m_\varphi(A)\,\delta\varphi(\vec{r},\vec{\ell},g\vartheta)\right] d\mu(\vartheta) =$$
$$= \int_{\Omega} \left[\oplus^2 m_\varphi(A)\right]\left[\otimes^2\delta\varphi(\vec{r},\vec{\ell},g\vartheta)\right] d\mu(\vartheta) =$$
$$= \oplus^2 m_\varphi(A) \int_{\Omega} \otimes^2 \delta\varphi(\vec{r},\vec{\ell},g\vartheta)\, d\mu(\vartheta) =$$
$$= \oplus^2 m_\varphi(A) \int_{\Omega} \otimes^2 \delta\varphi(\vec{r},\vec{\ell},\vartheta)\, d\mu(\vartheta) =$$
$$= \oplus^2 m_\varphi(A)\, \mathbf{D}_0(\vec{\ell}) = m_{\mathbf{D}_0}(A)\, \mathbf{D}_0(\vec{\ell}) \Rightarrow$$

$\Rightarrow \mathbf{D}_0$ is an isotropic tensor.

• 2 Assume spatial averaging

The alternative approach, if we assume only asymptotic local homogeneity and isotropy, is to integrate out the $\vec{r}$ dependence:

Definition :

a) The 2nd-order structure tensor is given by:

$$\boxed{\mathbf{D}_0(\vec{\ell}) = \int_{\mathbb{R}^n} \mathbf{D}(\vec{r}, \vec{\ell})\, d\vec{r}}$$

b) The 2nd-order structure function is given by:

$$\boxed{D_0(\ell) = \frac{1}{\gamma_n} \int_{SO(n)} \operatorname{tr} \mathbf{D}_0(\ell A \vec{e})\, d\underline{\Omega}}$$

A natural question is: when does the integral that defines the tensor $\mathbf{D}_0(\vec{\ell})$ converge? We will answer this question indirectly by focusing on $D_0(\ell)$ instead. First, we introduce the rms (root-mean-square) ensemble average of φ:

Definition : Let φ be a random field that is scalar or vector. Then $\varphi_{rms}(\vec{r})$ is a scalar field that is given by:

$$\boxed{\varphi_{rms}(\vec{r}) = \sqrt{\langle |\varphi(\vec{r})|^2 \rangle}}$$

where $|\cdot|$ is absolute value, if φ is scalar and a modulus, if φ is vector.

φ_{rms} can be evaluated from the energy spectrum using the following result:

<u>Proposition</u> : $$\boxed{\varphi^2_{rms}(\vec{r}) = \int_0^{+\infty} 2E(\vec{r},k)\,dk}$$

<u>Proof</u>

If we assume that the flow is ~~locally~~ loaded with finite energy, then at far distances φ vanishes, so $\hat{\varphi}(\vec{k})$ is not a generalized function. It follows that

$$\lim_{k\to 0^+} \varphi^<_k(\vec{r}) = 0 \Rightarrow \lim_{k\to 0^+} |\varphi^<_k(\vec{r})|^2 = 0 \qquad (1)$$

So, we get:

$$\varphi^2_{rms}(\vec{r}) = \langle|\varphi(\vec{r})|^2\rangle = \lim_{k\to+\infty}\langle|\varphi^<_k(\vec{r})|^2\rangle =$$

$$= \lim_{k\to+\infty}\langle|\varphi^<_k(\vec{r})|^2\rangle - \lim_{k\to 0}\langle|\varphi^<_k(\vec{r})|^2\rangle =$$

$$= \int_0^{+\infty}\frac{d}{dk}\langle|\varphi^k_<(\vec{r})|^2\rangle\,dk = \int_0^{+\infty} 2E(\vec{r},k)\,dk. \qquad \square$$

If the integral of $\varphi^2_{rms}(\vec{r})$ over $\vec{r}\in\mathbb{R}^n$ converges, then we call it the <u>total variance</u> of φ:

$$\boxed{\operatorname{var}[\varphi] = \int_{\mathbb{R}^n}\varphi^2_{rms}(\vec{r})\,d\vec{r}}$$

Now we can show the following convergence criterion for $D_0(\ell)$:

<u>Theorem</u> : Let φ be a random field with finite $\operatorname{var}[\varphi]$. Then the 2nd-order structure function is given by:

$$\boxed{D_0(\rho) = 2\operatorname{var}[\varphi] - 2B_0(\rho)}$$

Proof

$$\mathbf{D}(\vec{r},\vec{\ell}) = \langle \otimes^2 \delta\varphi(\vec{r},\vec{\ell}) \rangle = \langle \otimes^2 [\varphi(\vec{r}+\vec{\ell}) - \varphi(\vec{r})] \rangle =$$

$$= \langle \varphi(\vec{r}+\vec{\ell}) \otimes \varphi(\vec{r}+\vec{\ell}) - 2\varphi(\vec{r}) \otimes \varphi(\vec{r}+\vec{\ell}) + \varphi(\vec{r}) \otimes \varphi(\vec{r}) \rangle =$$

$$= \langle \otimes^2 \varphi(\vec{r}+\vec{\ell}) \rangle + \langle \otimes^2 \varphi(\vec{r}) \rangle - 2 \langle \varphi(\vec{r}) \otimes \varphi(\vec{r}+\vec{\ell}) \rangle \Rightarrow$$

$$\mathbf{D}_0(\vec{\ell}) = \int_{\mathbb{R}^n} d\vec{r}\, \mathbf{D}(\vec{r},\vec{\ell}) =$$

$$= \int_{\mathbb{R}^n} d\vec{r} \left[\langle \otimes^2 \varphi(\vec{r}+\vec{\ell}) \rangle + \langle \otimes^2 \varphi(\vec{r}) \rangle \right] - 2 \int_{\mathbb{R}^n} d\vec{r}\, \mathbf{B}(\vec{r}, \vec{r}+\vec{\ell})$$

$$= 2 \int_{\mathbb{R}^n} d\vec{r} \langle \otimes^2 \varphi(\vec{r}) \rangle - 2\mathbf{B}_0(\vec{\ell}) \Rightarrow$$

$$\Rightarrow$$

$$D_0(\rho) = \frac{1}{\gamma_n} \int_{SO(n)} \mathrm{tr}\, \mathbf{D}_0(\rho A\vec{e})\, d\Omega =$$

$$= \frac{1}{\gamma_n} \int_{SO(n)} \mathrm{tr} \left[2 \int_{\mathbb{R}^n} \langle \otimes^2 \varphi(\vec{r}) \rangle d\vec{r} - 2\mathbf{B}_0(\rho A\vec{e}) \right] d\Omega$$

$$= \frac{1}{\gamma_n} \int_{SO(n)} \left[2 \int_{\mathbb{R}^n} \langle |\varphi(\vec{r})|^2 \rangle d\vec{r} - 2B_0(\rho A\vec{e}) \right] d\Omega =$$

$$= 2\,\mathrm{var}[\varphi] \frac{1}{\gamma_n} \int_{SO(n)} d\Omega - 2 \frac{1}{\gamma_n} \int_{SO(n)} B(\rho A\vec{e})\, d\Omega =$$

$$= 2\,\mathrm{var}[\varphi] - 2B_0(\rho). \qquad \square$$

So we see that in order for $D_0(\rho)$ to converge, we need $\mathrm{var}[\varphi]$ to be finite. This is equivalent with the integral of the energy spectrum $E(k)$ being finite, since

$$\boxed{\mathrm{var}[\varphi] = \int_0^{+\infty} 2E(k)\, dk.}$$

Energy spectrum and structure functions

When the variance is finite we have the following relation between $E(k)$ and $D_0(\rho)$:

<u>Theorem</u>: Let φ be a random field with finite $\mathrm{var}[\varphi]$, and finite energy spectrum $E(k)$. Then:

$$\boxed{D_0(\rho) = 4\int_0^{+\infty} E(k)\left[1-\phi_n(k\rho)\right]dk}$$

<u>Proof</u>

$$\begin{aligned} D_0(\rho) &= 2\,\mathrm{var}[\varphi] - 2B_0(\rho) = 2\int_0^{+\infty} 2E(k)\,dk - 2\int_0^{+\infty} \mathcal{E}(k)\,\phi_n(k\rho)\,dk \\ &= 4\int_0^{+\infty} E(k)\,dk - 4\int_0^{+\infty} E(k)\,\phi_n(k\rho)\,dk = \\ &= 4\int_0^{+\infty} E(k)\left[1-\phi_n(k\rho)\right]dk. \qquad \square \end{aligned}$$

When $\mathrm{var}[\varphi]$ is finite, then it is possible to solve this integral equation for $E(k)$. First note the following lemma:

<u>Lemma</u>: $$\boxed{\frac{\gamma_n^2}{(2\pi)^n}\int_0^{+\infty} (k\rho)^{n-1}\phi_n(k\rho)\,d\rho = 2\delta(k)}$$

<u>Proof</u>

$$1 = \phi_n(0) = \int_0^{+\infty} 2\delta(k)\,\phi_n(k\rho)\,dk \Rightarrow \text{the result.} \qquad \square$$

Even though it is trivial, note carefully the factor of 2.

<u>Theorem</u>: Let φ be a random field with finite $\mathrm{var}[\varphi]$. Then:

$$\boxed{E(k) = \mathrm{var}[\varphi]\delta(k) - \frac{1}{4}\frac{\gamma_n^2}{(2\pi)^n}\int_0^{+\infty}(k\rho)^{n-1}D_0(\rho)\Phi_n(k\rho)\,d\rho}$$

<u>Proof</u>

$$E(k) = \frac{b(k)}{2} = \frac{\gamma_n^2}{(2\pi)^n}\int_0^{+\infty}(k\rho)^{n-1}\frac{B_0(\rho)}{2}\Phi_n(k\rho)\,d\rho$$

$$= \frac{\gamma_n^2}{(2\pi)^n}\int_0^{+\infty}(k\rho)^{n-1}\left[\frac{\mathrm{var}[\varphi]}{2} - \frac{D_0(\rho)}{4}\right]\Phi_n(k\rho)\,d\rho =$$

$$= \frac{\mathrm{var}[\varphi]}{2}\frac{\gamma_n^2}{(2\pi)^n}\int_0^{+\infty}(k\rho)^{n-1}\Phi_n(k\rho)\,d\rho - \frac{1}{4}\frac{\gamma_n^2}{(2\pi)^n}\int_0^{+\infty}(k\rho)^{n-1}D_0(\rho)\Phi_n(k\rho)\,d\rho$$

$$= \mathrm{var}[\varphi]\delta(k) - \frac{1}{4}\frac{\gamma_n^2}{(2\pi)^n}\int_0^{+\infty}(k\rho)^{n-1}D_0(\rho)\Phi_n(k\rho)\,d\rho \qquad \square$$

Now let $\langle \varphi_n \mid n \in \mathbb{N}\rangle$ be a sequence of random fields such that each φ_n has finite energy $E_n(k)$ per wavenumber k, and finite variance $\mathrm{var}[\varphi_n]$. Using the above relations, let $D_n(\rho)$ be the corresponding structure functions. It follows that for each n, the energy integral converges:

$$\int_0^{+\infty}E_n(k)\,dk = \frac{\mathrm{var}[\varphi_n]}{2}$$

Suppose however that

$$\lim\left[\mathrm{var}[\varphi_n]\right] = +\infty$$
$$\lim E_n(k) = E(k) = \text{finite}$$

So while energy per wavenumber is constant, the total variance becomes infinite. We would like to answer the following questions:

1) When does the limit $\lim D_n(\rho) = D(\rho)$ converge?
2) If it does converge, how do we find $E(k)$ from $D(\rho)$?

Recall the following results about the $\Phi_n(x)$ functions:

1) $\Phi_n(x)$ is bounded, because $|\Phi_n(x)| \leqslant 1$
2) $\lim_{x\to 0^+} \frac{1-\Phi_n(x)}{x^2} = \frac{1}{2n}$

First we prove a convergence result for $\lim D_n(\rho)$:

<u>Theorem</u>: Suppose that

a) $(\exists \varepsilon_1 > 0) \lim_{k\to+\infty} k^{1+\varepsilon_1} E(k) = 0$, b) $(\exists \varepsilon_2 > 0) \lim_{k\to 0^+} k^{3-\varepsilon_2} E(k) = \ell \in \mathbb{R}$

then for all sequences φ_n that yield $E(k)$, the limit $\lim D_n$ also converges and it is given by:

$$\boxed{\lim D_n(\rho) = 4\int_0^{+\infty} E(k)(1-\Phi_n(k\rho))\,dk}$$

<u>Proof</u>

$$\lim_{m\in\mathbb{N}} D_m(\rho) = 4 \lim_{m\in\mathbb{N}} \int_0^{+\infty} E_m(k)(1-\Phi_n(k\rho))\,dk =$$
$$= 4\int_0^{+\infty} \left[\lim_{m\in\mathbb{N}} E_m(k)\right](1-\Phi_n(k\rho))\,dk =$$
$$= 4\int_0^{+\infty} E(k)\left[1-\Phi_n(k\rho)\right] dk.$$

To justify taking the limit into the integral we must show that the resulting integral converges. We must show convergence both at <u>infrared</u> $(k\to 0^+)$ and at <u>ultraviolet</u> $(k\to+\infty)$.

At ultraviolet:

$$\left.\begin{array}{l}\lim_{k\to+\infty} k^{1+\varepsilon_1} E(k) = 0 \\ 1-\Phi_n(k\rho) \text{ bounded}\end{array}\right\} \Rightarrow \lim_{k\to+\infty}\left[k^{1+\varepsilon_1} E(k)(1-\Phi_n(k\rho))\right] = 0 \Rightarrow$$
$$\Rightarrow \int_a^{+\infty} E(k)(1-\Phi_n(k\rho))\,dk \quad \text{converges}$$

At infrared:

$$\lim_{k\to 0^+} k^{1-\varepsilon_2} E(k)\,[1-\Phi_n(k\rho)] = \lim_{k\to 0^+}\left[k^{3-\varepsilon_2} E(k)\,\frac{1-\Phi_n(k\rho)}{k^2}\right] =$$

$$= \lim_{k\to 0^+}\left[k^{3-\varepsilon_2} E(k)\right] \rho^2 \lim_{k\to 0^+} \frac{1-\Phi_n(k\rho)}{(k\rho)^2} =$$

$$= \ell \rho^2 \frac{1}{2n} \Rightarrow \int_0^a E(k)\,[1-\Phi_n(k\rho)]\,dk \text{ converges.}$$

It follows that the resulting integral converges. □

To answer the 2nd question, first we show a uniqueness result. Then we study a special case that is of interest.

<u>Theorem</u>: Let φ_n be a sequence of random fields such that

a) $\operatorname{var}[\varphi_n]$ is finite

b) $\lim_{n\in\mathbb{N}} D_n(\rho) = D(\rho)$ converges

Also let $E_0(k)$ be a function that satisfies the equation

$$D_0(\rho) = \int_0^{+\infty} 4\, E_0(k)\,[1-\Phi_n(k\rho)]\,dk$$

Then the sequence of energy spectra $E_n(k)$ also converges and it satisfies

$$\boxed{\lim_{n\in\mathbb{N}} E_n(k) = \lambda\delta(k) + E_0(k)}$$

where $\lambda = \lim_{\rho\to+\infty} \lim_{n\in\mathbb{N}} \left[\operatorname{var}[\varphi_n] - \frac{D_n(\rho)}{2}\right]$

<u>Proof</u>

Recall that $D_n(\rho) = 4\int_0^{+\infty} E_n(k)\,[1-\Phi_n(k\rho)]\,dk$

Since $|\Phi_n(k\rho)| \leqslant 1 \Rightarrow 1-\Phi_n(k\rho) \geqslant 0$, $D_n(\rho)$ is the integral of $E_n(k)$ weighted by a positive function. It follows that:

$$\lim_{n\in\mathbb{N}} E_n(k) \text{ diverges} \Rightarrow \lim_{n\in\mathbb{N}} D_n(\rho) \text{ diverges}$$

This is equivalent to the contrapositive statement:

$$\lim_{n\in\mathbb{N}} D_n(\rho) \text{ converges} \Rightarrow \lim_{n\in\mathbb{N}} E_n(k) \text{ converges.}$$

Since $\lim_{n\in\mathbb{N}} D_n(\rho)$ converges by hypothesis, we conclude that the limit $\lim_{n\in\mathbb{N}} E_n(k)$ also converges.

Let $E(k) = \lim_{n\in\mathbb{N}} E_n(k)$ and $f(k) = E(k) - E_0(k)$.
To find $f(k)$ we give the following argument:
We have:

$$D(\rho) = \int_0^{+\infty} 4E_0(k)\,[1-\Phi_n(k\rho)]\,dk \quad \text{(by hypothesis)}$$

$$D(\rho) = \int_0^{+\infty} 4E(k)\,[1-\Phi_n(k\rho)]\,dk \quad \text{(by theorem p. 122)}$$

therefore

$$I \equiv \int_0^{+\infty} f(k)\,[1-\Phi_n(k\rho)]\,dk = 0$$

In general, $f(k)$ can be a generalized function. Since every generalized function is the transform of an ordinary function, there is an $g(\rho)$ such that

$$f(k) = \frac{\gamma_n^2}{(2\pi)^n} \int_0^{+\infty} (k\rho)^{n-1} g(\rho)\,\Phi_n(k\rho)\,d\rho$$

Now we can evaluate I in terms of $g(\rho)$:

$$I = \int_0^{+\infty} f(k)\,[1-\Phi_n(k\rho)]\,dk =$$

$$= \int_0^{+\infty} \left\{ \frac{\gamma_n^2}{(2\pi)^n} \int_0^{+\infty} (k\rho_0)^{n-1} g(\rho_0)\,\Phi_n(k\rho_0)\,d\rho_0 \right\} [1-\Phi_n(k\rho)]\,dk =$$

$$= \int_0^{+\infty} d\rho_0\, g(\rho_0) \left\{ \frac{\gamma_n^2}{(2\pi)^n} \int_0^{+\infty} (k\rho_0)^{n-1}\,\Phi_n(k\rho_0)\,[1-\Phi_n(k\rho)]\,dk \right\}$$

$$= \int_0^{+\infty} d\rho_0\, g(\rho_0) \left\{ \frac{\gamma_n^2}{(2\pi)^n} \int_0^{+\infty} (k\rho_0)^{n-1}\Phi_n(k\rho_0)\,dk - \frac{\gamma_n^2}{(2\pi)^n} \int_0^{+\infty} (k\rho_0)^{n-1}\Phi_n(k\rho_0)\,\Phi_n(k\rho)\,dk \right\}$$

$$= \int_0^{+\infty} d\rho_0\, g(\rho_0)\,[2\delta(\rho_0) - \delta(\rho-\rho_0)] =$$

$$= \int_0^{+\infty} 2g(\rho_0)\delta(\rho_0)\,d\rho_0 - \int_0^{+\infty} g(\rho_0)\,\delta(\rho-\rho_0)\,d\rho_0 = g(0) - g(\rho)$$

Since $I = g(0) - g(\rho) = 0 \rightarrow g(\rho) = c = \text{constant}$ for $\rho \geq 0$
therefore

$$f(k) = \frac{\gamma_n^2}{(2\pi)^n} \int_0^{+\infty} (k\rho)^{n-1}\, c\, \Phi_n(k\rho)\,d\rho = 2c\,\delta(k) = \lambda\delta(k). \rightarrow$$

$$\Rightarrow \underline{E(k) = \lambda\delta(k) + E_0(k).}$$

To interpret λ, note that

$$\lim_{\rho\to+\infty} B(\rho) = \lim_{\rho\to+\infty} \int_0^{+\infty} 2[\lambda\delta(k) + E_0(k)]\,\Phi_n(k\rho)\,dk =$$

$$= \lim_{\rho\to+\infty} \left[2\lambda \int_0^{+\infty} \delta(k)\Phi_n(k\rho)\,dk + 2\int_0^{+\infty} E_0(k)\,\Phi_n(k\rho)\,dk \right]$$

$$= \lim_{\rho\to+\infty} \left[2\lambda\Phi_n(0)\,\frac{1}{2} + 2\int_0^{+\infty} E_0(k)\,\Phi_n(k\rho)\,dk \right] =$$

$$= \lambda + 2\lim_{\rho\to+\infty} \int_0^{+\infty} E_0(k)\,\Phi_n(k\rho)\,dk = \lambda + 2\cdot 0 = \lambda \Rightarrow$$

$$\Rightarrow \lambda = \lim_{\rho\to+\infty} B(\rho) = \lim_{\rho\to+\infty} \lim_{n\in\mathbb{N}} B_n(\rho) =$$

$$= \lim_{\rho\to+\infty} \lim_{n\in\mathbb{N}} \left[\mathrm{var}[\varphi_n] - \frac{D_n(\rho)}{2} \right] \qquad \square$$

The power-law theorem

The power-law theorem is an exact result that is of vital importance to the theory of turbulence.

Theorem: Let φ_n be a sequence of random fields such that the corresponding spectra $E_n(k)$ satisfy: $\lim_{n\in\mathbb{N}} E_n(k) = ak^{-m}$ with $1<m<3$. Then the limit of the structure functions converges and it is given by

$$\boxed{\lim_{n\in\mathbb{N}} D_n(\rho) = a\lambda_n \rho^{m-1}}$$

where λ_n is a constant given by

$$\boxed{\lambda_n = 4\int_0^{+\infty} x^{-m}[1-\Phi_n(x)]\,dx}$$

Proof

Since $1<m$, choose $\varepsilon_1>0$ such that $1+\varepsilon_1<m$.
Also, since $m<3$, choose $\varepsilon_2>0$ such that $m+\varepsilon_2<3$
It follows that $1+\varepsilon_1-m<0$ and $3-\varepsilon_2-m>0$, so we have

$$\lim_{k\to+\infty} k^{1+\varepsilon_1}E(k) = \lim_{k\to+\infty} k^{1+\varepsilon_1} a k^{-m} = a\lim_{k\to+\infty} k^{1+\varepsilon_1-m} = 0$$

$$\lim_{k\to0^+} k^{3-\varepsilon_2}E(k) = \lim_{k\to0^+} k^{3-\varepsilon_2} a k^{-m} = a\lim_{k\to0^+} k^{3-\varepsilon_2-m} = 0$$

This means that we can employ theorem p.122 to obtain the limit structure function:

$$\lim_{n\in\mathbb{N}} D_n(\rho) = 4\int_0^{+\infty} E(k)[1-\Phi_n(k\rho)]\,dk = 4a\int_0^{+\infty} k^{-m}[1-\Phi_n(k\rho)]\,dk$$

Substitute $x = k\rho \Rightarrow dx = \rho\, dk \Rightarrow dk = \rho^{-1} dx$

Also $k = \rho^{-1} x$ and $k = 0 \Rightarrow x = 0$

$k = +\infty \Rightarrow x = +\infty$

It follows that

$$\lim_{n \in \mathbb{N}} D_n(\rho) = 4a \int_0^{+\infty} (\rho^{-1} x)^{-m} \left[1 - \Phi_n(x)\right] \rho^{-1}\, dx =$$

$$= 4a\rho^{m-1} \int_0^{+\infty} x^{-m} \left[1 - \Phi_n(x)\right] dx = a\, \mathcal{J}_n(m) \rho^{m-1}$$

where $\mathcal{J}_n(m)$ is the integral:

$$\mathcal{J}_n(m) = 4 \int_0^{+\infty} x^{-m} \left[1 - \Phi_n(x)\right] dx. \qquad \square$$

We now show how to evaluate $\mathcal{J}_n(m)$ in closed form. First we obtain an integral expression for $1 - \Phi_n(x)$

<u>Lemma</u> : $$\boxed{1 - \Phi_n(x) = 2 \frac{\gamma_{n-1}}{\gamma_n} \int_0^{\pi} \sin^2(x \cos\varphi / 2) \sin^{n-2} \varphi \, d\varphi}$$

<u>Proof</u>

Recall that $\gamma_n = 2\pi \prod_{l=1}^{n-2} I(k)$ where $I(k) = \int_0^{\pi} (\sin\varphi)^k d\varphi$

It follows that

$$\frac{\gamma_{n-1}}{\gamma_n} = \frac{2\pi \prod_{l=1}^{n-3} I(k)}{2\pi \prod_{l=1}^{n-2} I(k)} = \frac{1}{I(n-2)} \Rightarrow \frac{\gamma_{n-1}}{\gamma_n} I(n-2) = 1$$

so:

$$1 - \Phi_n(x) = 1 - \frac{\gamma_{n-1}}{\gamma_n} \int_0^{\pi} \cos(x \cos\varphi) \sin^{n-2} \varphi \, d\varphi =$$

$$= \frac{\gamma_{n-1}}{\gamma_n} \left[I(n-2) - \int_0^{\pi} \cos(x\cos\varphi) \sin^{n-2}\varphi \, d\varphi \right] =$$

$$= \frac{\gamma_{n-1}}{\gamma_n} \int_0^{\pi} \left[1 - \cos(x\cos\varphi)\right] \sin^{n-2}\varphi \, d\varphi =$$

$$= 2\,\frac{\gamma_{n-1}}{\gamma_n}\int_0^{\pi}\sin^2(x\cos\varphi/2)\,\sin^{n-2}\varphi\,d\varphi \qquad \square$$

Using this expression we can begin evaluating $\mathcal{Z}_n(m)$:

$$\mathcal{Z}_n(m) = 4\int_0^{+\infty} x^{-m}\,[1-\Phi_n(x)]\,dx =$$

$$= 4\int_0^{+\infty} x^{-m}\left[2\,\frac{\gamma_{n-1}}{\gamma_n}\int_0^{\pi}\sin^2(x\cos\varphi/2)\,\sin^{n-2}\varphi\,d\varphi\right]dx =$$

$$= 8\,\frac{\gamma_{n-1}}{\gamma_n}\int_0^{\pi}d\varphi\,\sin^{n-2}\varphi\left\{\int_0^{+\infty}x^{-m}\sin^2(x\cos\varphi/2)\,dx\right\}$$

At this point, the greatest challenge is to evaluate integrals of the form:

$$\boxed{g(a) \equiv \int_0^{+\infty} x^{-m}\sin^2(ax)\,dx.}$$

First note that $\underline{g(-a)=g(a)}$ so suppose that $a>0$.
Let $y = 2ax \Rightarrow dy = 2a\,dx \Rightarrow dx = (2a)^{-1}dy$
For $x=0 \Rightarrow y=0$ and $x=+\infty \Rightarrow y=+\infty$
It follows that:

$$g(a) = \int_0^{+\infty} x^{-m}\sin^2(ax)\,dx = \int_0^{+\infty}(y/2a)^{-m}\sin^2(y/2)\,(2a)^{-1}dy =$$

$$= (2a)^{m-1}\int_0^{+\infty} y^{-m}\sin^2(y/2)\,dy = (2a)^{m-1}g(1/2) \Rightarrow$$

$$\Rightarrow \boxed{g(a) = (2a)^{m-1}g(1/2)},\quad \forall a>0$$

To evaluate $g(1/2)$ we rely on the following identity:

$$x^{-m} = \frac{1}{\Gamma(m)}\int_0^{+\infty} y^{m-1}e^{-xy}\,dy$$

We then find:

$$g(1/2) = \int_0^{+\infty} x^{-m} \sin^2(x/2)\,dx = \frac{1}{\Gamma(m)} \int_0^{+\infty} dy \int_0^{+\infty} dx \; y^{m-1} e^{-xy} \sin^2(x/2) =$$

$$= \frac{1}{\Gamma(m)} \int_0^{+\infty} dy\, y^{m-1} \left\{ \int_0^{+\infty} e^{-xy} \sin^2(x/2)\,dx \right\} =$$

$$= \frac{1}{\Gamma(m)} \int_0^{+\infty} dy\, y^{m-1} \frac{1}{2y(y^2+1)} = \frac{1}{2\Gamma(m)} \int_0^{+\infty} \frac{y^{m-2}}{y^2+1}\,dy .$$

The last integral can be evaluated with contour integration. The general result is that

$$\int_0^{+\infty} \frac{y^a}{1+y^2}\,dy = \frac{\pi}{2}\,\frac{1}{\cos(a\pi/2)} \quad , \quad -1 < a < 1$$

It follows that

$$g(1/2) = \frac{1}{2\Gamma(m)}\,\frac{\pi}{2}\,\frac{1}{\cos[(m-2)\pi/2]} = \frac{\pi}{4}\,\frac{1}{\Gamma(m)}\,\frac{1}{\cos(\pi - m\pi/2)} =$$

$$= \frac{\pi}{4}\,\frac{1}{\Gamma(m)}\,\frac{-1}{\cos(m\pi/2)} = \frac{\pi}{4}\,\frac{1}{\Gamma(m)}\,\frac{-1}{\sin\left(\frac{\pi}{2} - \frac{m\pi}{2}\right)} =$$

$$= \frac{\pi}{4}\,\frac{1}{\Gamma(m)}\,\frac{1}{\sin[(m-1)\pi/2]} \qquad \text{for } \underline{1 < m < 3}$$

Using this result we obtain, for arbitrary a:

$$\boxed{g(a) = (2|a|)^{m-1}\,\frac{\pi}{4}\,\frac{1}{\Gamma(m)}\,\frac{1}{\sin[(m-1)\pi/2]}}$$

where the absolute value for 'a' is based on $g(-a) = g(a)$ and allows our expression to be valid both for positive and negative a.

To evaluate $\mathcal{Z}_n(m)$ we use our expression for $g(a)$ and the following identity about the beta-function:

$$\boxed{\int_0^{\pi/2} \cos^a\varphi\,\sin^b\varphi\,d\varphi = \frac{1}{2}\,B\left(\frac{a+1}{2}, \frac{b+1}{2}\right)}$$

to get:

$$\lambda_n(m) = 8\,\frac{\gamma_{n-1}}{\gamma_n}\int_0^{\pi} d\varphi\,\sin^{n-2}\varphi\; g(\cos\varphi/2) =$$

$$= 8\,\frac{\gamma_{n-1}}{\gamma_n}\int_0^{\pi} d\varphi\,\sin^{n-2}\varphi\,\left(2\,\frac{|\cos\varphi|}{2}\right)^{m-1}\frac{\pi}{4}\,\frac{1}{\Gamma(m)}\,\frac{1}{\sin[(m-1)\pi/2]} =$$

$$= 2\pi\,\frac{\gamma_{n-1}}{\gamma_n}\,\frac{1}{\Gamma(m)\sin[(m-1)\pi/2]}\int_0^{\pi} d\varphi\,\sin^{n-2}\varphi\,|\cos\varphi|^{m-1}$$

$$= 4\pi\,\frac{\gamma_{n-1}}{\gamma_n}\,\frac{1}{\Gamma(m)\sin[(m-1)\pi/2]}\int_0^{\pi/2} d\varphi\,\sin^{n-2}\varphi\,\cos^{m-1}\varphi =$$

$$= 4\pi\,\frac{\gamma_{n-1}}{\gamma_n}\,\frac{1}{\Gamma(m)\sin[(m-1)\pi/2]}\,\frac{1}{2}\,B\left(\frac{n-1}{2},\frac{m}{2}\right)$$

The γ_{n-1}/γ_n constant can also be rewritten as:

$$\frac{\gamma_{n-1}}{\gamma_n} = I(n-2) = \int_0^{\pi}\sin^{n-2}\varphi\,d\varphi = 2\int_0^{\pi/2}\sin^{n-2}\varphi\,d\varphi$$

$$= 2\,\frac{1}{2}\,B\left(\frac{n-1}{2},\frac{1}{2}\right) = B\left(\frac{n-1}{2},\frac{1}{2}\right)$$

Putting it alltogether we have

$$\boxed{\lambda_n(m) = 2\pi\,\frac{B\left(\frac{n-1}{2},\frac{1}{2}\right)B\left(\frac{n-1}{2},\frac{m}{2}\right)}{\Gamma(m)\sin[(m-1)\pi/2]}}$$

This expression can be simplified when $n=2$ and $n=3$.

→ For $n=3$

$$B\left(\frac{n-1}{2},\frac{1}{2}\right) = B\left(\frac{3-1}{2},\frac{1}{2}\right) = B\left(1,\frac{1}{2}\right) = \frac{\Gamma(1)\Gamma(1/2)}{\Gamma(3/2)} =$$

$$= \frac{1\cdot\sqrt{\pi}}{\frac{1}{2}\sqrt{\pi}} = 2$$

and

$$B\left(\frac{n-1}{2},\frac{m}{2}\right) = B\left(1,\frac{m}{2}\right) = \frac{\Gamma(1)\Gamma(m/2)}{\Gamma\left(1+\frac{m}{2}\right)} = \frac{\Gamma(m/2)}{\Gamma\left(1+\frac{m}{2}\right)} =$$

$$= \frac{\Gamma(m/2)}{\frac{m}{2}\,\Gamma(m/2)} = \frac{2}{m}$$

It follows that:

$$\boxed{\mathcal{A}_3(m) = \frac{8\pi}{m}\left\{\Gamma(m)\sin\left[(m-1)\pi/2\right]\right\}^{-1}}$$

→ <u>For $n=2$</u>

$$B\left(\frac{n-1}{2},\frac{1}{2}\right) = B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)} = \Gamma^2(1/2) = \pi$$

$$B\left(\frac{n-1}{2},\frac{m}{2}\right) = B\left(\frac{1}{2},\frac{m}{2}\right) = \frac{\Gamma(1/2)\Gamma(m/2)}{\Gamma\left(\frac{1}{2}+\frac{m}{2}\right)} =$$

$$= \frac{\Gamma(1/2)\,\Gamma(m/2)}{\dfrac{\Gamma(m)}{\pi^{-1/2}\,2^{m-1}\,\Gamma(m/2)}} =$$

$$= \frac{\pi^{-1/2}\,2^{m-1}\,\Gamma(m/2)\,\Gamma(1/2)\,\Gamma(m/2)}{\Gamma(m)} = 2^{m-1}\,\frac{\Gamma^2(m/2)}{\Gamma(m)}$$

where we have employed the identity:

$$\Gamma(2x) = \pi^{-1/2}\,2^{2x-1}\,\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right)$$

It follows that:

$$\mathcal{A}_2(m) = 2\pi\,\frac{\pi\cdot 2^{m-1}\,\frac{\Gamma^2(m/2)}{\Gamma(m)}}{\Gamma(m)\sin\left[(m-1)\pi/2\right]} = 2^m\pi^2\left(\frac{\Gamma(m/2)}{\Gamma(m)}\right)^2\frac{1}{\sin\left[(m-1)\pi/2\right]}$$

so:

$$\boxed{\mathcal{A}_2(m) = \left[\frac{\Gamma(m/2)}{\Gamma(m)}\right]^2\frac{2^m\pi^2}{\sin\left[(m-1)\pi/2\right]}}$$

▼ The asymptotic power-law theorem

A power-law spectrum $E(k) = ak^{-m}$ corresponds to infinite total energy, so we can not use the power-low theorem in any physically realistic situations. Usually we have to deal with structure functions that are approximately power-law in the following sense:

$$D_0(\rho) = a\rho^{m-1} F(\rho/\rho_0) + 2A(\rho_0)[1 - F(\rho/\rho_0)]$$

where F is a matching function such that

$$F(x) \sim 1 \quad \text{for} \quad x < 1 - \delta_1$$
$$F(x) \ll 1 \quad \text{for} \quad x > 1 + \delta_2$$

and $A(\rho_0)$ is an increasing function of ρ_0.
We call ρ_0 an integral scale and it corresponds to the length scale where $D_0(\rho)$ ceases following a power-law.
It is easy to see that $D_0(\rho)$ is a matching of the following asymptotic expressions:

$$D_0(\rho) \sim a\rho^{m-1} \quad \text{as} \quad \rho \ll \rho_0$$
$$D_0(\rho) \sim 2A(\rho_0) \quad \text{as} \quad \rho \gg \rho_0.$$

The first expression is our hypothesis. How do we justify the second expression, and what is the meaning of $A(\rho_0)$?
To explain that recall the definitions of the quantities involved:

$$D_0(\rho) = \frac{1}{\gamma_n} \int_{SO(n)} d\underline{O} \int_{\mathbb{R}^n} d\vec{r} \; \mathrm{tr} \langle \otimes^2 \delta\varphi(\vec{r}, \rho A\vec{e}) \rangle$$

$$B_0(\rho) = \frac{1}{\gamma_n} \int_{SO(n)} d\underline{O} \int_{\mathbb{R}^n} d\vec{r} \; \mathrm{tr} \langle \varphi(\vec{r}) \otimes \varphi(\vec{r} + \rho A\vec{e}) \rangle$$

We have already shown that

$$D_0(\rho) = 2ms[\varphi] - 2B_0(\rho)$$

Since we assume that φ vanishes when $\vec{r} \to \infty$ we have

$$\lim_{\rho \to +\infty} \varphi(\vec{r} + \rho A\vec{e}) = 0$$

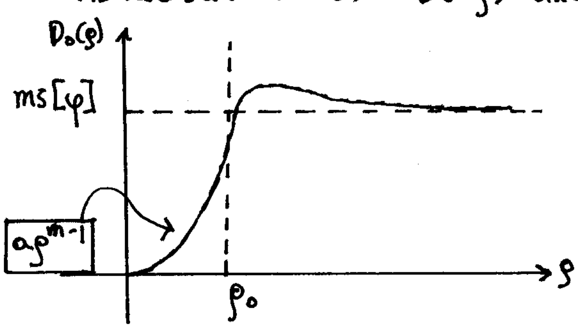
therefore

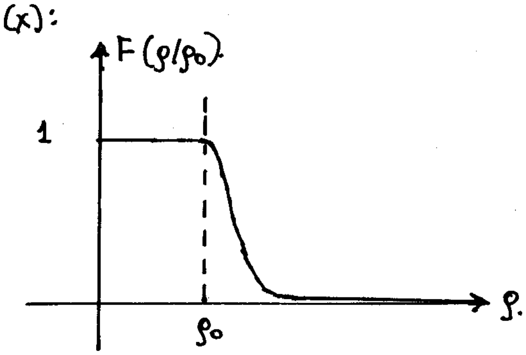
$$\lim_{\rho\to+\infty} B_0(\rho) = \lim_{\rho\to+\infty} \frac{1}{\gamma_n} \int_{SO(n)} dO \int_{\mathbb{R}^n} d\vec{r}\ \mathrm{tr}\langle \varphi(\vec{r}) \otimes \varphi(\vec{r} + \rho A\vec{e})\rangle =$$
$$= \frac{1}{\gamma_n} \int_{SO(n)} dO \int_{\mathbb{R}^n} d\vec{r}\ \mathrm{tr}\langle \varphi(\vec{r}) \otimes [\lim_{\rho\to+\infty} \varphi(\vec{r} + \rho A\vec{e})]\rangle =$$
$$= 0$$

So $\lim_{\rho\to+\infty} D_0(\rho) = 2ms[\varphi] - 2\lim_{\rho\to+\infty} B_0(\rho) = 2ms[\varphi] \Rightarrow$

$$\Rightarrow \underline{D_0(\rho) \sim 2ms[\varphi] \quad \text{as } \rho \gg \rho_0}$$

So $A_0(\rho_0) = ms[\varphi]$ is the mean-square itensity of φ.
Note that the crucial assumption behind this interpretation is that $B_0(\rho)$ vanishes as $\rho \to +\infty$. The following diagrams offer a visualization of $D_0(\rho)$ and $F(x)$:





Now, what can we say about the energy spectrum?

Theorem: Let $\langle \varphi_n \mid n \in \mathbb{N} \rangle$ be a sequence of random fields with structure function

$$D_n(\rho) = a\rho^{m-1} F(\rho/\rho_n) + 2\, ms[\varphi_n][1 - F(\rho/\rho_n)]$$

where F is the previously defined matching function and:

a) $ms[\varphi_n]$ is finite and equal to $A(\rho_n)$

b) $\lim_{n \in \mathbb{N}} \rho_n = +\infty$ (i.e. φ_n is a sequence of random fields with increasing integral scale).

Then the corresponding energy spectra are given by:

$$\boxed{E_n(k) = \frac{a}{\lambda_n(m)} k^{-m} G(k, \rho_n, A(\rho_n))}$$

where G is a universal function for fixed F of wavenumber k, integral scale ρ_n, and the mean-square of $\varphi_n \equiv A(\rho_n)$ and it satisfies:

$$\boxed{\lim_{\rho_0 \to +\infty} G(k, \rho_0, A(\rho_0)) = 1}$$

for any dependence $A(\rho_0)$.

Proof

To justify the existence of the G function as stated above, except for the limit result, we employ theorem p. 114 that relates $E_n(k)$ with $B_n(\rho)$. First note that:

$$B_n(\rho) = ms[\varphi_n] - \frac{D_n(\rho)}{2} =$$

$$= ms[\varphi_n] - \frac{a\rho^{m-1}}{2} F(\rho/\rho_n) - ms[\varphi_n][1 - F(\rho/\rho_n)] =$$

$$= F(\rho/\rho_n) \left\{ ms[\varphi_n] - \frac{a\rho^{m-1}}{2} \right\}$$

Then $E_n(k)$ is given by:

$$E_n(k) = \frac{1}{2}\,\frac{\gamma_{n_0}^2}{(2\pi)^{n_0}}\int_0^{+\infty}(k\rho)^{n-1}\left\{ms[\varphi_n] - \frac{a\rho^{m-1}}{2}\right\}F(\rho/\rho_n)\,\Phi_{n_0}'(k\rho)\,d\rho$$

where n_0 is the dimensionality of space (to avoid confusion with n). Upon inspection we see that $E_n(k)$ depends on k, ρ_n and $ms[\varphi_n]$, given a fixed function F, and a fixed constant a. So, anticipating an asymptotic power-law dependence, we may introduce a universal function $G(k, \rho_n, ms[\varphi_n])$ such that the following relation is satisfied:

$$E_n(k) = \frac{a}{\lambda_n(m)}\,k^{-m}\,G(k, \rho_n, ms[\varphi_n])$$

for all n <u>and</u> for all sequences φ_n that are consistent with the given a and F.

▶ Note that the integral representation of $E_n(k)$ does <u>not</u> suggest that G might have any similarity symmetries that might let us claim that G only depends on $k\rho_n$ and $ms[\varphi_n]$ or any other combination of two or one expressions. At least, not in any obvious way. ◀

To obtain the limit result we employ theorem, p. 123

First we establish the conditions of the theorem:

a) $ms[\varphi_n]$ are indeed finite by hypothesis.

b) $$\lim_{n\in\mathbb{N}} D_n(\rho) = a\rho^{m-1}\lim_{n\in\mathbb{N}} F(\rho/\rho_n) + 2\lim_{n\in\mathbb{N}}(ms[\varphi_n])\left[1 - \lim_{n\in\mathbb{N}} F(\rho/\rho_n)\right]$$

$$= a\rho^{m-1}\cdot 1 + 2\lim_{n\in\mathbb{N}}(ms[\varphi_n])[1-1] = a\rho^{m-1}$$

c) Define $E_0(k) = \dfrac{a}{\lambda_n(m)}\,k^{-m}$ (see notation on p. 123)

Then the power-law theorem says that

$$\int_0^{+\infty} 4E_0(k)\,[1-\Phi_n(k\rho)]\,dk = a\rho^{m-1} = \lim_{n\in\mathbb{N}} D_n(\rho)$$

so $E_0(k)$ satisfies the required constraint.

It follows that theorem p.123 applies. First we evaluate ∂:

$$\partial = \lim_{n\in\mathbb{N}} \lim_{\rho\to+\infty} \left[ms[\varphi_n] - \frac{D_n(\rho)}{2} \right] =$$

$$= \lim_{n\in\mathbb{N}} \left[ms[\varphi_n] - \lim_{\rho\to+\infty} \frac{D_n(\rho)}{2} \right] = \lim_{n\in\mathbb{N}} \left[ms[\varphi_n] - \frac{2ms[\varphi_n]}{2} \right] =$$

$$= 0$$

Then theorem p.123 yields:

$$\lim_{n\in\mathbb{N}} E_n(k) = E_0(k) = \frac{ak^{-m}}{\lambda_n(m)} \quad (1)$$

Also:

$$\lim_{n\in\mathbb{N}} E_n(k) = \lim_{n\in\mathbb{N}} \left\{ \frac{a}{\lambda_n(m)} k^{-m} G(k, \rho_n, ms[\varphi_n]) \right\} =$$

$$= \frac{a}{\lambda_n(m)} k^{-m} \lim_{n\in\mathbb{N}} G(k, \rho_n, A(\rho_n)) =$$

$$= \frac{a}{\lambda_n(m)} k^{-m} \lim_{\rho_0\to+\infty} G(k, \rho_0, A(\rho_0)) \quad (2)$$

From (1) and (2) we find that

$$\lim_{\rho_0\to+\infty} G(k, \rho_0, A(\rho_0)) = 1 \qquad \square$$

This is the asymptotic power-law theorem. Loosely speaking, it implies that if $D_0(\rho)$ is a power-law at $\rho \ll \rho_0$, then $E(k)$ is also a powerlaw for $k \gg k_0$, for some k_0.

The most challenging issue is the exact relation between k_0, ρ_0! It has to be settled either analytically or numerically.

► There is a claim that the assumption $k_0\rho_0 \sim 1$ may not be accurate; instead $k_0 \ll 1/\rho_0$? (Frisch p. 63-64). ◄