# RANDOM FUNCTIONS AND TURBULENCE

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# Probabilistic study of turbulence

## ▼ Motivation

Even though turbulence is governed by known physical laws, the Navier-Stokes equations, which are conjectured to be deterministic as $t \to +\infty$, it displays features that make it necessary to study it probabilistically:
1) Turbulent flow appears highly disorganized and presents structures on all scales.
2) Turbulent flow appears unpredictable in its detailed behaviour.
3) Some properties of turbulent flow are reproducible.

A probabilistic theory can allow us to focus on those features that are reproducable without demanding full knowledge of the evolution of turbulent flow.

## ▼ Configuration and state of a system

• The configuration of a system is the precise (but unknown) state of the system. In the case of turbulent flow this means the position and momentum of every particle. Although this viewpoint is idealized away by introducing the velocity / pressure field and the Navier-Stokes equations, it is relevant for the following reason: ultimately turbulent flow can be fully specified by a finite set of real numbers. (whereas the Navier-Stokes formulation would mislead you to assume that you need an uncountable set of real numbers, which is only an artifact of our "continuouization" of the fluid). This motivates the following assumption:

**Assumption**: The set of all configurations of fluid flow $\underline{\Omega}$ can be written as $\underline{\Omega} = \mathbb{R}^d$, where $d \in \mathbb{N}$ is a very large number.

## Measuring the configuration space

A consequence of this assumption is that $\underline{\Omega}$ can be measured with a Lebesgue measure as follows:

**Definition**: Let $A \in \mathcal{P}(\underline{\Omega})$ be a subset of $\underline{\Omega}$. Then:

a) $A$ is an <u>interval</u> iff there are vectors $a, b \in \underline{\Omega}$ with $a = \langle a_i \mid i \in d \rangle$ and $b = \langle b_i \mid i \in d \rangle$ such that

$$A = [a, b] = \{ \langle x_i \mid i \in d \rangle \mid (\forall i \in d)(a_i \leq x_i \leq b_i) \}.$$

b) $A$ is <u>simple</u> iff it is the countable union of intervals:

$$A = \bigcup_{i=1}^{n} [a_i, b_i] \quad , \quad \alpha_i, b_i \in \underline{\Omega}$$

□

In these two cases the Lebesgue measure is defined as follows:

**Definition**: Let $A \in \mathcal{P}(\underline{\Omega})$ be given.

a) If $A = [a, b]$ with $a = \langle a_i \mid i \in d \rangle$, $b = \langle b_i \mid i \in d \rangle$ then:

$$\boxed{a(A) \equiv \prod_{i \in d} (b_i - a_i)}$$

b) If $A = \bigcup_{i=1}^{n} I_i$ where $I_i$ are disjoint intervals, then:

$$\boxed{a(A) = \sum_{i=1}^{n} a(I_i)}$$

□

Note that there are many subsets of $\underline{\Omega}$ that can be approximated by a sequence of simple regions that refine themselves into shape as $n \to +\infty$. We call such sets <u>measurable</u>.

To effect this definition, we introduce the concept of characterisic functions:

<u>Definition</u>: Let $A \in \mathcal{P}(\Omega)$. The <u>characteristic function</u> of $A$ is $\chi_A : \Omega \to \{0,1\}$ with formula:

$$\chi_A(\vartheta) = \begin{cases} 1 & , \text{ if } \vartheta \in A \\ 0 & , \text{ if } \vartheta \notin A \end{cases} \quad \square$$

The convergence of sequences of regions of $\Omega$ can be defined in terms of the characteristic function

<u>Definition</u>: Let $\langle A_i \mid i \in \mathbb{N} \rangle$ with $A_i \in \mathcal{P}(\Omega)$ be a sequence of regions of $\Omega$. Then we say that the sequence <u>converges</u> to $A \in \mathcal{P}(\Omega)$ iff the charkdenstic functions $\chi_{A_i}$ uniformly converge to $\chi_A$.

$$\boxed{\lim_{i \in \mathbb{N}} A_i = A \longleftrightarrow (\forall \vartheta \in \Omega)\left( \lim_{i \in \mathbb{N}} \chi_{A_i}(\vartheta) = \chi_A(\vartheta)\right)} \quad \square$$

Now we can define measurability as follows:

<u>Definition</u>: Let $A \in \mathcal{P}(\Omega)$. We say that $A$ is <u>measurable</u> iff there is a unique number $\ell$ such that <u>for all</u> sequences $\langle A_i \mid i \in \mathbb{N} \rangle$ with $A_i \in \mathcal{P}(\Omega)$:

$$\left.\begin{array}{l} A_i \text{ simple}, \forall i \in \mathbb{N}. \\ \lim_{i \in \mathbb{N}} A_i = A \end{array}\right\} \Rightarrow \lim \partial(A_i) = \ell.$$

We define then $\partial(A) \equiv \ell$.

► <u>Notation</u>: We will denote the set of all measurable subsets of $\Omega$ as $\mathcal{B}$ (or $\mathcal{B}(\Omega)$ if not clear from the context). Note that $\mathcal{B}$ includes intervals and simple regions since those are trivially measurable. ◄

The function $a : \mathcal{B} \to [0, +\infty)$ is called the <u>Lebesgue measure</u> of $\Omega$. and it has the following properties:

a) $a(\emptyset) = 0$

b) $A \subset B \Rightarrow a(A) < a(B)$ , $\forall A, B \in \mathcal{B}$

c) Let $\langle A_i \mid i \in \mathbb{N} \rangle$ with $A_i \in \mathcal{B}$ be a sequence of regions. Then:

$$(\forall i, j \in \mathbb{N})(i \neq j \Rightarrow A_i \cap A_j = \emptyset) \Rightarrow a\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} a(A_i).$$

(i.e. $a$ is countably additive).

Any function $\mu : \mathcal{B} \to [0, +\infty)$ that satisfies these properties is called a <u>measure</u> on $\Omega$. However $a$ has one further property:

d) $a$ is <u>translation invariant</u>. That is: if for a given $c \in \Omega$:

$$\left.\begin{array}{l} A \in \mathcal{B} \\ B = \{x + c \mid x \in A\} = A + c \end{array}\right\} \Rightarrow a(A) = a(B).$$

This property makes ~~space~~ sense only when $\Omega$ is a vector space (as in our case $\Omega = \mathbb{R}^d$), and any measure that is translation invariant under the general setting, is called a Lebesgue measure.

## Thermodynamic/stochastic state

In practice the exact configuration of turbulent flow can not be known to us, so we should not use elements $\vartheta \in \Omega$ to represent our <u>knowledge of the state</u> of the flow. Instead we should use a <u>probability measure</u> $\mu : \mathcal{B} \to [0, 1]$ that assigns a <u>probability</u> for every region in $\mathcal{B}$. That probability is a statement of how likely it is for the configuration to be in any given measurable region. We give the following formal definition of probability measures:

<u>Definition</u>: A function $\mu : \mathcal{B} \to [0,1]$ is called a <u>probability measure</u> iff it satisfies the following properties:

a) $\mu(\emptyset) = 0$ and $\mu(\underline{\Omega}) = 1$

b) $(\forall A, B \in \mathcal{B})(A \subset B \Rightarrow \mu(A) < \mu(B))$.

c) If $\langle A_i \mid i \in \mathbb{N} \rangle$ with $A_i \in \mathcal{B}$ is a sequence of regions, then:

$$(\forall i,j \in \mathbb{N})(i \neq j \Rightarrow A_i \cap A_j = \emptyset) \Rightarrow \mu\Big(\bigcup_{i \in \mathbb{N}} A_i\Big) = \sum_{i \in \mathbb{N}} \mu(A_i).$$

Because turbulence is inherently governed by deterministic laws our knowledge of the present state implies knowledge about the future states of turbulence. We now turn to the formulation of a framework for this aspect of turbulence.

## <u>Evolution of the thermodynamic state</u>

The evolution of the turbulent flow configuration can be described in terms of a semi-group of configuration transformations $g : \underline{\Omega} \to \underline{\Omega}$.

<u>Definitions</u>:

a) A <u>configuration transformation</u> is a map $g : \underline{\Omega} \to \underline{\Omega}$ that maps configuration $\vartheta \in \underline{\Omega} \xrightarrow{g} g\vartheta \in \underline{\Omega}$.

b) The <u>inverse</u> $g^{-1} : \underline{\Omega} \to \mathcal{P}(\underline{\Omega})$ of the configuration transformation $g : \underline{\Omega} \to \underline{\Omega}$ maps $\vartheta_0 \in \underline{\Omega} \xrightarrow{g^{-1}} A = g^{-1}\vartheta_0 \in \mathcal{P}(\underline{\Omega})$ where $A$ is a region in $\underline{\Omega}$ that is defined as:
$$A = \{ \vartheta \in \underline{\Omega} \mid g\vartheta = \vartheta_0 \}$$
(i.e. $g^{-1}\vartheta_0$ is the set of points of origin that $g$ maps onto $\vartheta_0$)

► <u>Notation</u>: Because $g$ is thought of as an operator, we prefer to denote the image of $\vartheta \in \underline{\Omega}$ under $g$ as "$g\vartheta$" instead of $g(\vartheta)$. We also allow $g$ to operate on regions of $\underline{\Omega}$, according to the following definition:

$$gA = \{ g\vartheta \mid \vartheta \in A \}.$$

So $g$ induces a transformation $\tilde{g} : \mathcal{P}(\underline{\Theta}) \longrightarrow \mathcal{P}(\underline{\Theta})$ but we will abuse notation and use the same letter for both transformations. ◄

We are interested only in transformations that are <u>measurable</u> (which we will motivate shortly).

<u>Definition</u>: A coordinate transformation $g : \underline{\Theta} \to \underline{\Theta}$ is measurable iff. for every $A \in \mathcal{B}$, the points of origin $g^{-1}A \in \mathcal{B}$.

$$\boxed{g \text{ measurable} \Longleftrightarrow (\forall A \in \mathcal{B})(g^{-1}A \in \mathcal{B}).}$$ □

► <u>Notation</u>: For future use we will denote the set of all measurable transformations on $\underline{\Theta}$ as $\mathcal{T}(\underline{\Theta})$. Ultimately, the transformations in $\mathcal{T}(\underline{\Theta})$ should be thought of as $g : \mathcal{B} \to \mathcal{B}$ because $\{\vartheta\}$ is measurable for all $\vartheta \in \underline{\Theta}$, therefore $\{\vartheta\} \in \mathcal{B}$, so $\underline{\Theta}$ is "embedded" in $\mathcal{B}$ via the singleton regions $\{\vartheta\}$. Note however that there is a consistency restriction: every $\vartheta \in \underline{\Theta}$ must be mapped to a unique $\vartheta' \in \underline{\Theta}$ under any region $A$ that contains $\vartheta$. So there maybe transformations $\mathcal{B} \longrightarrow \mathcal{B}$ which do not belong in $\mathcal{T}(\underline{\Theta})$ because they do not map configurations consistently. ◄

<u>Definition</u>: Let $g_1, g_2 \in \mathcal{T}(\underline{\Theta})$ be two measurable transformations. Then we define their product $g_1 g_2$ by the following equivalence:

$$\boxed{g = g_1 g_2 \Longleftrightarrow (\forall \vartheta \in \underline{\Theta})(g\vartheta = g_1(g_2(\vartheta)))}$$

► <u>Notation</u>: We will use "id" to denote the identity transformation:

$$(id)\vartheta = \vartheta, \quad \forall \vartheta \in \underline{\Theta}$$

Clearly $g^{-1}g = id$. ◄

Given these preliminary ideas, the deterministic natural laws that govern turbulence can be expressed in terms of a family of transformations, called <u>flows</u>, which we define as follows:

<u>Definition</u> : A <u>flow</u> $g = \langle g(t) \mid t \in [0,+\infty) \rangle$ with $g(t) \in \mathcal{T}(\underline{\Omega})$ is a family of measurable transformations on $\underline{\Omega}$ that satisfy the following properties:

a) $g(0) = id$

b) $g(t_1+t_2) = g(t_1)g(t_2)$, $\forall t_1, t_2 \in [0,+\infty)$

Now we are ready to state our second assumption in this study:

<u>Assumption</u> : There is a flow $g$ that governs the evolution of the turbulent flow configuration $\omega(t)$ such that given an initial condition $\omega(0) = \omega_0 \in \underline{\Omega}$, $\omega(t)$ is given by

$$\boxed{\omega(t) = g_t \omega_0 \quad , \forall t > 0}$$

This assumption should not be taken lightly:

a) We assume that the underlying natural laws are deterministic whereas in fact they are quantum mechanical.

b) Even given the viewpoint of the Navier-Stokes equations, the existence and uniqueness of a solution as $t \to +\infty$ for any initial condition is, so far as we know, a conjecture.

► <u>Remark</u> : Let $\mu_0 : \mathcal{B} \longrightarrow [0,1]$ be a probability measure that corresponds to our present knowledge $(t=0)$ about the macroscopic state of the system. What do we know about $t>0$? Let $\mu : [0,+\infty) \times \mathcal{B} \longrightarrow [0,1]$ with $\mu_t(A) \equiv \mu(t,A)$ the probability measure that corresponds to the future state of the system at a given time $t$. Then $\mu$ can be written in terms of $\mu_0$ ◄

<u>Proposition :</u> $\boxed{\mu(t, A) = \mu_0(g_t^{-1} A) \;, \forall t > 0, \forall A \in \mathcal{B}}$

<u>Proof</u>

Recall that $g_t^{-1} A = \{ \vartheta \in \Omega \mid g_t \vartheta \in A \}$ therefore: $g_t \vartheta \in A \Leftrightarrow \vartheta \in g_t^{-1} A$. It follows that

$$\mu(t, A) = \text{Prob}(\vartheta(t) \in A) = \text{Prob}(g_t \vartheta(0) \in A) =$$

$$= \text{Prob}(\vartheta(0) \in g_t^{-1} A) = \mu_0(g_t^{-1} A) \qquad \square$$

This result motivates our requirement that $A \in \mathcal{B} \to g_t^{-1} A \in \mathcal{B}$. Essentially we say that if $\mu$ corresponds to $t = 0$ then future states must be projected back to $t = 0$ (hence $g_t^{-1}$) where we can evaluate probabilities with $\mu_0$.

An assumption that we may make about $\mu$ is that it is invariant wrt the flow $g_t$:

<u>Definition</u> : We say that $\mu_0$ is <u>invariant</u> wrt the flow $g_t$ iff

$$\boxed{(\forall t > 0)(\mu_0(g_t^{-1} A) = \mu_0(A))} \qquad \square$$

This assumption trivializes $\mu(t, A)$ which becomes constant wrt $t$ with value $\mu_0(A)$. We will see further consequences of this assumption later.

<u>Assumption</u> : Turbulent flow has a unique invariant measure

Note that strictly speaking this assumption is a conjecture. Moreover, it is a poor conjecture because turbulence is dissipative. Many dissipative systems have more than one invariant measure, each of which is associated with a basin ~~of~~ and an attractor. Then the appropriate measure is the one that corresponds to the basin where the initial condition belongs.

## ▼ Integration in $\underline{\Omega}$

Given a measure $\mu$ on $(\underline{\Omega}, \mathcal{B})$ (which may or may not be a probability measure) we can define integration over regions of $\underline{\Omega}$ in $\mathcal{B}$ as follows:

### Convergent nets

**Definition**: A directed set is a pair $(D, <)$ where $<$ is a relation on $D$ such that:

a) $a < a \ , \ \forall a \in D$

b) $a < b \wedge b < c \Rightarrow a < c \ , \ \forall a, b, c \in D$

c) $(\forall a, b \in D)(\exists c \in D)(a < c \wedge b < c)$ □

**Definition**: A net is a function $\varphi : D \to \mathbb{R}$, denoted as $\varphi = \langle x_a \mid a \in D \rangle$ where $D$ is a directed set. □

**Definition**: (A net $x_a$ converges to $\ell$)

$$\boxed{\lim_{a \in D} x_a = \ell \Leftrightarrow (\forall \varepsilon > 0)(\exists a_0 \in D)(\forall a \in D)(a > a_0 \Rightarrow |x_a - \ell| < \varepsilon)}$$ □

Limits of nets are a generalization upon which we may get limits of sequences, functions, and integrals under ~~th~~ a unified framework. We use nets to define the Lebesgue integral over $\underline{\Omega}$

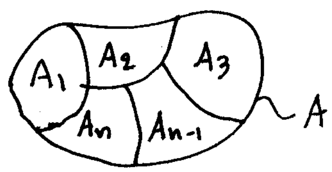
## Partitions of $A \in \mathcal{B}$

<u>Definition</u>: Let $A \in \mathcal{B}$ be a measurable region of $\underline{\Omega}$. A <u>partition</u> $P$ of $A$ is a sequence of regions $P = \langle A_i \mid i \in [n] \rangle$ such that the following is true:

a) $(\forall i,j \in [n])(i \neq j \Rightarrow A_i \cap A_j = \emptyset)$

b) $\bigcup_{i \in [n]} A_i = A$. $\square$

A schematic example:



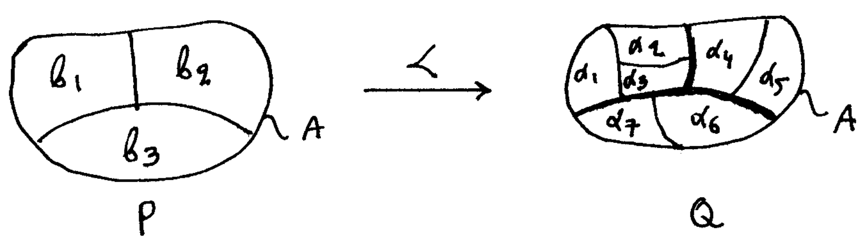

▸ <u>notation</u>: We will denote the set of all partitions of $A$ as $\underline{\pi(A)}$. ◂

Now we define $<$ such that $\pi(A)$ becomes a directed set.

<u>Definition</u>: Let $P, Q \in \pi(A)$ be two partitions of $A \in \mathcal{B}$. We say that $Q$ is a <u>refinement</u> of $P$ (and we write $P < Q$) iff

$$(\forall a \in Q)(\exists b \in P)(a \subseteq b) \qquad \square$$

A schematic example:




It is easy to show that $(\pi(A), <)$ is a directed set. so we can take limits over the space of all partitions.

## The Lebesgue integral

Definition: Let $f : \Omega \to \mathbb{R}$ be a function, $\mu$ a measure on $(\Omega, \mathcal{B})$ and $P \in \pi(A)$ for some $A \in \mathcal{B}$ be a partition.
Then we define the left and right Riemann sums as follows:

$$\boxed{L(f, P, \mu) \equiv \sum_{a \in P} \mu(a) \inf \{ f(\varpi) \mid \varpi \in a \}.}$$

$$\boxed{U(f, P, \mu) \equiv \sum_{a \in P} \mu(a) \sup \{ f(\varpi) \mid \varpi \in a \}}$$

Definition: A function $f : \Omega \to \mathbb{R}$ is integrable on $A \in \mathcal{B}$ wrt a measure $\mu$ in $(\Omega, \mathcal{B})$ iff

$$\boxed{(\exists \ell \in \mathbb{R}) \left( \lim_{P \in \pi(A)} L(f, P, \mu) = \lim_{P \in \pi(A)} U(f, P, \mu) = \ell \right)}$$

We denote the unique number $\ell$ as:

$$\boxed{\ell = \int_A f(\varpi)\, d\mu(\varpi).}$$

These integrals can be defined over the Lebesgue measure $\lambda(\varpi)$ of $\Omega$, OR over any probability measure $\mu(\varpi)$.

### Random variables and ensemble averaging.

A random variable corresponds to a measurement whose outcome is unpredictable because it is sensitive to the configuration $\varpi \in \Omega$ of the system which is unknown to us. We define random variables, then, as functions $\Omega \to \mathbb{R}$:

**Definition**: A random variable is a function $\tilde{\varphi} : \Omega \longrightarrow \mathbb{R}$ that maps every configuration $\vartheta \in \Omega$ into a measurement $x \in \mathbb{R}$ such that:

a) $\tilde{\varphi}^n$ is integrable over $(\Omega, \mathcal{B}, \mu)$, $\forall n \in \mathbb{N}$

b) $(\forall A \in \mathcal{B}(\mathbb{R}))\ (\{\vartheta \in \Omega \mid \tilde{\varphi}(\vartheta) \in A\} \in \mathcal{B})$

where $\mathcal{B}(\mathbb{R})$ is the set of all Lebesgue-measurable subsets of $\mathbb{R}$.

**Definition**: The ensemble average $\overline{\tilde{\varphi}}$ of a random variable wrt the state of the system $\mu : \mathcal{B} \rightarrow [0,1]$ is defined as:

$$\boxed{\overline{\tilde{\varphi}} = \int_\Omega \tilde{\varphi}(\vartheta)\, d\mu(\vartheta)}$$

**Definition**: The fluctuation $\varphi : \Omega \longrightarrow \mathbb{R}$ of a random variable $\tilde{\varphi}$ is defined as:

$$\boxed{\varphi(\vartheta) = \tilde{\varphi}(\vartheta) - \overline{\tilde{\varphi}} \quad , \forall \vartheta \in \Omega}$$

▶ notation: We will adopt the convention of placing a tilde ~ on every random variable. Removing the tilde will mean that we refer to the corresponding fluctuation. ◀

**Definition**: The $n^{th}$-order moment of a random variable $\tilde{\varphi}$ is given, in terms of the corresponding fluctuation, by:

$$\boxed{\overline{\varphi^n} = \int_\Omega \varphi^n(\vartheta)\, d\mu(\vartheta)}$$

i.e. it is equal to the ensemble average of $\varphi^n$.

↳ The requirements that we state in our definition of random variable are used to (a) guarantee the existence of moments AND (b) guarantee the existence of "reasonable" probability density functions.

## ▼ Probability density functions

In this section we will explain how random variables can be characterized by probability density functions.

Note that so far we have set $\Omega = \mathbb{R}^d$ and defined

1) The Lebesgue measure $\lambda : \mathcal{B} \longrightarrow \mathbb{R}$
2) Probability measures $\mu : \mathcal{B} \longrightarrow [0,1]$
   (on the same $\mathcal{B}$ as $\lambda$).

To avoid reproving the same stuff we need to abstract these definitions. We do that as follows:

Definition: Let $\Omega$ be an arbitrary set. A set $\mathcal{M} \subseteq \mathcal{P}(\Omega)$ is a $\sigma$-algebra on $\Omega$ iff:

a) $\emptyset \in \mathcal{M}$ and $\Omega \in \mathcal{M}$
b) $(\forall A_1, A_2 \in \mathcal{M})(A_1 - A_2 \in \mathcal{M})$
c) $(\forall \langle A_i \mid i \in \mathbb{N} \rangle : A_i \in \mathcal{M})\left(\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{M} \wedge \bigcap_{i \in \mathbb{N}} A_i \in \mathcal{M}\right)$

► It can be shown that $\mathcal{B}$, the measurable sets of $\Omega = \mathbb{R}^d$, is a $\sigma$-algebra on $\Omega$. In fact this definition motivates a constructive process for building $\mathcal{B}$:

Let $\sigma(A)$ be the sigma-extension of $A$:

$$\sigma(A) = \{A_1 - A_2 \mid A_1, A_2 \in A\} \cup \{\bigcup_{i \in \mathbb{N}} A_i, \bigcap_{i \in \mathbb{N}} A_i \mid (\forall i \in \mathbb{N})(A_i \in A)\}$$

Let $\mathcal{B}_0$ = set of all intervals on $\Omega$, and define:

$$\mathcal{B}_{a+1} = \sigma(\mathcal{B}_a)$$
$$\mathcal{B}_\lambda = \bigcup_{a<\lambda} \mathcal{B}_a \quad , \text{if } \lambda \text{ a limit ordinal}$$

Then it can be shown that $\mathcal{B}$ will be reached in $\omega_1$ steps:
$\mathcal{B} = \mathcal{B}_{\omega_1}$

This is also a cute characterization of $\aleph_1$. ◄

<u>Definition</u>: A <u>measure</u> is a function $\mu : \mathfrak{M} \to [0, +\infty)$ such that

a) $\mathfrak{M}$ is a $\sigma$-algebra on a domain $\Omega$

b) $\mu(\varnothing) = 0$

c) $(\forall A, B \in \mathfrak{M})(A \subset B \Rightarrow \mu(A) < \mu(B))$

d) If $\langle A_i \mid i \in \mathbb{N} \rangle$ is a sequence of sets $A_i \in \mathfrak{M}$ such that

$$(\forall i, j \in \mathbb{N})(i \neq j \Rightarrow A_i \cap A_j = \varnothing)$$

then

$$\mu\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} \mu(A_i)$$

<u>Definition</u>: A <u>probability measure</u> is a function $\mu : \mathfrak{M} \to [0, 1]$ such that

a) $\mu$ is a measure

b) $\mu(\Omega) = 1$

Now we can start our development

## The Radon-Nikodym theorem

We state the following definitions in a general setting but they are most easily interpreted in the context where $\mu$ is a probability measure and $\lambda$ a Lebesgue measure that captures a notion of volume.

<u>Definition</u>: Let $\mu, \lambda$ be two measures on $\mathfrak{M}$ over $\Omega$.

a) $\mu$ is <u>absolutely continuous</u> wrt $\lambda$ iff $\lambda(A) = 0 \Rightarrow \mu(A) = 0$.
i.e.

$$\boxed{\mu \ll \lambda \Leftrightarrow (\forall A \in \mathfrak{M})(\lambda(A) = 0 \Rightarrow \mu(A) = 0)}$$

b) $\mu$ and $\lambda$ are <u>mutually singular</u> iff there is an $A \in \mathfrak{M}$ such that $\mu(A) = 0$ and $\lambda(\Omega - A) = 0$.

$$\boxed{\mu \perp \lambda \Leftrightarrow (\exists A \in \mathfrak{M})(\mu(A) = 0 \land \lambda(\Omega - A) = 0)}$$

▶ interpretation :

$\mu \ll \lambda$ means that the "mass" of $\mu$ is ~~evenly~~ smoothly spread over $\Omega$ such that all regions of volume $0$ have probability $0$.
$\mu \perp \lambda$ means that the "mass" of $\mu$ is concetrated on a region with volume $0$. ◀

Theorem : (Radon – Nikodym)
If $\mu \ll \lambda$, then there is a function $\Delta_\mu : \Omega \to [0,+\infty)$ such that

$$\boxed{\mu(A) = \int_A \Delta_\mu(\omega)\, d\lambda(\omega) \quad , \forall A \in \mathcal{M}}$$

▶ The function $\Delta_\mu(\omega)$ is called the Radon – Nikodym derivative of $\mu$ wrt $\lambda$ and it is denoted as:

$$\Delta_\mu = d\mu / d\lambda$$

It can be interpreted as a density distribution function for $\mu$. ◀

Theorem : (Lebesgue decomposition)
Let $\mu, \lambda$ be any two measures on $\Omega$. Then $\mu$ can be written as a sum $\mu = \mu_0 + \mu_1$ with $\mu_0 \ll \lambda$ and $\mu_1 \perp \lambda$

Proof

Define $\Lambda = \mu + \lambda \Rightarrow \mu \ll \Lambda$ and $\lambda \ll \Lambda$
Let $f = d\lambda / d\Lambda : \Omega \to [0, +\infty)$
Define $A = \{\omega \in \Omega \mid f(\omega) > 0\}$ and $B = \{\omega \in \Omega \mid f(\omega) = 0\}$.
Then $A \cap B = \varnothing$ and $A \cup B = \Omega$.
Construct $\mu_0, \mu_1$ as : $\mu_0(E) = \mu(E \cap A)$ and $\mu_1(E) = \mu(E \cap B)$, $\forall E \in \mathcal{M}$.
Now we will show that $\mu_0 \ll \lambda$ and $\mu_1 \perp \lambda$.
a) Assume $\lambda(E) = 0 \Rightarrow \int_E f(\omega)\, d\Lambda(\omega) = 0 \Rightarrow$

$$\Rightarrow \Lambda(\{\omega \in E \mid f(\omega) \neq 0\}) = 0 \quad (1).$$

It follows that

$$\mu_0(E) = \mu(E\cap A) \leqslant \lambda(E\cap A) = \lambda(E\cap\{\vartheta\in\Omega \mid f(\vartheta)>0\}) =$$
$$= \lambda(\{\vartheta\in E \mid f(\vartheta)>0\}) = \lambda(\{\vartheta\in E \mid f(\vartheta)\neq 0\}) = 0 \Rightarrow \mu_0(E)=0.$$

therefore $(\forall E\in\mathfrak{m})(\lambda(E)=0 \Rightarrow \mu_0(E)=0) \Rightarrow \mu_0 \ll \lambda$.

b) $\left.\begin{array}{l} \mu_1(A) = \mu(A\cap B) = \mu(\emptyset) = 0 \\ \lambda(\Omega - A) = \lambda(B) = \int_B f(\vartheta)\,d\lambda(\vartheta) = \int_B 0\,d\lambda(\vartheta) = 0 \end{array}\right\} \Rightarrow \mu_1 \perp \lambda.$

□

- <u>Probability density function of a random variable</u>

<u>Definition</u>: Let $\tilde{\varphi}$ be a random variable. The <u>cumulative distribution</u> of $\tilde{\varphi}$ is a function $Q : \mathbb{R} \to [0,+\infty)$ such that

$$\boxed{Q(x) = \mu(\{\vartheta\in\Omega \mid \varphi(\vartheta) < x\})}$$

<u>Definition</u>: Assume $Q(x)$ is differentiable.
The <u>probability density function</u> of $\tilde{\varphi}$ is given by:

$$\boxed{p(x) = dQ(x)/dx}$$

Now we will state a property that $\tilde{\varphi}$ needs to have in order for the pdf to exist. It is intuitevely understood that the probability that $\tilde{\varphi}(\vartheta)$ is <u>exactly</u> equal to $x$ is zero. This motivates the following:

<u>Definition</u>: A random variable $\tilde{\varphi}$ is <u>smoothly distributed</u> iff
$$\mu(\{\vartheta\in\Omega \mid \tilde{\varphi}(\vartheta) = x\}) = 0 \ , \ \forall x\in\mathbb{R}.$$

<u>Theorem</u>: If $\tilde{\varphi}$ is smoothly distributed $\Rightarrow$ $Q(x)$ is differentiable

<u>Proof</u>

Let $\mathcal{B}(\mathbb{R})$ be the $\sigma$-algebra of all measurable ~~sets~~ subsets of $\mathbb{R}$ as defined (more generally) in p. 13

Let $\ell : \mathcal{B}(\mathbb{R}) \longrightarrow [0, +\infty)$ be the Lebesgue measure on $\mathbb{R}$.

Define $q : \mathcal{B}(\mathbb{R}) \longrightarrow [0, +\infty)$ as:

$$q(A) = \mu(\{\vartheta \in \Omega \mid \tilde{\varphi}(\vartheta) \in A\})$$

It is trivial to show that $q$ is a measure also.

▸ Claim : $q \ll \ell$

To show the claim, assume $\ell(A) = 0 \Rightarrow A$ countable. (state with no proof).

Let $a = \langle a_k \mid k \in \mathbb{N} \rangle$ such that $A = \bigcup_{k \in \mathbb{N}} \{a_k\}$

Then

$$q(A) = q\left(\bigcup_{k \in \mathbb{N}} \{a_k\}\right) = \sum_{k \in \mathbb{N}} q(\{a_k\}) =$$

$$= \sum_{k \in \mathbb{N}} \mu(\{\vartheta \in \Omega \mid \tilde{\varphi}(\vartheta) \in \{a_k\}\}) =$$

$$= \sum_{k \in \mathbb{N}} \mu(\{\vartheta \in \Omega \mid \tilde{\varphi}(\vartheta) = a_k\}) = \sum_{k \in \mathbb{N}} 0 = 0$$, bc. $\tilde{\varphi}$ smoothly distributed.

Since $(\forall A \in \mathcal{B}(\mathbb{R}))(\ell(A) = 0 \Rightarrow q(A) = 0) \Rightarrow q \ll \ell \Rightarrow$

$\Rightarrow$ the Radon - Nikodym theorem applies $\Rightarrow$

$\Rightarrow$ we can define $f = dq/d\ell : \mathbb{R} \longrightarrow [0, +\infty)$

Then we have:

$$Q(x) = \mu(\{\vartheta \in \Omega \mid \tilde{\varphi}(\vartheta) < x\}) = \mu(\{\vartheta \in \Omega \mid \tilde{\varphi}(\vartheta) \in (-\infty, x)\}) =$$

$$= q((-\infty, x)) = \int_{(-\infty, x)} f(y)\, d\ell(y) = \int_{-\infty}^{x} f(y)\, dy \Rightarrow$$

$\Rightarrow Q(x)$ differentiable with $dQ/dx = f$. $\square$

We will also show that the converse holds. However, first let us consider what happens if we do not assume that $\tilde{\varphi}$ is smoothly distributed.

<u>Theorem</u>: Let $\tilde{\varphi}$ be a random variable with cumulative distribution $Q(x)$. Then there is a function $p_0 : \mathbb{R} \to [0, +\infty)$ and two countable sequences $a = \langle a_k \mid k \in \mathbb{N} \rangle$ and $b = \langle b_k \mid k \in \mathbb{N} \rangle$ such that:

$$\boxed{Q(x_0) = \int_{-\infty}^{x_0} \left[ p_0(x) + \sum_{k \in \mathbb{N}} b_k \delta(x - a_k) \right] dx}$$

<u>Proof</u>

Let $\mathcal{B}(\mathbb{R})$ and $\ell : \mathcal{B}(\mathbb{R}) \to [0, +\infty)$ be defined as in the previous proof and let $q : \mathcal{B}(\mathbb{R}) \to [0, +\infty)$ be given by:

$$q(A) = \mu(\{\omega \in \Omega \mid \tilde{\varphi}(\omega) \in A\})$$

Let $q = q_0 + q_1$ be the Lebesgue decomposition of $q$ with $q_0 \ll \ell$ and $q_1 \perp \ell$.

▶ Construct $p_0$ as $\underline{p_0 = dq_0 / d\ell}$. (1)

To construct the sequences $a, b$ note that

$$q_1 \perp \ell \Rightarrow (\exists C \in \mathcal{B}(\mathbb{R}))(\underline{q_1(C) = 0} \wedge \underline{\ell(\Omega - C) = 0}) \quad (2)$$

Then,

$$\ell(\Omega - C) = 0 \Rightarrow \Omega - C \text{ countable} \Rightarrow$$
$$\Rightarrow (\exists a = \langle a_k \mid k \in \mathbb{N} \rangle)(\Omega - C = \bigcup_{k \in \mathbb{N}} \{a_k\})$$

▶ Construct $b_k$ as $\underline{b_k = q_1(\{a_k\})}$. (3)

For our future convenience define $K(x_0) = \{k \in \mathbb{N} \mid a_k < x_0\}$ as the set of all indices $k$ that yield points $a_k$ that lie before $x_0$.

Then note that:

$$C \cap (-\infty, x_0) \subseteq C \Rightarrow q_1(C \cap (-\infty, x_0)) \leq q_1(C) = 0 \Rightarrow$$
$$\Rightarrow q_1(C \cap (-\infty, x_0)) = 0 \quad (4).$$

It follows that

$$q_1((-\infty, x)) = q_1\left[(-\infty, x) \cap \left(C \cup \bigcup_{k \in \mathbb{N}} \{a_k\}\right)\right] =$$
$$= q_1[(-\infty, x) \cap C] + q_1\left[(-\infty, x) \cap \bigcup_{k \in \mathbb{N}} \{a_k\}\right] =$$

$$= 0 + q_1\Big(\bigcup_{k\in K(x_0)} \{a_k\}\Big) = \sum_{k\in K(x_0)} q_1(\{a_k\}) =$$

$$= \sum_{k\in K(x_0)} b_k = \int_{-\infty}^{x} \sum_{k\in\mathbb{N}} b_k \delta(x-a_k)\,dx$$

We can now show the expression for $Q(x_0)$:

$$Q(x_0) = \mu(\{\vartheta\in\Omega \mid \tilde{\varphi}(\vartheta)<x_0\}) = q((-\infty,x_0)) = q_0((-\infty,x_0)) + q_1((-\infty,x_0)) =$$

$$= \int_{(-\infty,x_0)} p_0(x)\,d\ell(x) + \int_{-\infty}^{x} \sum_{k\in\mathbb{N}} \big[b_k\delta(x-a_k)\big]\,dx =$$

$$= \int_{-\infty}^{x_0} \Big[p_0(x) + \sum_{k\in\mathbb{N}} b_k\delta(x-a_k)\Big]\,dx \qquad \square$$

## Properties of probability density functions

**Proposition**: Let $p(x)$ be the pdf of a random variable $\tilde{\varphi}$.
Then:

$$\boxed{\int_{-\infty}^{+\infty} p(x)\,dx = 1}$$

**Proof**

$$\int_{-\infty}^{+\infty} p(x)\,dx = \lim_{x\to+\infty} Q(x) = \lim_{x\to+\infty} \mu(\{\vartheta\in\Omega \mid \tilde{\varphi}(\vartheta)<x\}) =$$

$$= \mu(\{\vartheta\in\Omega \mid \tilde{\varphi}(\vartheta)\in\mathbb{R}\}) = \mu(\Omega) = 1 \qquad \square$$

The most important result is the relation between $p(x)$ and the moments of the random variable.

**Theorem**: Let $\tilde{\varphi}$ be a random variable with $x_0 = \overline{\tilde{\varphi}}$ ensemble average. Then, if $p(x)$ is the pdf of $\tilde{\varphi}$:

$$\boxed{x_0 = \int_{-\infty}^{+\infty} x\,p(x)\,dx} \quad \text{and} \quad \boxed{\overline{\varphi^n} = \int_{-\infty}^{+\infty} (x-x_0)^n\,p(x)\,dx}$$

Proof

Split $\underline{\Omega}$ into slices given by

$$A(x,\varepsilon) = \{\vartheta \in \underline{\Omega} \mid x \leqslant \varphi(\vartheta) < x+\varepsilon\}$$

Then, by definition, we have that

$$\varphi(\vartheta) = x + E(\vartheta, x, \varepsilon) \quad , \forall \vartheta \in A(x,\varepsilon)$$

where $\lim_{\varepsilon \to 0} E(\vartheta, x, \varepsilon) = 0$ because $|E(\vartheta, x, \varepsilon)| \leqslant \varepsilon$ , $\forall \vartheta \in A(x,\varepsilon)$.
Let $p(x)$ be the pdf of $\tilde{\varphi}$, as stated in the hypothesis.
Then,

$$\begin{aligned}
\mu(A(x,\varepsilon)) &= \mu(\{\vartheta \in \underline{\Omega} \mid x \leqslant \varphi(\vartheta) < x+\varepsilon\}) = \\
&= \mu(\{\vartheta \in \underline{\Omega} \mid x + x_0 \leqslant \varphi(\vartheta) + x_0 < x + x_0 + \varepsilon\}) = \\
&= Q(x + x_0 + \varepsilon) - Q(x + x_0) = \\
&= \int_{x+x_0}^{x+x_0+\varepsilon} p(x)\,dx = \int_{x}^{x+\varepsilon} p(x+x_0)\,dx \qquad (1).
\end{aligned}$$

Then, the $n^{th}$-order moment is given by:

$$\langle \varphi^n \rangle = \int_{\underline{\Omega}} \varphi^n(\vartheta)\,d\mu(\vartheta) = \sum_{k\in\mathbb{Z}} \int_{A(k\varepsilon,\varepsilon)} \varphi^n(\vartheta)\,d\mu(\vartheta) =$$

$$= \lim_{\varepsilon\to 0} \sum_{k\in\mathbb{Z}} \int_{A(k\varepsilon,\varepsilon)} \varphi^n(\vartheta)\,d\mu(\vartheta) = \lim_{\varepsilon\to 0} \sum_{k\in\mathbb{Z}} \int_{A(k\varepsilon,\varepsilon)} [k\varepsilon + E(\vartheta, k\varepsilon, \varepsilon)]^n\,d\mu(\vartheta)$$

$$\begin{aligned}
&= \lim_{\varepsilon\to 0} \sum_{k\in\mathbb{Z}} \int_{A(k\varepsilon,\varepsilon)} (k\varepsilon)^n\,d\mu(\vartheta) = \lim_{\varepsilon\to 0} \sum_{k\in\mathbb{Z}} (k\varepsilon)^n \mu(A(k\varepsilon,\varepsilon)) = \\
&= \lim_{\varepsilon\to 0} \sum_{k\in\mathbb{Z}} (k\varepsilon)^n \int_{k\varepsilon}^{k\varepsilon+\varepsilon} p(x+x_0)\,dx = \\
&= \int_{-\infty}^{+\infty} x^n p(x+x_0)\,dx = \int_{-\infty}^{+\infty} (x-x_0)^n p(x)\,dx
\end{aligned}$$

To obtain the ensemble average, consider the case $n=1$.

$$\overline{\varphi} = \int_{-\infty}^{+\infty} (x-x_0)\, p(x)\, dx = \int_{-\infty}^{+\infty} x p(x) dx - x_0 \int_{-\infty}^{+\infty} p(x) dx =$$

$$= \int_{-\infty}^{+\infty} x p(x)\, dx - x_0$$

However $\overline{\varphi} = 0$, because $\varphi$ is the fluctuation $\Rightarrow$

$$\Rightarrow \quad x_0 = \int_{-\infty}^{+\infty} x\, p(x)\, dx \qquad \square$$

→ The above result gives a useful characterization of the following properties of random variables:

<u>Definition</u>: Let $\tilde{\varphi}$ be a random variable with fluctuation $\varphi$. Then:

a) The <u>variance</u> of $\tilde{\varphi}$ is: $\boxed{\mathrm{var}(\tilde{\varphi}) = \langle \varphi^2 \rangle}$
and it measures the amount with which $\tilde{\varphi}$ fluctuates away from its average.

b) The <u>skewness</u> of $\tilde{\varphi}$ is: $\boxed{sk(\tilde{\varphi}) = \dfrac{\langle \varphi^3 \rangle}{\langle \varphi^2 \rangle^{3/2}}}$
and it measures whether the variable fluctuates to the right of the average ($sk(\tilde{\varphi}) > 0$) or to the left of the average ($sk(\tilde{\varphi}) < 0$).

c) The <u>Kurtosis</u> of $\tilde{\varphi}$ is $\boxed{ku(\tilde{\varphi}) = \dfrac{\langle \varphi^4 \rangle}{\langle \varphi^2 \rangle^2}}$
and it measures whether the flow is intermittent ($ku(\varphi) \ll 1$) or not ($ku(\varphi) \gg 1$)

## ▼ Random functions and their characteristics

Definition: A random function is a function $\tilde{f} : V \times \Omega \to \mathbb{R}$ such that

a) $V$ is an appropriate space of elements.

b) $(\forall x_0 \in V)(\tilde{\varphi} = \langle f(x_0, \vartheta) \mid \vartheta \in \Omega \rangle$ is a random variable)

► The usual choices for $V$ are:

1) $V = [0, +\infty) \longrightarrow$ represents time ($V = \mathbb{R}$ may also be appropriate)
2) $V = \mathbb{R}^2 \longrightarrow$ represents 2d space
3) $V = \mathbb{R}^3 \longrightarrow$ represents 3d space.

or a combination of the above.

The approach of this study is to treat all the relevant fluid flow fields (velocity, pressure, vorticity, etc.) as random functions and postulate certain properties about them from which we can deduce other statistical properties of turbulence. ◄

► notation: Because, when $x_0 \in V$ is fixed, $\tilde{\varphi}(\vartheta) = \tilde{f}(x_0, \vartheta), \forall \vartheta \in \Omega$ is a random variable, all of our definitions and results about random variables are inherited by random functions. To facilitate this, we will denote $\tilde{\varphi} \equiv \tilde{f}(x)$. So by dropping $\vartheta$ from the argument list of $\tilde{f}$, now $\tilde{f}(x) : \Omega \to \mathbb{R}$, i.e. it represents the random variable that ~~is~~ gives the value of $\tilde{f}$ at a given $x$.

This allows us to define ensemble average and fluctuations as follows:

$$\overline{\tilde{f}(x)} \equiv \int_{\Omega} \tilde{f}(x, \vartheta)\, d\mu(\vartheta). \quad , \forall x \in V \longrightarrow \text{ensemble average}$$

$$f(x, \vartheta) \equiv \tilde{f}(x, \vartheta) - \overline{\tilde{f}(x)} \qquad \longrightarrow \text{fluctuation.}$$

Since $f(x) : \Omega \to \mathbb{R}$ will be the random variable that evaluates the fluctuation at a given $x$, we can also define moments:

$$\overline{f^n(x)} = \int_{\underline{\Omega}} f^n(x,\vartheta)\, d\mu(\vartheta).$$

Similarly we can define probability density functions. We will adopt the convention that the last argument, $y$, is the value of the function and that all the previews arguments correspond to $x \in V$. So if $p(x,y) = \mathrm{pdf}(\tilde{f})$ then:

$$y_0 = \overline{\tilde{f}(x)} = \int_{-\infty}^{+\infty} y\, p(x,y)\, dy \quad , \quad \overline{f^n(x)} = \int_{-\infty}^{+\infty} (y-y_0)^n\, p(x,y)\, dy.$$

## Correlation function and structure functions

The following notions generalize the idea of moments and take into account the <u>function</u> aspect of random functions.

<u>Definition</u>: Let $\tilde{f} : V \times \underline{\Omega} \longrightarrow \mathbb{R}$ be a random function with fluctuation $f$. Then the <u>correlation function</u> $\Gamma : V \times V \longrightarrow \mathbb{R}$ is given by:

$$\boxed{\Gamma(x_1, x_2) = \overline{f(x_1) f(x_2)} \quad , \quad \forall\, x_1, x_2 \in V}$$

- A trivial consequence of the definition is that

$$\Gamma(x_1, x_2) = \Gamma(x_2, x_1) \quad , \quad \forall\, x_1, x_2 \in V$$

The following is a less obvious property:

<u>Proposition</u>: Let $\Gamma$ be the correlation function of a random function $\tilde{f}$, with fluctuation $f$. Then:

$$\boxed{|\Gamma(x_1, x_2)| \leqslant \frac{1}{2}\left[\, \overline{f^2(x_1)} + \overline{f^2(x_2)}\, \right]}$$

<u>Proof</u>

Evaluate the following integral:

$$
\begin{aligned}
I_{\pm} &= \int_{\Omega} [f(x_1,\vartheta) \pm f(x_2,\vartheta)]^2 \, d\mu(\vartheta) = \\
&= \int_{\Omega} [f^2(x_1,\vartheta) \pm 2 f(x_1,\vartheta) f(x_2,\vartheta) + f^2(x_2,\vartheta)] \, d\mu(\vartheta) = \\
&= \int_{\Omega} f^2(x_1,\vartheta) \, d\mu(\vartheta) + \int_{\Omega} f^2(x_2,\vartheta) \, d\mu(\vartheta) \pm 2 \int_{\Omega} f^2(x_2,\vartheta) \, d\mu(\vartheta) = \\
&= \overline{f^2(x_1)} + \overline{f^2(x_2)} \pm 2\Gamma(x_1,x_2) \qquad (1)
\end{aligned}
$$

However, note that $[f(x_1,\vartheta) \pm f(x_2,\vartheta)]^2 \geq 0$, $\forall \vartheta \in \Omega \Rightarrow$

$$
\Rightarrow I = 0 \Rightarrow \begin{cases} \overline{f^2(x_1)} + \overline{f^2(x_2)} - 2\Gamma(x_1,x_2) \geq 0 \\ \overline{f^2(x_1)} + \overline{f^2(x_2)} + 2\Gamma(x_1,x_2) \geq 0 \end{cases}
$$

$$
\Rightarrow |\Gamma(x_1,x_2)| \leq \frac{1}{2}\left[\overline{f^2(x_1)} + \overline{f^2(x_2)}\right] \qquad \square
$$

Correlations can be generalized to $n^{th}$-order as follows:

<u>Definition</u>: Let $\tilde{f}: V \times \Omega \longrightarrow \mathbb{R}$ be a random ~~variable~~ function with fluctuation $f$. Then the <u>$n^{th}$-order correlation function</u> $\Gamma: V^n \longrightarrow \mathbb{R}$ is given by:

$$
\boxed{\Gamma(x_1,x_2,\ldots,x_n) = \overline{f(x_1) f(x_2) \cdots f(x_n)}}
$$

▶ <u>notation</u>: In the subsequent discussion we will denote ensemble averaging with brackets $\langle \cdot \rangle$ and reserve the bar $\overline{\cdot}$ for averaging over $x$. ◀

The following theorem offers an interpretation of the correlation function:

<u>Theorem</u> : Let $\tilde{f}$ be a random function and $\Gamma$ its correlation function. Then:

$$\boxed{\langle \tilde{f}(x_1)\tilde{f}(x_2)\rangle = \langle \tilde{f}(x_1)\rangle\langle \tilde{f}(x_2)\rangle + \Gamma(x_1,x_2) \quad , \forall x_1,x_2 \in V}$$

<u>Proof</u>

Let $f$ be the fluctuation of $\tilde{f}$, and define the following random variables:

$$\tilde{\varphi}_1 = \tilde{f}(x_1) \quad , \quad \tilde{\varphi}_2 = \tilde{f}(x_2) \, , \quad \varphi_1 = f(x_1) \quad , \quad \varphi_2 = f(x_2)$$

Then, note that

$\varphi_1$ fluctuation of $\tilde{\varphi}_1 \Rightarrow \tilde{\varphi}_1 = \varphi_1 + \langle \tilde{\varphi}_1 \rangle$ (1) $\langle \varphi_1 \rangle = 0$ (2)

$\varphi_2$ fluctuation of $\tilde{\varphi}_2 \Rightarrow \tilde{\varphi}_2 = \varphi_2 + \langle \tilde{\varphi}_2 \rangle$. $\langle \varphi_2 \rangle = 0$

It follows that

$$\begin{aligned}
\langle \tilde{f}(x_1)\tilde{f}(x_2)\rangle &= \langle \tilde{\varphi}_1 \tilde{\varphi}_2 \rangle = \langle (\varphi_1 + \langle\tilde{\varphi}_1\rangle)(\varphi_2 + \langle\tilde{\varphi}_2\rangle)\rangle = \\
&= \langle \varphi_1\varphi_2 + \varphi_1\langle\tilde{\varphi}_2\rangle + \varphi_2\langle\tilde{\varphi}_1\rangle + \langle\tilde{\varphi}_1\rangle\langle\tilde{\varphi}_2\rangle\rangle = \\
&= \langle\tilde{\varphi}_1\rangle\langle\tilde{\varphi}_2\rangle + \langle\varphi_1\rangle\langle\tilde{\varphi}_2\rangle + \langle\varphi_2\rangle\langle\tilde{\varphi}_1\rangle + \langle\varphi_1\varphi_2\rangle = \\
&= \langle\tilde{\varphi}_1\rangle\langle\tilde{\varphi}_2\rangle + 0 + 0 + \langle\varphi_1\varphi_2\rangle = \\
&= \langle\tilde{f}(x_1)\rangle\langle\tilde{f}(x_2)\rangle + \Gamma(x_1,x_2) \qquad \square
\end{aligned}$$

► The correlation function $\Gamma(x_1,x_2)$ measures whether the values of $f$ at $x_1$ are <u>dependent</u> on the values of $f$ at $x_2$.
If the values were completely independent, then we would expect

$$\langle \tilde{f}(x_1)\tilde{f}(x_2)\rangle = \langle\tilde{f}(x_1)\rangle\langle\tilde{f}(x_2)\rangle$$

The correlation function measures the deviation between the RHS and LHS of this equation. The relation can be generalized to

$$\langle \prod_{k=1}^{n} \tilde{f}(x_k)\rangle = \prod_{k=1}^{n}\langle\tilde{f}(x_k)\rangle + \Gamma(x_1,x_2,\ldots,x_n)$$

which can be shown by induction ◄

The structure functions of a random variable can be defined as the moments of its increments.

<u>Definition</u>: Let $\tilde{f}$ be a random function with fluctuation $f$.

a) The <u>increment</u> function of $\tilde{f}$ is defined by

$$\boxed{\delta f(x_1, x_2, \omega) = f(x_1,\omega) - f(x_2,\omega), \ \forall x_1, x_2 \in V, \ \forall \omega \in \Omega}$$

b) The <u>p-structure function</u> of $\tilde{f}$ is defined as the $p^{th}$-order moment of its increment, and it is given by:

$$\boxed{S_p(f, x_1, x_2) = \langle (\delta f(x_1, x_2))^p \rangle}$$

where $p$ is taken as an integer. $p \geq 2$.

There are interesting results for structure functions only in the context of symmetry.

## ▼ <u>Symmetric transformations and symmetries.</u>

In the context of turbulence we are interested in properties of the flow such as homogeneity, isotropy, local homogeneity, local isotropy and their consequences.

Part of this discussion can be done in the more general setting of symmetry. We will carry that as far as it goes and then look at specific symmetries in more detail later.

### <u>Definition of symmetry</u>

We begin with symmetries over ordinary functions, as opposed to random functions.

<u>Definition</u>: Let $f: V \to W$ be an arbitrary function, where $V$ and $W$ are spaces of values.

a) A <u>transformation</u> of $f$ is a pair $(A,B)$, where $A: V \leftrightarrow V$ and $B: W \leftrightarrow W$ are bijections (i.e. they have inverses $A^{-1}$ and $B^{-1}$). such that $AV = V$ and $BW = W$

b) If $(A,B)$ is a transformation of $f$, then we defined the <u>transformed function</u> $(A,B)f$ as:
$$(A,B)f = \langle B^{-1} f(Ax) \mid x \in V \rangle$$

c) A function $f$ is <u>symmetric</u> wrt to the transformation $(A,B)$ iff $(A,B)f = f$.

d) Let $\mathcal{S}$ be a space of functions $f: V \to W$. We say that $\mathcal{S}$ is <u>symmetric</u> wrt the transformation $(A,B)$ iff
$$(\forall f \in \mathcal{S})((A,B)f \in \mathcal{S}).$$

▸ <u>interpretation</u>: What we call "transformation" usually is one of the following:

1) A translation
2) A rotation
3) A similarity transform
4) A reflection
5) Galilean transformation

We require from our transformations two properties:

a) They must have a corresponding inverse transformation.

b) They must map the entire $V$ to $V$ and $W$ to $W$.
For example, when we rotate the entire plane we get the same plane. We do not want our transformation to "lose" points; we do not ~~po~~ want points in $V$ that are inaccessible from all points in $V$ after an application of $A$.
We will give physical justification for this ~~per~~ symmetry later.

If a function $f$ is symmetric wrt $(A,B)$ then:
$$(A,B)f = f \leftrightarrow B^{-1} f(Ax) = f(x) \quad , \forall x \in V \leftrightarrow$$
$$\leftrightarrow \underline{f(Ax) = Bf(x).} \quad , \forall x \in V$$

This yields the following interpretation: when $x$ becomes $Ax$, then $f(x)$ becomes $Bf(x)$. So if we set $y=f(x)$ then when $x\to Ax \Rightarrow y\to By$. This motivates the following notation:

$$g=(A,B) \Leftrightarrow g: x\to Ax\ ,\ f\longrightarrow Bf.$$

The content of this statement is that the change of $x$ caused by $A$ can be countered by another transformation $B$ on the value of $f$ such that that value remains invariant.

Usually $\mathcal{S}$ is the solution space of a system of PDEs. When $\mathcal{S}$ is symmetric under $(A,B)$, it means that if we apply the transformation on any solution of the PDEs, what we get is also a solution. We say then that the PDEs are <u>invariant</u> under that transformation and, equivalently, that they have a <u>symmetry</u>. Note that the individual solutions may not be symmetric themselves, for if they are then $(A,B)f=f$ which trivially ~~means~~ implies that $(A,B)f=f$. ◄

► <u>example</u>: The Navier-Stokes equations are invariant with respect to the following transformations:

| | | |
|---|---|---|
| a) Spatial-translations | $Sp(\rho): t,r,v \mapsto t, r+\rho, v$ | $\rho\in\mathbb{R}^3$ |
| b) Temporal-translations | $g(\tau): t,r,v \mapsto t+\tau, r, v$ | $\tau\in\mathbb{R}$ |
| c) Galilean-transformations | $Gal(V): t,r,v \mapsto t, r+Vt, v+V$ | $V\in\mathbb{R}^3$ |
| d) Parity | $P: t,r,v \mapsto t,-r,-v$ | |
| e) Rotations | $Rot(A): t,r,v \mapsto t, Ar, Av$ | $A\in SO(3)$ |
| f) Scaling | $Scal(\lambda,h): t,r,v \mapsto \lambda^{1-h}t, \lambda r, \lambda^h v$ | $(\lambda,h)\in[0,\infty)\times\mathbb{R}$ |

where $t$ = time
$r=(x,y,z)$ = position in space
$v(t,x,y,z)\in\mathbb{R}^3 \longrightarrow$ velocity field.

Note that in all of those <u>$B$ is linear</u> and with translations it is <u>$B=id$</u>! ◄

## Statistical symmetries

As we pointed out, when a system of PDEs is invariant wrt a transformation, it does not follow that the solutions themselves will also be symmetric. However, if we treat the solution as a random function over the space $\underline{\Omega}$ of configurations, then we may recover statistical symmetry.

For every symmetry we may define a local and global statistical symmetry as follows:

Definition: Let $\tilde{f}: V\times\underline{\Omega} \to W$ be some arbitrary random function, where $V$ and $W$ are spaces of values and $\underline{\Omega}$ is a space of configurations. Let $(A,B)$ be a transformation of $f$, as before.

a) We say that a transformation $g: \underline{\Omega} \to \underline{\Omega}$ is ~~[illegible]~~ a symmetry transformation iff the following hold:

i) The measure $\mu$ is invariant wrt $g$: $\mu(g^{-1}A) = \mu(A)$, $\forall A \in \mathcal{B}$.

ii) $g^{-1}\underline{\Omega}$ and $\underline{\Omega}$ differ only by a set of measure $0$:

$$\mu(g^{-1}\underline{\Omega} \,\Delta\, \underline{\Omega}) = 0$$

b) We say that $\tilde{f}$ is (globally) symmetric wrt $(A,B)$ iff there is a symmetry transformation $g: \underline{\Omega} \to \underline{\Omega}$ such that

$$\boxed{B^{-1} f(Ax, \vartheta) = f(x, g\vartheta)}\,, \quad \forall (x,\vartheta) \in V\times\underline{\Omega}$$

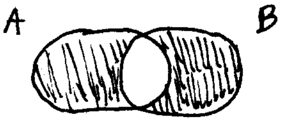
c) We say that $\tilde{f}$ is locally symmetric wrt $(A,B)$ iff there is a symmetry transformation $g: \underline{\Omega} \to \underline{\Omega}$ such that:

$$\boxed{B^{-1}\delta f(Ax_1, Ax_2, \vartheta) = \delta f(x_1, x_2, g\vartheta)}\,, \quad \forall x_1, x_2 \in V,\ \forall \vartheta \in \underline{\Omega}$$

Note that in the above definition $f$ may be the random variable itself or the fluctuation, so we drop the tilde.

▶ <u>remarks</u>: For motivation for requiring $\mu(g^{-1}A)=\mu(A)$ see p.
The difference between two sets $A, B$ is defined by

$$A \Delta B = (A-B) \cup (B-A)$$




The statement $\mu(g^{-1}\underline{\Omega} \Delta \underline{\Omega})=0$ means that $g^{-1}\underline{\Omega}$ and $\underline{\Omega}$ are almost equal. This requirement is analogous to $AV=V$ and $BW=W$ that we required of the transformation $(A,B)$.
The motivation of this requirement for space-translations, rotations, and scaling transformations is obvious: Recall that $\omega \in \underline{\Omega}$ is all the 3d vectors of the velocities and positions of every particle of the fluid. The $g$ that corresponds to a rotation, for example, simply rotates all of those vectors correspondingly.
The motivation for time-translations is less obvious: Let $\omega \in \underline{\Omega}$ be a configuration. Then $g\omega$ will be the configuration at a later point in time. When we say that $g\underline{\Omega}$ and $\underline{\Omega}$ are almost equal, for such $g$, we mean that our options for an initial configuration do <u>not</u> depend on time since the entire space $\underline{\Omega}$ of those options does not change after an application of $g$.
This is an instance of saying that <u>the laws of nature do not change with time.</u>; time is irrelevant in defining the set of all possible initial conditions. for fluid flow.
We could make the stronger requirement $g^{-1}\underline{\Omega}=\underline{\Omega}$. We don't do that however because the weaker condition $\mu(g^{-1}\underline{\Omega} \Delta \underline{\Omega})=0$ is sufficient for our proofs.
Kolmogorov's K41 theory of turbulence postulates that turbulence has certain statistical symmetries and derives experimentally verified facts about turbulence from these assumptions. We now turn to consequences of statistical symmetries that motivated K41. ◀

## Consequences of statistical symmetry.

We begin the development with a few technical lemmas:

▸ **notation**: Let $A, B$ be two regions in $(\Omega, \mathcal{B}, \mu)$. We define equivalence mod$\mu$ as:

$$\boxed{A = B \bmod \mu \Leftrightarrow \mu(A \triangle B) = 0}$$

In words we say that $A, B$ are almost equal. ◂

▸ **notation**: Let $L^n(\Omega, \mathcal{B}, \mu)$ be the space of all functions $\varphi : \Omega \to \mathbb{R}$ such that $\varphi^n$ is integrable over every region in $\mathcal{B}$. ◂

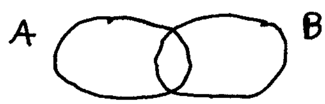
**Lemma**:

$$\boxed{A = B \bmod \mu \Leftrightarrow (\forall \varphi \in L^1(\Omega, \mathcal{B}, \mu)) \left( \int_A \varphi \, d\mu = \int_B \varphi \, d\mu \right)}$$

**Proof**

Note that:

$$A = (A - B) \cup (A \cap B)$$
$$B = (B - A) \cup (A \cap B)$$




are decompositions of $A, B$ to disjoints sets.

($\Rightarrow$): Assume that $A = B \bmod \mu$. Then:

$$\mu(A \triangle B) = 0 \Rightarrow \mu(A - B) + \mu(B - A) = 0 \Rightarrow$$
$$\Rightarrow \mu(A - B) = 0 \land \mu(B - A) = 0.$$
$$\Rightarrow \int_{A-B} \varphi \, d\mu = \int_{B-A} \varphi \, d\mu = 0, \quad \forall \varphi \in L^1(\Omega, \mathcal{B}, \mu).$$

It follows that:

$$\int_A \varphi \, d\mu = \int_{A-B} \varphi \, d\mu + \int_{A \cap B} \varphi \, d\mu = \int_{B-A} \varphi \, d\mu + \int_{A \cap B} \varphi \, d\mu = \int_B \varphi \, d\mu$$

for all $\varphi \in L^1(\Omega, \mathcal{B}, \mu)$.

($\Leftarrow$): Assume $\int_A \varphi d\mu = \int_B \varphi d\mu$, $\forall \varphi \in L^1(\Omega, \mathcal{B}, \mu)$.

Then $\int_{A-B} \varphi d\mu = \int_A \varphi d\mu - \int_{A\cap B} \varphi d\mu = \int_B \varphi d\mu - \int_{A\cap B} \varphi d\mu = \int_{B-A} \varphi d\mu$, $\forall \varphi \in L^1(\Omega, \mathcal{B}, \mu)$.

It follows that

$$\mu(A-B) = \int_{A-B} \chi_{A-B}(\vartheta)\, d\mu(\vartheta) = \int_{B-A} \chi_{B-A}(\vartheta)\, d\mu(\vartheta) =$$

$$= \int_{B-A} 0\, d\mu(\vartheta) = 0 \quad , \text{ because } \chi_{B-A}(\vartheta) = 0, \ \forall \vartheta \in A-B.$$

Similarly $\mu(B-A) = 0$. Therefore

$$\mu(A \triangle B) = \mu(A-B) + \mu(B-A) = 0 + 0 = 0 \Rightarrow A = B \mod \mu. \ \square$$

<u>Lemma</u>: Let $g: \Omega \to \Omega$ be an invariant transformation over $(\Omega, \mathcal{B}, \mu)$, i.e. $\mu(g^{-1}A) = \mu(A)$, $\forall A \in \mathcal{B}$.

Then the following equivalence is true:

$$\boxed{g^{-1}\Omega = \Omega \mod \mu \Leftrightarrow \int_\Omega \varphi(g\vartheta)\, d\mu(\vartheta) = \int_\Omega \varphi(\vartheta)\, d\mu(\vartheta)., \ \forall \varphi \in L^1(\Omega, \mathcal{B}, \mu).}$$

<u>Proof</u>

Note that

$$\int_{g^{-1}\Omega} \varphi(g\vartheta)\, d\mu(\vartheta) = \lim_{P \in \pi(g^{-1}\Omega)} \sum_{A \in P} \mu(A) \sup_{\vartheta \in A} [f(g\vartheta)] =$$

$$= \lim_{P \in \pi(\Omega)} \sum_{A \in P} \mu(g^{-1}A) \sup_{\vartheta \in g^{-1}A} [f(g\vartheta)] = \quad \text{bc. } \mu(g^{-1}A) = \mu(A)$$

$$= \lim_{P \in \pi(\Omega)} \sum_{A \in P} \mu(A) \sup_{\vartheta \in g^{-1}A} [f(g\vartheta)] =$$

$$= \lim_{P \in \pi(\Omega)} \sum_{A \in P} \mu(A) \sup_{g\vartheta \in A} [f(g\vartheta)] =$$

$$= \lim_{P \in \pi(\Omega)} \sum_{A \in P} \mu(A) \sup_{\vartheta \in A} [f(\vartheta)] = \int_\Omega \varphi(\vartheta)\, d\mu(\vartheta). \quad (1)$$

It follows that

$$g^{-1}\underline{\Omega} = \underline{\Omega} \ \mathrm{mod}\ \mu \Leftrightarrow \int_{\underline{\Omega}} \varphi(g\vartheta)\, d\mu(\vartheta) = \int_{g^{-1}\underline{\Omega}} \varphi(g\vartheta)\, d\mu(\vartheta), \ \forall \varphi \in L^1(\underline{\Omega}, \mathcal{B}, \mu)$$

(by previous lemma)

$$\Leftrightarrow \int_{\underline{\Omega}} \varphi(g\vartheta)\, d\mu(\vartheta) = \int_{\underline{\Omega}} \varphi(\vartheta)\, d\mu(\vartheta), \ \forall \varphi \in L^1(\underline{\Omega}, \mathcal{B}, \mu). \quad \square$$

(by (1)).

Now we show a few results about statistical symmetries:

<u>Proposition:</u> Let $(A, B)$ be a transformation and <u>assume $B$ is linear.</u>
Then: $f$ symmetric wrt $(A,B) \Rightarrow f$ locally symmetric wrt $(A,B)$.

<u>Proof</u>

$f$ symmetric wrt $(A,B) \Rightarrow \underline{f(Ax, \vartheta) = Bf(x, g\vartheta)}$, $\forall x \in V$, $\forall \vartheta \in \underline{\Omega}$.
Let $x_1, x_2 \in V$ and $\vartheta \in \underline{\Omega}$ be given. Then:

$$\begin{aligned}
\delta f(Ax_1, Ax_2, \vartheta) &= f(Ax_1, \vartheta) - f(Ax_2, \vartheta) = \\
&= Bf(x_1, g\vartheta) - Bf(x_2, g\vartheta) = \\
&= B[f(x_1, g\vartheta) - f(x_2, g\vartheta)] = B\delta f(x_1, x_2, g\vartheta) \Rightarrow
\end{aligned}$$

$\Rightarrow B^{-1}\delta f(Ax_1, Ax_2, \vartheta) = \delta f(x_1, x_2, g\vartheta)$, $\forall x_1, x_2 \in V$, $\forall \vartheta \in \underline{\Omega} \Rightarrow$
$\Rightarrow f$ locally symmetric wrt $(A,B)$. $\square$

The converse is not true. Therefore local symmetry is a weakening of symmetry.

<u>Proposition</u>: Let $\tilde{f}$ be a random function with fluctuation $f$.

a) $\tilde{f}$ symmetric wrt $(A,B) \Rightarrow f$ symmetric wrt $(A,B)$
b) $\tilde{f}$ locally symmetric wrt $(A,B) \Rightarrow f$ locally symmetric wrt $(A,B)$.

where we assume $B$ linear.

Proof

a) $$f(Ax,\vartheta) = \tilde{f}(Ax,\vartheta) - \int_{\Omega} \tilde{f}(Ax,\vartheta)\, d\mu(\vartheta) =$$ ($\tilde{f}$ symmetric)
$$= B\tilde{f}(x,g\vartheta) - \int_{\Omega} B\tilde{f}(x,g\vartheta)\, d\mu(\vartheta) =$$ ($B$ linear)
$$= B\tilde{f}(x,g\vartheta) - B\int_{\Omega} \tilde{f}(x,g\vartheta)\, d\mu(\vartheta) =$$ ($B$ linear)
$$= B\left[\tilde{f}(x,g\vartheta) - \int_{\Omega} \tilde{f}(x,g\vartheta)\, d\mu(\vartheta)\right] =$$ (lemma, p. 31).
$$= B\left[\tilde{f}(x,g\vartheta) - \int_{\Omega} \tilde{f}(x,\vartheta)\, d\mu(\vartheta)\right] =$$
$$= B\, f(x,g\vartheta).\quad, \forall x \in V, \forall \vartheta \in \Omega. \Rightarrow f \text{ symmetric wrt } (A,B).$$

b) Can be shown using an analogous argument. □

If we specialize to $B = id$ we obtain the following theorem that connects symmetry with the correlation function and local symmetry with the structure function:

Theorem: Let $\tilde{f}$ be a random function with fluctuation $f$, correlation function $\Gamma$ and structure function $S_p$. Then:

a) $f$ symmetric wrt $(A, id) \Rightarrow \Gamma(Ax_1, Ax_2) = \Gamma(x_1, x_2).\ , \forall x_1, x_2 \in V$

b) $f$ locally symmetric wrt $(A, id) \Rightarrow S_p(Ax_1, Ax_2) = S_p(x_1, x_2)\ , \forall x_1, x_2 \in V$

Proof

a) $f$ symmetric wrt $(A, id) \Rightarrow f(Ax,\vartheta) = f(x, g\vartheta)\ , \forall (x,\vartheta) \in V \times \Omega$.
It follows that for given $x_1, x_2 \in V$:

$$\Gamma(Ax_1, Ax_2) = \int_{\Omega} f(Ax_1,\vartheta) \cdot f(Ax_2,\vartheta)\, d\mu(\vartheta) =$$ ($f$ symmetric)
$$= \int_{\Omega} f(x_1, g\vartheta) f(x_2, g\vartheta)\, d\mu(\vartheta) =$$ (lemma, p. 31).

$$= \int_{\Omega} f(x_1,\vartheta) f(x_2,\vartheta)\, d\mu(\vartheta) = \Gamma(x_1,x_2).$$

b) $f$ locally symmetric wrt $(A, id)$ $\Rightarrow$

$\Rightarrow \delta f(Ax_1, Ax_2, \vartheta) = \delta f(x_1, x_2, g\vartheta)$, $\forall x_1, x_2 \in V$, $\forall \vartheta \in \Omega$.

For given $x_1, x_2 \in V$ and $\vartheta \in \Omega$ we have:

$$\begin{aligned} S_p(Ax_1, Ax_2) &= \int_{\Omega} (\delta f(Ax_1, Ax_2, \vartheta))^p\, d\mu(\vartheta) = && (f \text{ locally symmetric}) \\ &= \int_{\Omega} (\delta f(x_1, x_2, g\vartheta))^p\, d\mu(\vartheta) = && (\text{lemma, p. 31}) \\ &= \int_{\Omega} (\delta f(x_1, x_2, \vartheta))^p\, d\mu(\vartheta) = S_p(x_1, x_2). \quad \square \end{aligned}$$

$\hookrightarrow$ Thanks to the previous proposition, this theorem still applies if we assume that $\tilde{f}$ is (locally) symmetric instead of the fluctuation $f$.

▶ <u>Remark:</u> Recall that the variance of $\tilde{f}$ is given by:

$$\langle f^2(x)\rangle = \int_{\Omega} f(x,\vartheta) f(x,\vartheta)\, d\mu(\vartheta) = \Gamma(x,x).$$

From the theorem above we see that if $f$ is symmetric wrt $A$ then:

$$\langle f^2(Ax)\rangle = \Gamma(Ax, Ax) = \Gamma(x,x) = \langle f^2(x)\rangle$$

In other words, symmetry requires that the variance remain constant during the symmetric symmetry. Local symmetry weakens this restriction and allows the variance to change ◀

## Stationary and locally stationary functions.

In this section we concentrate on random functions of the form:

$$\tilde{\varphi} : (t, \vartheta) \in [0, +\infty) \times \underline{\Omega} \longmapsto \tilde{\varphi}(t, \vartheta) \in \mathbb{R}$$

The variable $t \in [0, +\infty)$ can be interpreted as time but we do not demand this interpretation, therefore our discussion remains fairly abstract.

Let $g(t) = g_t : \underline{\Omega} \to \underline{\Omega}$ be a flow on $\underline{\Omega}$ as defined on p.7 which, we also assume, satisfies

$$g_t^{-1} \underline{\Omega} = \underline{\Omega} \mod \mu \quad , \forall t \in [0, +\infty).$$

Given this flow, we are interested in the following symmetries:

Definition: Let $\varphi : [0, +\infty) \times \underline{\Omega} \to \mathbb{R}$ be a random function.

a) We ~~say~~ say that $\varphi$ is stationary iff:

$$\boxed{\varphi(t+\tau, \vartheta) = \varphi(t, g_\tau \vartheta) \quad , \forall \tau, t \in [0, +\infty) \, , \forall \vartheta \in \underline{\Omega}}$$

b) We say that $\varphi$ is locally stationary (or that it has stationary increments) iff:

$$\boxed{\varphi(t_1+\tau, \vartheta) - \varphi(t_2+\tau, \vartheta) = \varphi(t_1, \vartheta) - \varphi(t_2, \vartheta) \quad , \forall \tau, t_1, t_2 \in [0, +\infty) \, , \forall \vartheta \in \underline{\Omega}}$$

It is easy to see that stationarity is a symmetry and local stationarity the corresponding local symmetry.

Our general results specialize as follows:

a) $\varphi$ stationary $\Rightarrow$ $\varphi$ locally stationary.

b) $\tilde{\varphi}$ stationary $\Rightarrow$ $\varphi$ stationary,
$\tilde{\varphi}$ locally stationary $\Rightarrow$ $\varphi$ locally stationary,
if $\varphi$ is the fluctuation of $\tilde{\varphi}$.

We also have that

$$\varphi \text{ stationary} \Rightarrow \Gamma(t_1+\tau, t_2+\tau) = \Gamma(t_1, t_2) \quad , \forall \tau \in [0,+\infty).$$

where

$$\Gamma(t_1,t_2) \equiv \langle \varphi(t_1)\varphi(t_2) \rangle$$

is the correlation function. It follows that $\Gamma$ depends only on the time difference $t_1 - t_2$ so we can write:

$$\langle \varphi(t+\tau)\varphi(t) \rangle = \Gamma(\tau) \quad , \forall t \in [0,+\infty).$$

Note that $\Gamma(0)$ gives the variance of $\tilde{\varphi}$:

$$\langle \varphi^2(t) \rangle = \langle \varphi(t+0)\varphi(t) \rangle = \Gamma(0).$$

which is independent of $t$.

Assuming stationarity we show further properties of $\Gamma(\tau)$:

## Properties of $\Gamma(\tau)$

Theorem: Let $\varphi(t,\vartheta)$ be a fluctuation that is stationary. Then, if $\varphi$ differentiable wrt $t \Rightarrow \Gamma(\tau)$ differentiable with:

$$\boxed{\Gamma'(\tau) = \langle \varphi(t)\varphi'(t+\tau) \rangle}$$

Proof

Let $\tau$ be given. Then:

$$\lambda_\Gamma(\tau,h) \equiv \frac{\Gamma(\tau+h)-\Gamma(\tau)}{h} = \frac{1}{h}\left[\langle \varphi(t)\varphi(t+\tau+h)\rangle - \langle \varphi(t)\varphi(t+\tau+h)\rangle\right] =$$

$$= \frac{1}{h}\langle \varphi(t)\varphi(t+\tau+h) - \varphi(t)\varphi(t+\tau) \rangle =$$

$$= \langle \varphi(t) \frac{\varphi(t+\tau+h)-\varphi(t+\tau)}{h} \rangle$$

Therefore, we have:

$$\Gamma'(\tau) = \lim_{h\to 0} \lambda_\Gamma(\tau,h) = \lim_{h\to 0} \left\langle \varphi(t)\, \frac{\varphi(t+\tau+h)-\varphi(t+\tau)}{h} \right\rangle$$

$$= \left\langle \varphi(t) \lim_{h\to 0} \frac{\varphi(t+\tau+h)-\varphi(t+\tau)}{h} \right\rangle = \langle \varphi(t)\, \varphi'(t+h) \rangle \quad \square$$

If we restrict $t\in[0,+\infty)$ then we cannot obtain differentiability at $\tau=0$. However if we extend the domain of $\varphi$ to $t\in\mathbb{R}$ then our theorem yields differentiability at $\tau=0$, and allows in fact $\Gamma(\tau)$ to be defined over $\tau\in\mathbb{R}$. Since the restriction to $t\in[0,+\infty)$ is imposed artificially, in physical contexts, it is reasonable to assume that if we extend $\Gamma$ by:

$$\Gamma(-\tau) \equiv \Gamma(\tau) \quad , \forall \tau\in[0,+\infty)$$

then $\Gamma$ is differentiable at $\tau=0$.

This extention is motivated by noting that if $\varphi$ is a stationary fluctuation with $t\in\mathbb{R}$ then:

$$\Gamma(-\tau) = \Gamma(t,t-\tau) = \Gamma(t+\tau, t-\tau+\tau) = \Gamma(t+\tau, t) =$$
$$= \Gamma(t,t+\tau) = \Gamma(\tau).$$

With this assumption we have:

**Theorem**: If $f$ is a stationary, differentiable on $\mathbb{R}$, fluctuation, then:

$$\boxed{\begin{aligned} &\langle f(t) f'(t)\rangle = \Gamma'(0) = 0 \\ &\langle f(t) f''(t)\rangle = -\langle (f'(t))^2\rangle = \Gamma''(0). \end{aligned}}$$

**Proof**

a) From the theorem on p. 23 we have:

$$|\Gamma(\tau)| = |\Gamma(t,t+\tau)| \leqslant \frac{1}{2}\left[\langle f^2(t)\rangle + \langle f^2(t+\tau)\rangle\right] =$$

$$= \frac{1}{2}\left[\Gamma(0)+\Gamma(0)\right] = \Gamma(0) \Longrightarrow$$

$$\left.\begin{array}{l}\Rightarrow (\forall \tau \in \mathbb{R})\,(\Gamma(\tau) \leqslant \Gamma(0)) \\ \Gamma \text{ differentiable at } \mathbb{R}\end{array}\right\} \Rightarrow \Gamma \text{ has a global maximum at } \tau = 0$$

$\Rightarrow \Gamma'(0) = 0 \qquad (1).$

From the previous theorem it follows that

$$\langle f(t) f'(t) \rangle = \langle f(t) f'(t+0) \rangle = \Gamma'(0) = 0$$

b) Note that $(ff')' = f'f' + ff''$ and that

$$\begin{aligned}\langle (f(t) f'(t))' \rangle &= \int_\Omega \frac{\partial}{\partial t}\left[ f(t,\omega) f'(t,\omega) \right] d\mu(\omega) = \\ &= \frac{\partial}{\partial t} \int_\Omega \left[ f(t,\omega) f'(t,\omega) \right] d\mu(\omega) = \\ &= \frac{\partial}{\partial t} \langle f(t) f'(t) \rangle .\end{aligned}$$

Since $\langle f(t) f'(t) \rangle = 0, \forall t \in \mathbb{R} \Rightarrow \langle (f(t) f'(t))' \rangle = 0, \forall t \in \mathbb{R}$.

It follows that

$$\begin{aligned}\langle f(t) f''(t) \rangle &= \langle (f(t) f'(t))' - f'(t) f'(t) \rangle = \\ &= \langle (f(t) f'(t))' \rangle - \langle (f'(t))^2 \rangle = \\ &= -\langle (f'(t))^2 \rangle .\end{aligned}$$

Also:

$$\Gamma''(0) = \langle f(t) f''(t+0) \rangle = \langle f(t) f''(t) \rangle$$

therefore:

$$\langle f(t) f''(t) \rangle = -\langle (f'(t))^2 \rangle = \Gamma''(0). \qquad \square$$

↳ This result characterizes $-\Gamma''(0)$ as the variance $\langle (f'(t))^2 \rangle$ of the derivative of a stationary ~~variable~~ function $\tilde{f}$ (or a function $\tilde{f}$ with stationary fluctuation).

~~By applying this theorem recursively on $g(t,\omega) = f'(t,\omega)$ we can show that:~~

$$\cancel{\langle f(t) f^{(2n+1)}(t) \rangle = \Gamma^{(2n+1)}(0) = 0}$$

$$\cancel{\langle f(t) f^{(2n)}(t) \rangle}$$

## Time averaging and the ergodic theorem.

If a function $\tilde{\varphi}$ is stationary, then under certain assumptions we may measure its ensemble average by measuring a time average. To show this, we begin with defining time averaging:

**Definition**: Let $\tilde{\varphi} : [0, +\infty) \times \underline{\Omega} \longrightarrow \mathbb{R}$ be some random function.

a) Then the "time"-average of $\tilde{\varphi}$ is a random variable $\mathring{\varphi} : \underline{\Omega} \to \mathbb{R}$ defined by:

$$\boxed{\mathring{\varphi}(\varpi) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \tilde{\varphi}(t, \varpi)\, dt}$$

b) The partial "time"-average for some period $T > 0$ of $\tilde{\varphi}$ is a random variable $\mathring{\varphi}_T : \underline{\Omega} \to \mathbb{R}$ defined by:

$$\boxed{\mathring{\varphi}_T(\varpi) = \frac{1}{T} \int_0^T \tilde{\varphi}(t, \varpi)\, dt}$$

► The ergodic hypothesis states that for almost all $\varpi \in \underline{\Omega}$

$$\mathring{\varphi}(\varpi) = \langle \tilde{\varphi}(0) \rangle$$

where $\langle \varphi(0) \rangle$ is the ensemble average at $t = 0$.
By almost all $\varpi \in \underline{\Omega}$ we mean that there is an $A = \underline{\Omega} \bmod \mu$ such that the hypothesis is true for all $\varpi \in A$. ◄

Assume only that $\tilde{\varphi}$ is stationary. Then, recall that

$$\tilde{\varphi}(t + \tau, \varpi) = \tilde{\varphi}(t, g_\tau \varpi), \quad \forall \tau > 0$$

and $\quad g_\tau^{-1} \underline{\Omega} = \underline{\Omega} \bmod \mu, \quad \forall \tau > 0.$

and $\quad \mu(g_\tau^{-1} A) = \mu(A), \quad \forall A \in \mathcal{B}, \forall \tau > 0.$

Then the closest we can get to the ergodic hypothesis with only these assumptions is given by the following theorem:

<u>Theorem</u>: If $\tilde{\varphi} : [0,+\infty) \times \underline{\Omega} \longrightarrow \mathbb{R}$ is a stationary random function with correlation $\Gamma(\tau)$ and partial time average $\mathring{\varphi}_T : \underline{\Omega} \longrightarrow \mathbb{R}$ then the following are true:

$$\boxed{\langle \mathring{\varphi}_T \rangle = \langle \tilde{\varphi}(0) \rangle \ , \forall T > 0}$$

$$\boxed{\langle (\mathring{\varphi}_T - \langle \tilde{\varphi}(0) \rangle)^2 \rangle = \frac{2}{T^2} \int_0^T dt_1 \int_0^{t_1} dt_2 \, \Gamma(t_2)}$$

<u>Proof</u>

a) 
$$\langle \mathring{\varphi}_T \rangle = \int_{\underline{\Omega}} \mathring{\varphi}_T(\omega)\, d\mu(\omega) \underset{\text{(definitions)}}{=} \int_{\underline{\Omega}} \left[ \frac{1}{T} \int_0^T \tilde{\varphi}(t,\omega)\, dt \right] d\mu(\omega) =$$

$$= \frac{1}{T} \int_0^T \left[ \int_{\underline{\Omega}} \tilde{\varphi}(t,\omega)\, d\mu(\omega) \right] dt = \qquad (\tilde{\varphi} \text{ stationary})$$

$$= \frac{1}{T} \int_0^T \left[ \int_{\underline{\Omega}} \tilde{\varphi}(0, g_t \omega)\, d\mu(\omega) \right] dt = \qquad (g_t^{-1} \underline{\Omega} = \underline{\Omega} \bmod \mu)$$

$$= \frac{1}{T} \int_0^T \left[ \int_{\underline{\Omega}} \tilde{\varphi}(0, \omega)\, d\mu(\omega) \right] dt = \qquad (\text{integrand constant wrt } t)$$

$$= \left[ \int_{\underline{\Omega}} \tilde{\varphi}(0,\omega)\, d\mu(\omega) \right] \frac{1}{T} \int_0^T dt =$$

$$= \int_{\underline{\Omega}} \tilde{\varphi}(0,\omega)\, d\mu(\omega) = \langle \tilde{\varphi}(0) \rangle$$

b) Let $\varphi$ be the fluctuation of $\tilde{\varphi}$. Then $\langle \varphi(t_1)\varphi(t_2) \rangle = \Gamma(t_2 - t_1)$.
Now we have: $\tilde{\varphi}(t,\omega) = \langle \tilde{\varphi}(t) \rangle + \varphi(t,\omega) = \langle \tilde{\varphi}(0) \rangle + \varphi(t,\omega)$, bc. $\tilde{\varphi}$ stationary.
therefore:

$$\langle (\mathring{\varphi}_T - \langle \tilde{\varphi}(0) \rangle)^2 \rangle = \left\langle \left( \frac{1}{T} \int_0^T \tilde{\varphi}(t)\, dt - \langle \tilde{\varphi}(0) \rangle \right)^2 \right\rangle =$$

$$= \left\langle \left( \frac{1}{T} \int_0^T (\tilde{\varphi}(t) - \langle \tilde{\varphi}(0) \rangle) dt \right)^2 \right\rangle$$

$$= \left\langle \left( \frac{1}{T} \int_0^T \varphi(t)\, dt \right)^2 \right\rangle =$$

$$= \int_{\Omega} \left[ \frac{1}{T} \int_0^T \varphi(t,\vartheta)\,dt \right]^2 d\mu(\vartheta) =$$

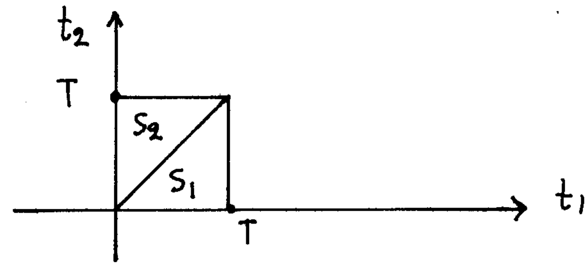
$$= \frac{1}{T^2} \int_{\Omega} d\mu(\vartheta) \int_0^T dt_1 \int_0^T dt_2 \;\; \varphi(t_1,\vartheta)\,\varphi(t_2,\vartheta)$$

$$= \frac{1}{T^2} \iint_{[0,T]^2} dt_1\,dt_2 \left[ \int_{\Omega} \varphi(t_1,\vartheta)\,\varphi(t_2,\vartheta)\,d\mu(\vartheta) \right] =$$

$$= \frac{1}{T^2} \iint_{[0,T]^2} dt_1\,dt_2\; \Gamma(t_2-t_1)$$

Let $S_1 = \bigcup_{t_1 \in [0,T]} \{t_1\} \times [0,t_1]$

$S_2 = \bigcup_{t_2 \in [0,T]} [0,t_2] \times \{t_2\}$




be a partition of $[0,T]^2$ as shown in the figure.
Then, because $\Gamma(t_2-t_1) = \Gamma(t_1-t_2)$ it follows that

$$\iint_{S_1} dt_1\,dt_2\,\Gamma(t_2-t_1) = \iint_{S_2} dt_1\,dt_2\,\Gamma(t_2-t_1) = \frac{1}{2}\iint_{[0,T]^2} dt_1\,dt_2\,\Gamma(t_2-t_1)$$

therefore:

$$\langle (\mathring{\varphi}_T - \langle \tilde{\varphi}(0)\rangle)^2 \rangle = \frac{2}{T^2} \iint_{S_1} dt_1\,dt_2\;\Gamma(t_2-t_1) =$$

$$= \frac{2}{T^2} \int_0^T dt_1 \int_0^{t_1} dt_2\;\Gamma(t_2-t_1) =$$

$$= \frac{2}{T^2} \int_0^T dt_1 \int_{-t_1}^{0} dt_2\,\Gamma(t_2) =$$

$$= \frac{2}{T^2} \int_0^T dt_1 \int_0^{t_1} dt_2\,\Gamma(t_2) \qquad \square$$

↳ This theorem tells us that the ensemble average of $\mathring{\varphi}_T(\vartheta)$ equals $\langle \tilde{\varphi}(0) \rangle$, however $\mathring{\varphi}(\vartheta)$ itself (as $T \to +\infty$) may not equal $\langle \tilde{\varphi}(0) \rangle$ unless it is constant over almost all $\Omega$. Then we can take it out of the average and obtain the ergodic hypothesis. The second part of the theorem gives a relation between $\Gamma(\tau)$ and the variance of

$$r(\varpi) = \mathring{\varphi}_T(\varpi) - \langle \tilde{\varphi}(0) \rangle$$

which is the residual error in the ergodic hypothesis for patrtial time averaging. At the limit $T \to +\infty$ we obtain the error of the ergodic hypothesis itself.

The closest we can get to saying that $\mathring{\varphi}(\varpi)$ is constant is the following proposition that says that it is constant along the orbit $\{g_t \varpi \mid t > 0\}$ :

Proposition : Let $\mathring{\varphi}$ be the time average of a stationary function $\tilde{\varphi}$, and let $g_t$, $t>0$ be the corresponding flow. Then

$$\boxed{\mathring{\varphi}(g_t \varpi) = \mathring{\varphi}(\varpi) \ , \ \forall t > 0}$$

Proof

$$\mathring{\varphi}(g_\tau \varpi) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \tilde{\varphi}(t, g_\tau \varpi)\,dt \qquad \text{(stationarity)}$$

$$= \lim_{T \to +\infty} \frac{1}{T} \int_0^T \tilde{\varphi}(t+\tau, \varpi)\,dt \qquad \text{(change of variables)}$$

$$= \lim_{T \to +\infty} \frac{1}{T} \int_\tau^{T+\tau} \tilde{\varphi}(t, \varpi)\,dt = \qquad \text{(add term that vanishes as } 1/T\text{)}$$

$$= \lim_{T \to +\infty} \frac{1}{T} \left( \int_0^\tau \tilde{\varphi}(t, \varpi)\,dt + \int_\tau^{T+\tau} \tilde{\varphi}(t, \varpi)\,dt \right) =$$

$$= \lim_{T \to +\infty} \frac{1}{T} \int_0^{T+\tau} \tilde{\varphi}(t, \varpi)\,dt =$$

$$= \lim_{T \to +\infty} \left[ \frac{T+\tau}{T} \frac{1}{T+\tau} \int_0^{T+\tau} \tilde{\varphi}(t, \varpi)\,dt \right] =$$

$$= \lim_{T \to +\infty} \frac{T+\tau}{T} \lim_{T \to +\infty} \frac{1}{T} \int_0^T \tilde{\varphi}(t, \varpi)\,dt =$$

$$= \lim_{T\to+\infty} \frac{1}{T}\int_0^T \tilde{\varphi}(t,\vartheta)dt = \mathring{\varphi}(\vartheta) \quad , \forall \tau > 0. \quad \square$$

Both this result and part 2 of the theorem on p.40 can be used to introduce additional assumptions that can get us to the ergodic hypothesis, or close enough.

There are two approaches:

● $_1$ Assume ergodic measure

Definition: Let $g: \Omega \to \Omega$ be a transformation on $\Omega$. We say that a measure $\mu$ on $\Omega$ is ergodic wrt the transformation $g$ iff:

$$\boxed{(\forall A \in \mathcal{B})(g^{-1}A = A \bmod \mu \Rightarrow \mu(A) = 0 \lor \mu(A) = 1)}$$

So we require that the transformation $g$ be such that any region that remains invariant under $g$ has either measure 1 (e.g. $\Omega$) or measure 0 (e.g. $\varnothing$).

▶ notation: Let $f(\vartheta), g(\vartheta)$ be two random variables. We say that $f(\vartheta) = g(\vartheta) \bmod \mu \Leftrightarrow \mu(\{\vartheta \in \Omega \mid f(\vartheta) = g(\vartheta)\}) = 1.$ ◀

The main theorem is the following result:

Theorem: Let $f: \Omega \to \mathbb{R}$ be a random variable and $\mu$ an ergodic probability measure wrt $g: \Omega \to \Omega$. Then the following is true:

$$\boxed{(\forall \vartheta \in \Omega)(f(g^{-1}\vartheta) = f(\vartheta)) \Rightarrow (\exists A \in \mathcal{B})(\mu(A) = 1 \land f(\vartheta) = \text{const on } A)}$$

Proof

Define $f_+(\vartheta) = \begin{cases} f(\vartheta) & , \text{if } f(\vartheta) > 0 \\ 0 & , \text{otherwise.} \end{cases}$

and $f_-(\vartheta) = \begin{cases} -f(\vartheta) & \text{, if } f(\vartheta) \leqslant 0 \\ 0 & \text{, otherwise} \end{cases}$

Then $f(\vartheta) = f_+(\vartheta) - f_-(\vartheta) \ , \forall \vartheta \in \underline{\Omega}$

Since $f_+, f_-$ are integrable they can be constructed by a countable sum of characteristic functions:

$$f_+(\vartheta) = \sum_{i=0}^{+\infty} a_i \chi_{A_i}(\vartheta) \ , \forall \vartheta \in \underline{\Omega} \qquad \text{with } a_i > 0$$

$$f_-(\vartheta) = \sum_{i=0}^{+\infty} b_i \chi_{B_i}(\vartheta) \ , \forall \vartheta \in \underline{\Omega} \qquad \text{with } b_i > 0$$

(Both $f_+, f_-$ can be approximated from bellow by accumulating strictly positive simple functions countable times. Each such function in turn is at most a countable sum of characteristic functions.)

From the hypothesis:

$$f(g^{-1}\vartheta) = f(\vartheta), \forall \vartheta \in \underline{\Omega} \Rightarrow \begin{cases} f_+(g^{-1}\vartheta) = f_+(\vartheta) \ , \forall \vartheta \in \underline{\Omega} \\ f_-(g^{-1}\vartheta) = f_-(\vartheta) \end{cases}$$

Note that

$$f_+(\vartheta) - f_+(g^{-1}\vartheta) = \sum_{i=0}^{+\infty} a_i \chi_{A_i}(\vartheta) - \sum_{i=0}^{+\infty} a_i \chi_{A_i}(g^{-1}\vartheta) =$$

$$= \sum_{i=0}^{+\infty} a_i \left( \chi_{A_i}(\vartheta) - \chi_{A_i}(g^{-1}\vartheta) \right) = 0 \ , \forall \vartheta \in \underline{\Omega} \Rightarrow$$

$$\Rightarrow \underline{\chi_{A_i}(\vartheta) = \chi_{A_i}(g^{-1}\vartheta) \ , \forall \vartheta \in \underline{\Omega}} \qquad (1)$$

Then:

$$g^{-1}A_i = g^{-1}\{\vartheta \mid \chi_{A_i}(\vartheta) = 1\} = \{g^{-1}\vartheta \mid \chi_{A_i}(\vartheta) = 1\} =$$

(Because of (1))

$$= \{g^{-1}\vartheta \mid \chi_{A_i}(g^{-1}\vartheta) = 1\} =$$

(Because $g^{-1}\underline{\Omega} = \underline{\Omega} \bmod \mu$)

$$= \{\vartheta \mid \chi_{A_i}(\vartheta) = 1\} \bmod \mu =$$

$$= A_i \bmod \mu \Rightarrow \underline{\mu(A_i) = 0 \vee \mu(A_i) = 1} \qquad (2)$$

By a similar argument we find that $\mu(B_i)=0 \vee \mu(B_i)=1$
There are two cases:
a) If $\mu(A_i)=\mu(B_i)=0$, $\forall i$ then let $\Gamma = \underline{\Omega} - \bigcup_{i=1}^{+\infty}(A_i \cup B_i)$
Then

$$\left.\begin{array}{l} f(\vartheta)=0, \forall \vartheta \in \Gamma \\ \mu(\Gamma)=\mu(\underline{\Omega})=1 \end{array}\right\} \Rightarrow f(\vartheta)=0 \bmod \mu \Rightarrow f \text{ constant } \bmod \mu.$$

b) Assume that there are regions of measure 1.
Let $\langle \Gamma_i \mid i \in a \rangle$ be a sequence of those regions among $A_i, B_i$ that have $\mu(\Gamma_i)=1$.
Let $\langle \Delta_i \mid i \in b \rangle$ be all other regions that have $\mu(\Delta_i)=0$.
Let $\gamma_i, \delta_i$ be the coefficients that correspond to these regions for $f(\vartheta)$. Finally, define:

$$\Gamma = \left(\bigcap_{i\in a} \Gamma_i\right) - \left(\bigcup_{i \in b} \Delta_i\right)$$

Then
$$f(\vartheta) = \sum_{i\in a} \gamma_i \chi_{\Gamma_i}(\vartheta) + \sum_{i \in b} \delta_i \chi_{\Delta_i}(\vartheta) =$$
$$= \sum_{i\in a} \gamma_i \cdot 1 + \sum_{i\in b} \delta_i \cdot 0 = \sum_{i \in a} \gamma_i = \text{const.}, \forall \vartheta \in \Gamma.$$

Also
$$\mu(\Gamma) \geq \mu\left(\bigcap_{i\in a} \Gamma_i\right) - \mu\left(\bigcup_{i\in b} \Delta_i\right) = \mu\left(\bigcap_{i\in a}\Gamma_i\right) - \sum_{i\in b}\mu(\Delta_i) =$$
$$= \mu\left(\bigcap_{i\in a}\Gamma_i\right) = \mu\left(\underline{\Omega} - \bigcup_{i\in a}(\underline{\Omega}-\Gamma_i)\right) \geq \mu(\underline{\Omega}) - \mu\left(\bigcup_{i\in a}(\underline{\Omega}-\Gamma_i)\right) =$$
$$= \mu(\underline{\Omega}) - \sum_{i\in a}\mu(\underline{\Omega}-\Gamma_i) = 1 - \sum_{i\in a} 0 = 1 \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow \mu(\Gamma)=1 \\ f(\vartheta) \text{ constant on } \Gamma \end{array}\right\} \Rightarrow f \text{ constant } \bmod \mu. \quad \square$$

<u>Corollary:</u> Assume that $\mu$ is ergodic wrt $g_t$, $\forall t>0$. Then the ergodic hypothesis is true.

<u>Proof</u>

Note that

$$\mathring{\varphi}(g_t^{-1}\vartheta) = \mathring{\varphi}(g_t g_t^{-1}\vartheta) =$$

(proposition p. 42)

$$= \mathring{\varphi}(id\,\vartheta) = \mathring{\varphi}(\vartheta), \quad \forall \vartheta \in \underline{\Omega} \Rightarrow$$

$\Rightarrow \mathring{\varphi}$ is constant mod $\mu \Rightarrow (\exists A \in \mathcal{B})(\mu(A) = 1 \wedge (\mathring{\varphi}(\vartheta) = \text{const}, \forall \vartheta \in A))$.

It follows that

$$\mathring{\varphi}(\vartheta_0) = \mathring{\varphi}(\vartheta_0)\mu(A) = \mathring{\varphi}(\vartheta_0)\int_A d\mu(\vartheta) = \int_A \mathring{\varphi}(\vartheta_0)\, d\mu(\vartheta) =$$

($\mathring{\varphi}$ constant mod $\mu$)

$$= \int_A \mathring{\varphi}(\vartheta)\, d\mu(\vartheta) = \int_{\underline{\Omega}} \mathring{\varphi}(\vartheta)\, d\mu(\vartheta) = \langle \mathring{\varphi} \rangle =$$

(theorem, p. 40)

$$= \langle \tilde{\varphi}(0) \rangle, \quad \forall \vartheta_0 \in A \Rightarrow \mathring{\varphi}(\vartheta) = \langle \tilde{\varphi}(0) \rangle \text{ mod } \mu. \quad \square$$

The assumption that $\mu$ is ergodic wrt $g_t$ means, as it can be shown, that the orbit $\{g_t\vartheta \mid t>0\}$ is a Monte-Carlo sampling of $\underline{\Omega}$. In that sense the time averaging is a Monte-Carlo integration of the ensemble average integral.

In a sense this result is not satisfying. We prove the ergodic hypothesis by almost assuming it. There is no proof that turbulence satisfies this condition and it is unlikely to be true.

● 2 <u>Assume rapidly vanishing $\Gamma(\tau)$.</u>

<u>Definition:</u> Assume $\Gamma(\tau) \sim \tau^{-a}$ with $a>1$. Then we say that $\tilde{\varphi}$ has a finite <u>integral time-scale</u> $T_I$ given by:

$$\boxed{T_I = \frac{1}{\Gamma(0)}\int_0^{+\infty} |\Gamma(\tau)|\, d\tau}$$

This assumption is reasonable since we do expect $\Gamma(\tau)$ to vanish as $\tau \to +\infty$, however it can not prove the ergodic hypothesis. We can only prove that $\mathring{\varphi}(\omega)$ equals $\langle \tilde{\varphi}(0) \rangle$ in the "mean-square" sense:

<u>Theorem</u>: Assume $\Gamma(\tau) \sim \tau^{-a}$ with $a > 1$. Then

$$\boxed{\langle (\mathring{\varphi} - \langle \tilde{\varphi}(0) \rangle)^2 \rangle = 0}$$

<u>Proof</u>

From theorem, p. 40 note that

$$\langle (\mathring{\varphi}_T - \langle \tilde{\varphi}(0) \rangle)^2 \rangle = \frac{2}{T^2} \int_0^T dt_1 \int_0^{t_1} dt_2 \, \Gamma(t_2) \leqslant$$

$$\leqslant \frac{2}{T^2} \int_0^T dt_1 \int_0^{t_1} dt_2 |\Gamma(t_2)| \leqslant \frac{2}{T^2} \int_0^T dt_1 \int_0^{+\infty} dt_2 |\Gamma(t_2)|$$

$$= \frac{2}{T^2} \int_0^T dt_1 \, \Gamma(0) T_I = \Gamma(0) T_I \frac{2}{T^2} \int_0^T dt_1 =$$

$$= 2\Gamma(0) \frac{T_I}{T}$$

Since $\lim_{T \to +\infty} \left[ 2\Gamma(0) \frac{T_I}{T} \right] = 0 \Rightarrow \langle (\mathring{\varphi} - \langle \tilde{\varphi}(0) \rangle)^2 \rangle = \lim_{T \to +\infty} \langle (\mathring{\varphi}_T - \langle \tilde{\varphi}(0) \rangle)^2 \rangle = 0$

$\square$

↳ The proof of the theorem also yields an estimate for the mean square error of partial time averaging in terms of the integral time scale:

$$\boxed{\langle (\mathring{\varphi}_T - \langle \tilde{\varphi}(0) \rangle)^2 \rangle \leqslant 2\Gamma(0) \frac{T_I}{T}}$$

## Spectrum of random functions in 1d

In the following analysis we will define and study the spectrum of stationary and locally stationary functions and show the connection between the spectrum and the variable's statistical properties.

Before we begin our analysis we recall some of our conventions. Our time-dependent random functions are assumed to be of the following form:

$$\tilde{\varphi} : \mathbb{R} \times \underline{\Omega} \to \mathbb{R}$$

Note that to simplify Fourier analysis we assume that time $t$ is defined in $t \in \mathbb{R}$.

The increment of the fluctuation of $\tilde{\varphi}$ is defined as:

$$\delta\varphi(t, \Delta t, \vartheta) = \varphi(t, \vartheta) - \varphi(t - \Delta t, \vartheta)$$

To state our definitions of stationarity we introduce a new notation which we call <u>equivalence in law</u>

<u>Definition</u>: Let $\varphi : V \times \underline{\Omega} \to \mathbb{R}$ and $\psi : V \times \underline{\Omega} \to \mathbb{R}$ be any two random functions. We say that $\varphi$ and $\psi$ are <u>equivalent in law</u>, which we denote $\underline{\underline{\varphi(t) \simeq \psi(t), \forall t \in V}}$ iff there is a transformation $g : \underline{\Omega} \to \underline{\Omega}$ that satisfies the following conditions:

a) $\mu(g^{-1} A) = \mu(A) \quad , \forall A \in \mathcal{B}$

b) $g^{-1} \underline{\Omega} = \underline{\Omega} \mod \mu$

c) $\varphi(t, \vartheta) = \psi(t, g\vartheta) \quad , \forall t \in V, \forall \vartheta \in \underline{\Omega}.$

When two random functions are equivalent in law they cannot be distinguished by any statistical measurements.

Using this notation we give the following concise definitions:

$$\boxed{\begin{array}{lcl} \varphi \text{ stationary} & \leftrightarrow & \varphi(t+\tau) \simeq \varphi(t), \forall t \in \mathbb{R}, \forall \tau \in \mathbb{R}. \\ \varphi \text{ locally stationary} & \leftrightarrow & \delta\varphi(t, \Delta t) \text{ stationary wrt } t \quad , \forall \Delta t > 0 \end{array}}$$

Except for a change in the domain of $t$ these definitions are logically equivalent to the ones we gave on page 35.

## Defining the spectrum

Definition: Let $\varphi(t,\vartheta)$ with $t \in \mathbb{R}$ be the fluctuation of a random function. Then we define:

a) The Fourier transform of $\varphi$ by

$$\boxed{\hat{\varphi}(\omega,\vartheta) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(t,\vartheta)\, e^{-i\omega t}\, dt}$$

b) The low-pass-filtered random function $\varphi^{<}_{\omega_0}(t,\vartheta)$ by

$$\boxed{\varphi^{<}_{\omega_0}(t,\vartheta) = \int_{-\omega_0}^{\omega_0} \hat{\varphi}(\omega,\vartheta)\, e^{i\omega t}\, d\omega}$$

c) The high-pass-filtered random function $\varphi^{>}_{\omega_0}(t,\vartheta)$ by

$$\boxed{\varphi^{>}_{\omega_0}(t,\vartheta) = \int_{\mathbb{R}-(-\omega_0,\omega_0)} \hat{\varphi}(\omega,\vartheta)\, e^{i\omega t}\, d\omega}$$

d) The cumulative spectrum of $\varphi$ by

$$\boxed{\mathcal{E}(t,\omega) = \frac{1}{2} \langle [\varphi^{<}_{\omega}(t)]^2 \rangle}$$

e) The spectrum of $\varphi$ by

$$\boxed{E(t,\omega) = \frac{d}{d\omega} \mathcal{E}(t,\omega)}$$

## Spectrum and variance

The spectrum of a random function is related with its variance with the following relation:

**Proposition** :

$$\boxed{\langle \varphi^2(t) \rangle = \int_0^{+\infty} 2E(t,\omega)\, d\omega}$$

**Proof**

Let $\mathcal{E}(t,\omega)$ be the cumulative spectrum. Then

$$\varphi_0^<(t) = 0 \Rightarrow \mathcal{E}(t,0) = \frac{1}{2}\langle \varphi_0^<(t)\, \varphi_0^<(t)\rangle = 0 \qquad (1)$$

Also:

$$\lim_{\omega\to+\infty} \varphi_\omega^<(t) = \lim_{\omega\to+\infty} \int_{-\omega}^{\omega} \hat{\varphi}(\omega_0)\, e^{i\omega_0 t}\, d\omega_0 =$$

$$= \int_{-\infty}^{+\infty} \hat{\varphi}(\omega_0)\, e^{i\omega_0 t}\, d\omega_0 = \varphi(t). \Rightarrow$$

$$\Rightarrow \lim_{\omega\to+\infty} \mathcal{E}(t,\omega) = \lim_{\omega\to+\infty} \frac{1}{2}\langle [\varphi_\omega^<(t)]^2\rangle =$$

$$= \frac{1}{2}\langle [\lim_{\omega\to+\infty} \varphi_\omega^<(t)]^2\rangle = \frac{1}{2}\langle \varphi^2(t)\rangle. \qquad (2).$$

It follows that

$$\int_0^{+\infty} E(t,\omega)\, d\omega = \int_0^{+\infty} \frac{d}{d\omega}\mathcal{E}(t,\omega)\, d\omega = \lim_{\omega\to+\infty} \mathcal{E}(t,\omega) - \mathcal{E}(t,0) =$$

$$= \frac{1}{2}\langle \varphi^2(t)\rangle - 0 = \frac{1}{2}\langle \varphi^2(t)\rangle. \qquad \square$$

~~Recall~~

In light of this result the ~~energy~~ spectrum $E(t,\omega)$ can be interpreted as a multiresolution analysis of the variance $\langle \varphi^2(t)\rangle$ that measures the contribution of each frequency to the variance. For variables with finite variance, $E(t,\omega)$ must be integrable over $\omega \in [0,+\infty)$.

## Filtering and equivalence in law

The motivation for the following analysis is to show that when a variable $\varphi$ is stationary its spectrum is also constant wrt $t$. This is a consequence of a general result about filtering:

<u>Theorem</u>: Let $\varphi(t,\vartheta)$ and $\psi(t,\vartheta)$ be the fluctuations of two random functions. Then:

$$\boxed{\varphi(t) \simeq \psi(t)\,,\ \forall t \in \mathbb{R} \Rightarrow (\forall \omega_0 > 0)\left(\varphi^{<}_{\omega_0}(t) \simeq \psi^{<}_{\omega_0}(t)\,,\ \forall t \in \mathbb{R}\right)}$$

<u>Proof</u>

$$\varphi(t) \simeq \psi(t)\,,\ t \in \mathbb{R} \Rightarrow (\exists g : \Omega \to \Omega)\left(\varphi(t,\vartheta) = \psi(t, g\vartheta)\,,\ \forall (t,\vartheta) \in \mathbb{R}\times\Omega\right).$$

Let $\hat{\varphi}(\omega)$ and $\hat{\psi}(\omega)$ be the Fourier transforms of $\varphi(t)$ and $\psi(t)$ respectively. Then

$$\hat{\varphi}(\omega,\vartheta) = \frac{1}{2\pi}\int_{\mathbb{R}} \varphi(t,\vartheta)\, e^{-i\omega t}\, dt =$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \psi(t, g\vartheta)\, e^{-i\omega t}\, dt = \hat{\psi}(\omega, g\vartheta). \qquad (1)$$

It follows from (1) that

$$\varphi^{<}_{\omega_0}(t,\vartheta) = \int_{-\omega_0}^{\omega_0} \hat{\varphi}(\omega,\vartheta)\, e^{i\omega t}\, d\omega = \int_{-\omega_0}^{\omega_0} \hat{\psi}(\omega, g\vartheta)\, e^{i\omega t}\, d\omega = \psi^{<}_{\omega_0}(t, g\vartheta)$$

$$\Rightarrow \varphi^{<}_{\omega_0}(t) \simeq \psi^{<}_{\omega_0}(t). \qquad \square$$

Applying this result on the definition of stationarity and local stationarity gives the following corollary:

<u>Corollary</u>:

$$\boxed{\begin{array}{lcl} \varphi \text{ stationary} & \Rightarrow & \varphi_\omega^< \text{ stationary} \\ \varphi \text{ locally stationary} & \Rightarrow & \varphi_\omega^< \text{ locally stationary} \end{array}}$$

From this it follows that:

<u>Theorem</u>:

$$\boxed{\varphi \text{ stationary} \Rightarrow E(t,\omega) = E(\omega)}$$

<u>Proof</u>

$$\varphi \text{ stationary} \Rightarrow \varphi_\omega^< \text{ stationary} \Rightarrow \frac{\partial}{\partial t} \langle [\varphi_\omega^<(t)]^2 \rangle = 0. \qquad (1)$$

It follows that

$$\frac{\partial E(t,\omega)}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{2}\frac{\partial}{\partial \omega} \langle [\varphi_\omega^<(t)]^2 \rangle\right] =$$

$$= \frac{1}{2}\frac{\partial}{\partial \omega}\left[\frac{\partial}{\partial t} \langle [\varphi_\omega^<(t)]^2 \rangle\right] = 0, \text{ By } (1)$$

Therefore $E(t,\omega) = E(\omega)$ $\square$

## <u>Local stationarity ~~can~~ with $E(t,\omega) = E(\omega)$</u>

In general, if a random function is locally stationary, its variance does not have to be constant in time. For example, <u>Brownian motion</u> does not have constant variance and yet it is locally stationary. From this it follows that local stationarity does not imply $E(t,\omega) = E(\omega)$. Nevertheless in the subsequent analysis we will study the case where

a) $\varphi$ locally stationary AND

b) $E(t,\omega) = E(\omega)$.

Incidently note that the assumption $E(t,\omega)=E(\omega)$ does <u>not</u> imply that $\varphi$ is stationary. It only guarantees that the variance is constant. Stationarity requires that ~~no~~ all moments be constant and we can not bring that out from $E(t,\omega)=E(\omega)$. So we really are studying a situation more general than just stationarity. We would like to relate the statistics of $\varphi$ with $E(\omega)$ under these assumptions.

Recall the definitions of the correlation and the structure functions:

$$\boxed{\begin{aligned}\Gamma(t_1,t_2) &= \langle \varphi(t_1)\varphi(t_2)\rangle \\ S_p(t,\Delta t) &= \langle [\delta\varphi(t,\Delta t)]^p \rangle\end{aligned}}$$

where $p>1$ is an integer. It is easy to see that $S_p(t,\Delta t)$. is independent of $t$. To bring that out note that:

$$\begin{aligned}\varphi \text{ stationary locally} &\Rightarrow \delta\varphi(t,\Delta t) \text{ stationary wrt } t \Rightarrow \\ &\Rightarrow S_p(t+\tau,\Delta t) = S_p(t,\Delta t)\end{aligned}$$

This allows us to introduce the p-order structure function as a function only of $\Delta t$: $S_p(\Delta t)$.

Local stationarity by itself is not sufficient to yield a similar result for $\Gamma(t_1,t_2)$. However the additional condition $E(t,\omega)=E(\omega)$ changes that.

<u>Proposition</u>: If $\varphi$ is locally stationary and $E(t,\omega)=E(\omega)$ then the correlation function satisfies

$$\Gamma(t_1,t_2) = \Gamma(t_1-t_2).$$

where $\Gamma(\tau)$ is given by

$$\boxed{\Gamma(\tau) = \langle \varphi^2 \rangle - \frac{1}{2} S_2(\tau)}$$

Proof

First note that $E(t,\omega) = E(\omega) \Rightarrow \langle \varphi^2(t) \rangle = \langle \varphi^2 \rangle = \text{const.}$ Local stationarity allows us to define $S_2(\tau)$ which is given by:

$$\begin{aligned} S_2(\tau) &= \langle \delta\varphi^2(t,\tau) \rangle = \langle [\varphi(t) - \varphi(t-\tau)]^2 \rangle = \\ &= \langle \varphi^2(t) - 2\varphi(t)\varphi(t-\tau) + \varphi^2(t-\tau) \rangle = \\ &= \langle \varphi^2 \rangle - 2\langle \varphi(t)\varphi(t-\tau) \rangle + \langle \varphi^2 \rangle = \\ &= 2\langle \varphi^2 \rangle - 2\Gamma(t,t-\tau). \Rightarrow \end{aligned}$$

$$\Rightarrow \Gamma(t,t-\tau) = \langle \varphi^2 \rangle - \frac{1}{2} S_2(\tau).$$

Because the RHS of this equation depends only on $\tau$ we have $\Gamma(t_1+\tau, t_2+\tau) = \Gamma(t_1,t_2)$. and
$\Gamma(\tau) = \Gamma(t-(t-\tau)) = \Gamma(t,t-\tau) = \langle \varphi^2 \rangle - \frac{1}{2} S_2(\tau).$ □

So even though we are not in the context of stationarity we may still talk about a correlation function $\Gamma(\tau)$. Because $\Gamma(\tau) = \Gamma(-\tau)$, it has a cosine expansion and a corresponding spectrum $\hat{\Gamma}(\omega)$:

$$\boxed{\begin{aligned} \Gamma(\tau) &= \int_0^{+\infty} \hat{\Gamma}(\omega)\cos(\omega\tau)\,d\omega \\ \hat{\Gamma}(\omega) &= \frac{2}{\pi}\int_0^{+\infty} \Gamma(\tau)\cos(\omega\tau)\,d\tau. \end{aligned}}$$

A result of fundamental importance is that $\hat{\Gamma}(\omega) = E(\omega)$. This is called the Wrenier-Khinchin formula. To bring it out we need to prove some preliminary results.

## Energy spectrum and frequency correlations.

Theorem : Assume $E(t,\omega)=E(\omega)$. Then the spectrum $E(\omega)$ is related with the frequency correlation $\langle \hat{\varphi}(\omega_1)\hat{\varphi}^*(\omega_2)\rangle$ by the following relation:

$$\boxed{E(\omega)=\int_{-\omega}^{\omega} 2\,\mathrm{Re}\,\langle \hat{\varphi}(\omega)\hat{\varphi}^*(\omega_0)\rangle \, d\omega_0}$$

Proof

First note that $\varphi_\omega^<(0,\vartheta)$ can be written in terms of its Fourier transform as follows:

$$\varphi_\omega^<(0)=\int_{-\omega}^{\omega}\hat{\varphi}(\omega_0)\,d\omega_0=\int_{-\omega}^{\omega}\hat{\varphi}^*(\omega_0)\,d\omega_0 \qquad (1)$$

where $\hat{\varphi}(-\omega_0)=\hat{\varphi}^*(\omega)$ because $\varphi(t)$ is real.
Its derivative with respect to $\omega$ is given by:

$$\frac{d}{d\omega}\varphi_\omega^<(0)=\frac{d}{d\omega}\int_{-\omega}^{\omega}\hat{\varphi}(\omega_0)\,d\omega_0=\frac{d}{d\omega}\int_{0}^{\omega}[\hat{\varphi}(\omega_0)+\hat{\varphi}^*(\omega_0)]\,d\omega_0=$$
$$=\hat{\varphi}(\omega)+\hat{\varphi}^*(\omega). \qquad (2)$$

Using (1) and (2) we now evaluate $E(\omega)$ from its definition:

$$E(\omega)=\frac{1}{2}\frac{d}{d\omega}\langle \varphi_\omega^<(0)\varphi_\omega^<(0)\rangle=\frac{1}{2}\left\langle \frac{d}{d\omega}\left[\varphi_\omega^<(0)\varphi_\omega^<(0)\right]\right\rangle=$$
$$=\frac{1}{2}\left\langle 2\varphi_\omega^<(0)\frac{d}{d\omega}\varphi_\omega^<(0)\right\rangle=\langle \varphi_\omega^<(0)[\hat{\varphi}(\omega)+\hat{\varphi}^*(\omega)]\rangle=$$
$$=2\,\mathrm{Re}\,\langle \varphi_\omega^<(0)\hat{\varphi}(\omega)\rangle$$

If we use the conjugate form of (1) onto our outcome for

$E(\omega)$, we have

$$E(\omega) = 2\mathrm{Re}\int_{\Theta} d\mu(\theta)\left[\int_{-\omega}^{\omega} d\omega_0\, \hat{\varphi}^*(\omega_0)\right]\hat{\varphi}(\omega) =$$

$$= 2\mathrm{Re}\int_{-\omega}^{\omega} d\omega_0 \int_{\Theta} d\mu(\theta)\, \hat{\varphi}(\omega)\hat{\varphi}^*(\omega_0) =$$

$$= 2\mathrm{Re}\int_{-\omega}^{\omega} \langle \hat{\varphi}(\omega)\hat{\varphi}^*(\omega_0)\rangle\, d\omega_0 = \int_{-\omega}^{\omega} 2\mathrm{Re}\langle \hat{\varphi}(\omega)\hat{\varphi}^*(\omega_0)\rangle\, d\omega_0. \quad \square$$

In general this is the most we can say about constant spectra. However, if $\Gamma(\tau)$ exists then we may compute the correlation $\langle \hat{\varphi}(\omega)\hat{\varphi}^*(\omega_0)\rangle$. To bring out that, first we derive the addition theorem.

## Addition theorem for 1d Fourier

By definition of $\delta(x)$ we have.

$$\int_{\mathbb{R}} \delta(\omega) e^{i\omega t}\, d\omega = e^{i0t} = 1 \Rightarrow \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t}\, dt = \delta(\omega)$$

Since $\delta(\omega) = \delta(-\omega)$ it follows that:

$$\boxed{\int_{\mathbb{R}} e^{i\omega t}\, dt = 2\pi\,\delta(\omega)}$$

The addition theorem is an extension of this result.

**Theorem**: $$\boxed{\int_{-\infty}^{+\infty} \cos[\omega_0(t_1 - t_2)]\, e^{-i\omega t_1}\, dt_1 = \pi\,\delta(\omega_0 - |\omega|)\, e^{-i\omega t_2}, \ \forall \omega_0 > 0}$$

Proof

By definition of $\delta(\omega-\omega_0)$, for $\omega_0>0$

$$\int_0^{+\infty}\delta(\omega-\omega_0)\cos(\omega t)\,d\omega = \cos(\omega_0 t) \Rightarrow \delta(\omega-\omega_0)=\frac{2}{\pi}\int_0^{+\infty}\cos(\omega_0 t)\cos(\omega t)\,dt$$

Note that $\omega_0>0$ is given positive, ~~here~~ however $\omega$ is not. Our expression above assumes that $\omega>0$, because we obtained it via inverting a cosine transform. To generalize for all $\omega$, we must place it in absolute value:

$$\delta(\omega_0-|\omega|)=\frac{2}{\pi}\int_0^{+\infty}\cos(\omega_0 t)\cos(\omega t)\,dt =$$

$$=\frac{2}{\pi}\int_0^{+\infty}\cos(\omega_0 t)\,\frac{e^{i\omega t}+e^{-i\omega t}}{2}\,dt =$$

$$=\frac{1}{\pi}\int_{-\infty}^{+\infty}\cos(\omega_0 t)\,e^{-i\omega t}\,dt = \qquad (\text{substitute } t_1=t+t_2)$$

$$=\frac{1}{\pi}\int_{-\infty}^{+\infty}\cos[\omega_0(t_1-t_2)]\exp[-i\omega(t_1-t_2)]\,dt_1$$

$$=\frac{1}{\pi}e^{i\omega t_2}\int_{-\infty}^{+\infty}\cos[\omega_0(t_1-t_2)]\,e^{-i\omega t_1}\,dt_1 \Rightarrow$$

$$\Rightarrow \int_{-\infty}^{+\infty}\cos[\omega_0(t_1-t_2)]\,e^{-i\omega t_1}\,dt_1 = \pi\,\delta(\omega_0-|\omega|)\,e^{-i\omega t_2} \qquad \square$$

Note that when $\omega_0=0$, the addition theorem reduces to an equation that is missing a factor of 2. The reason for this is that the range of applicability of the theorem is $\omega_0>0$. For $\omega_0=0$ the step in the above proof that breaks down is that

$$\int_0^{+\infty}\delta(\omega)\cos(\omega t)\,d\omega = \frac{1}{2}\cos(0)=\frac{1}{2}\neq 1$$

## The frequency correlations

**Theorem**: Assume $\Gamma(\tau)$ exists. If $\hat{\Gamma}(\omega)$ is the cosine transform of $\Gamma(\tau)$ then

$$\boxed{\langle \hat{\varphi}(\omega_1)\hat{\varphi}^*(\omega_2)\rangle = \frac{1}{2}\hat{\Gamma}(|\omega_1|)\delta(\omega_2-\omega_1)}, \quad \forall \omega_1 \neq 0$$

**Proof**

Recall that

$$\hat{\varphi}(\omega_1) = \frac{1}{2\pi}\int_{\mathbb{R}} \varphi(t_1)e^{-i\omega_1 t_1}\,dt_1$$

$$\hat{\varphi}^*(\omega_2) = \frac{1}{4\pi}\int_{\mathbb{R}} \varphi(t_2)e^{+i\omega_2 t_2}\,dt_2$$

It follows that:

$$\langle \hat{\varphi}(\omega_1)\hat{\varphi}^*(\omega_2)\rangle = \frac{1}{4\pi^2}\int_{\Omega} d\mu(\vartheta)\int_{\mathbb{R}^2} dt_1 dt_2\, \varphi(t_1)\varphi(t_2)e^{-i\omega_1 t_1}e^{+i\omega_2 t_2} =$$

$$= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} dt_1 dt_2\, e^{-i\omega_1 t_1}e^{+i\omega_2 t_2}\left\{\int_{\Omega} d\mu(\vartheta)\varphi(t_1,\vartheta)\varphi(t_2,\vartheta)\right\} =$$

$$= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} dt_1 dt_2\, \Gamma(t_1,t_2)e^{-i\omega_1 t_1}e^{+i\omega_2 t_2} =$$

$$= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} dt_1 dt_2\left[\int_0^{+\infty}\hat{\Gamma}(\omega)\cos[\omega(t_1-t_2)]\,d\omega\right]e^{-i\omega_1 t_1}e^{+i\omega_2 t_2}$$

$$= \frac{1}{4\pi^2}\int_0^{+\infty} d\omega\,\hat{\Gamma}(\omega)\left\{\int_{\mathbb{R}^2} dt_1 dt_2\cos[\omega(t_1-t_2)]e^{-i\omega_1 t_1}e^{+i\omega_2 t_2}\right\}$$

Now we use the addition lemma to evaluate the integral in the braces:

$$I = \int_{\mathbb{R}^2} dt_1 dt_2\cos[\omega(t_1-t_2)]e^{-i\omega_1 t_1}e^{+i\omega_2 t_2} =$$

$$= \int_{\mathbb{R}} dt_2\, e^{i\omega_2 t_2}\left\{\int_{\mathbb{R}} dt_1\cos[\omega(t_1-t_2)]e^{-i\omega_1 t_1}\right\} =$$

$$= \int_{\mathbb{R}} dt_2 \, e^{i\omega_2 t_2} \left[ \pi \delta(\omega - |\omega_1|) e^{-i\omega_1 t_2} \right] =$$

$$= \pi \delta(\omega - |\omega_1|) \int_{\mathbb{R}} dt_2 \, e^{i(\omega_2 - \omega_1) t_2} =$$

$$= \pi \delta(\omega - |\omega_1|) 2\pi \delta(\omega_2 - \omega_1) = 2\pi^2 \delta(\omega - |\omega_1|) \delta(\omega_2 - \omega_1).$$

We substitute this integral to our original expression and we get:

$$\langle \hat{\varphi}(\omega_1) \hat{\varphi}^*(\omega_2) \rangle = \frac{1}{4\pi^2} \int_0^{+\infty} \hat{\Gamma}(\omega) \left[ 2\pi^2 \delta(\omega - |\omega_1|) \delta(\omega_2 - \omega_1) \right] d\omega =$$

$$= \frac{\delta(\omega_2 - \omega_1)}{2} \int_0^{+\infty} \hat{\Gamma}(\omega) \delta(\omega - |\omega_1|) \, d\omega =$$

$$= \frac{\Gamma(|\omega_1|)}{2} \delta(\omega_2 - \omega_1) \qquad \square$$

► Note that this result holds only when $\omega_1 \neq 0$. For $\omega_1 = 0$ we get instead that:

$$\langle \hat{\varphi}(0) \hat{\varphi}^*(\omega_2) \rangle = \frac{1}{4\pi^2} \int_0^{+\infty} \hat{\Gamma}(\omega) \left[ \delta(\omega) \delta(\omega_2) \right] 2\pi^2 \, d\omega =$$

$$= \frac{\delta(\omega_2)}{2} \int_0^{+\infty} \hat{\Gamma}(\omega) \delta(\omega) \, d\omega = \frac{\hat{\Gamma}(0)}{4} \delta(\omega_2).$$

The general relation then is:

$$\boxed{\langle \hat{\varphi}(\omega_1) \hat{\varphi}^*(\omega_2) \rangle = \hat{\Gamma}(|\omega_1|) \delta(\omega_2 - \omega_1) \begin{cases} 1/2 & , \ \omega_1 \neq 0 \\ 1/4 & , \ \omega_1 = 0 \end{cases}}$$

A consequence of this result is that when $\Gamma(\tau)$ exists, different frequencies $\omega_1 \neq \omega_2$ are uncorrelated.

## The Wiener-Khinchin relation

<u>Theorem</u> : Let $\varphi$ be a fluctuation such that
a) $E(t,\omega) = E(\omega)$
b) $\varphi$ locally stationary
Then the following relation is true:

$$\boxed{E(\omega) = \frac{1}{\pi}\int_0^{+\infty} \Gamma(\tau)\cos(\omega\tau)\,d\tau}$$

$$\boxed{\Gamma(\tau) = \int_0^{+\infty} 2E(\omega)\cos(\omega\tau)\,d\omega}$$

<u>Proof</u>

From theorem, p.55 we compute $E(\omega)$:

$$E(\omega) = \int_{-\omega}^{\omega} d\omega_0\, 2\,\mathrm{Re}\,\langle\hat\varphi(\omega)\hat\varphi^*(\omega_0)\rangle\, d\omega_0 =$$

(theorem, p. 58)

$$= \int_{-\omega}^{\omega} d\omega_0\, 2\mathrm{Re}\left[\frac{\hat\Gamma(|\omega|)}{2}\delta(\omega_0-\omega)\right] =$$

($\hat\Gamma(\omega)$ is real-valued)

$$= \int_{-\omega}^{\omega} d\omega_0\, \hat\Gamma(|\omega|)\,\delta(\omega_0-\omega) =$$

($\omega>0$).

$$= \frac{\hat\Gamma(\omega)}{2}$$

It follows that

$$E(\omega) = \frac{\hat\Gamma(\omega)}{2} = \frac{1}{2}\,\frac{2}{\pi}\int_0^{+\infty}\Gamma(\tau)\cos(\omega\tau)\,d\tau = \frac{1}{\pi}\int_0^{+\infty}\Gamma(\tau)\cos(\omega\tau)\,d\tau$$

and

$$\Gamma(\tau) = \int_0^{+\infty}\hat\Gamma(\omega)\cos(\omega\tau)\,d\omega = \int_0^{+\infty} 2E(\omega)\cos(\omega\tau)\,d\omega. \qquad \square$$

<u>Corollary</u>: Let $\varphi$ be a fluctuation such that
a) $E(t,\omega) = E(\omega)$
b) $\varphi$ locally stationary
Then the following relation is true:

$$\boxed{\begin{array}{l} S_2(\tau) = 8\displaystyle\int_0^{+\infty} E(\omega)\sin^2(\omega\tau/2)\,d\omega \\ \hline E(\omega) = \langle\varphi^2\rangle\delta(\omega) - \dfrac{1}{2\pi}\displaystyle\int_0^{+\infty} S_2(\tau)\cos(\omega\tau)\,d\tau \end{array}}$$

<u>Proof</u>

Recall that

$$\langle\varphi^2\rangle = \int_0^{+\infty} 2E(\omega)\,d\omega \qquad \text{(proposition, p. 50)}$$

$$\Gamma(\tau) = \int_0^{+\infty} 2E(\omega)\cos(\omega\tau)\,d\omega \qquad \text{(Wienier - khinchin)}$$

$$S_2(\tau) = 2(\langle\varphi^2\rangle - \Gamma(\tau)) \qquad \text{(proposition, p. 53)}$$

It follows from all that:

$$S_2(\tau) = 2\left[\langle\varphi^2\rangle - \Gamma(\tau)\right] = 4\int_0^{+\infty} E(\omega)\,d\omega - 4\int_0^{+\infty} E(\omega)\cos(\omega\tau)\,d\omega =$$

$$= 8\int_0^{+\infty} E(\omega)\,\frac{1-\cos(\omega\tau)}{2}\,d\omega = 8\int_0^{+\infty} E(\omega)\sin^2(\omega\tau/2)\,d\omega.$$

To show the converse:

$$E(\omega) = \frac{1}{\pi}\int_0^{+\infty} \Gamma(\tau)\cos(\omega\tau)\,d\tau = \frac{1}{\pi}\int_0^{+\infty}\left[\langle\varphi^2\rangle - \frac{1}{2}S_2(\tau)\right]\cos(\omega\tau)\,d\cancel{\omega}\tau$$

$$= \langle\varphi^2\rangle\,\frac{1}{\pi}\int_0^{+\infty}\cos(\omega\tau)\,d\cancel{\omega}\tau - \frac{1}{2\pi}\int_0^{+\infty} S_2(\tau)\cos(\omega\tau)\,d\tau$$

$$= \langle\varphi^2\rangle\,\frac{1}{\pi}\,\pi\,\delta(\omega) - \frac{1}{2\pi}\int_0^{+\infty} S_2(\tau)\cos(\omega\tau)\,d\tau$$

$$= \langle\varphi^2\rangle\delta(\omega) - \frac{1}{2\pi}\int_0^{+\infty} S_2(\tau)\cos(\omega\tau)\,d\tau. \quad \square$$

## Comments and extensions.

The significance of the Wienier-Khinchin relations call for some comments:

1) First note that all of the development (p. 48–61) depends on the assumption that the variance is finite. In light of theorem, p.50, this means that we may assume that the integral

$$\langle \varphi^2(t) \rangle = \int_0^{+\infty} E(\omega)\, d\omega$$

converges. We will consider the limit of infinite variance in the next section.

2) The assumption $E(t,\omega) = E(\omega)$ requires justification. It is wrong to assume that it arises from the limit $t \to +\infty$. In fact it can be shown that if $E(t,\omega) \neq E(\omega)$ then that will remain so even as $t \to +\infty$. The correct justification is that there is no time dependence because we <u>averaged time out.</u> Thus we may trade this assumption with a redefinition of $\langle \varphi(t) \rangle$ as ensemble + time average. In the analysis of turbulence we work with ensemble + spatial average and retain time dependence.

3) When the variance $\langle \varphi^2 \rangle$ is finite, the Wienier-Khinchin relations tell us that a given $\Gamma(\tau)$ corresponds to a unique spectrum $E(\omega)$ and vice-versa. Similarly a given $S_2(\tau)$ corresponds to a unique spectrum and vice-versa. This result needs to be reexamined in the limit $\langle \varphi^2 \rangle \to +\infty$.

4) We know, intuitively, that $\lim_{\tau \to +\infty} \Gamma(\tau) = 0$. We can put this on a more firm basis by noting that it has to be true in order to get a finite energy spectrum $E(\omega)$. Let $\Gamma_0(\tau)$ be a physically correct correlation function. with corresponding spectrum $E_0(\omega)$. Shifting $\Gamma_0(\tau)$ by a constant $\lambda$

implies: $\Gamma(\tau) = \Gamma_0(\tau) + \lambda \Rightarrow E(\omega) = E_0(\omega) + \lambda\delta(\omega)$

So a non-vanishing correlation function appears in the spectrum in the form of $\delta$-peaks.

5) The relation between $S_2$ and $\Gamma$ was shown to be (see p. 53)

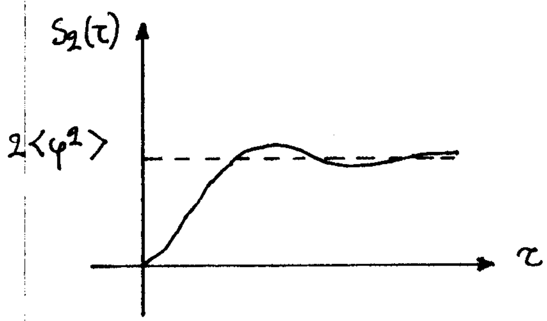
$$S_2(\tau) = 2\langle\varphi^2\rangle - 2\Gamma(\tau).$$

It follows from this that

$$S_2(0) = 2\langle\varphi^2\rangle - 2\Gamma(0) = 2\langle\varphi^2\rangle - 2\langle\varphi^2\rangle = 0$$

$$\lim_{\tau\to+\infty} S_2(\tau) = 2\langle\varphi^2\rangle - 2\lim_{\tau\to+\infty}\Gamma(\tau) = 2\langle\varphi^2\rangle$$

This suggests that the "typical" picture for $S_2(\tau)$ is as follows:



## The limit of infinite variance

We describe our approach to $\langle\varphi^2\rangle \to +\infty$ by giving a sequence of spectra $\langle E_n(\omega) \mid n \in \mathbb{N} \rangle$ that converge uniformly to some limit $E(\omega)$.

Let $S_2^n(\tau)$ be the structure function that corresponds to $E_n(\omega)$.

We ask the following questions.

1) When does the limit $\lim S_2^n(\tau)$ converge?
2) What does it converge to?
3) * If we know that $\lim S_2^n(\tau)$ converges what can we say about $\lim\limits_{n \in \mathbb{N}} E_n(\omega)$ ?

The last question is a crucial step in the development of K41 theory.

<u>Theorem</u>: Assume that

a) $\lim\limits_{\omega \to +\infty} \omega^{1+\varepsilon_1} E(\omega) = 0$ for some $\varepsilon_1 > 0$

b) $\lim\limits_{\omega \to 0+} \omega^{3+\varepsilon_2} E(\omega) = \ell \in \mathbb{R}$ for some $\varepsilon_2 > 0$

Then $\lim S_2^n$ converges and

$$\boxed{S_2(\tau) = \lim_{n \in \mathbb{N}} S_2^n(\tau) = \int_0^{+\infty} 8 E(\omega) \sin^2(\omega\tau/2)\, d\omega}$$

<u>Proof</u>

Recall that (p. 61)

$$S_2^n(\tau) = \int_0^{+\infty} 8 E_n(\omega) \sin^2(\omega\tau/2)\, d\omega$$

It follows that

$$S_2(\tau) = \lim_{n\in\mathbb{N}} S_2^n(\tau) = \lim_{n\in\mathbb{N}} \int_0^{+\infty} 8E_n(\omega)\sin^2(\omega\tau/2)\,d\omega =$$

$$= \int_0^{+\infty} 8\left[\lim_{n\in\mathbb{N}} E_n(\omega)\right]\sin^2(\omega\tau/2)\,d\omega =$$

$$= \int_0^{+\infty} 8E(\omega)\sin^2(\omega\tau/2)\,d\omega \qquad (1)$$

The last step can be justified only if the resulting integral converges. To show that note that

$$\left.\begin{array}{l}\lim\limits_{\omega\to+\infty} \omega^{1+\varepsilon_1}E(\omega) = 0 \\ \sin^2(\omega\tau/2)\ \text{bounded}\end{array}\right\} \Rightarrow \lim_{\omega\to+\infty} \omega^{1+\varepsilon_1}E(\omega)\sin^2(\omega\tau/2) = 0$$

$$\Rightarrow \int_a^{+\infty} E(\omega)\sin^2(\omega\tau)\,d\omega \quad \text{converges for } a>0.$$

Moreover,

$$\lim_{\omega\to0^+} \omega^{1-\varepsilon_2}E(\omega)\sin^2(\omega\tau/2) = \lim_{\omega\to0^+}\left[\omega^{3-\varepsilon_2}E(\omega)\,\frac{\sin^2(\omega\tau/2)}{\omega^2}\right] =$$

$$= \lim_{\omega\to0^+}\left[\omega^{3-\varepsilon_2}E(\omega)\right]\lim_{\omega\to0^+}\frac{\sin^2(\omega\tau/2)}{\omega^2} =$$

$$= \ell\cdot(\tau/2)^2 \Rightarrow$$

$$\Rightarrow \int_0^a E(\omega)\sin^2(\omega\tau/2)\,d\omega \quad \text{converges.}$$

Therefore (1) converges □

This answers questions 1 and 2. To answer 3 we use the following result:

<u>Theorem</u>: Suppose that we have a sequence of structure functions $\langle S_2^n(\tau) \mid n \in \mathbb{N} \rangle$ each of which has finite variance. Let $S_2 = \lim S_2^n$ the uniform limit of that sequence and let $E_0(\omega)$ be a spectrum that satisfies the relation

$$S_2(\tau) = \int_0^{+\infty} E_0(\omega) \sin^2(\omega\tau/2)\, d\omega \quad , \forall \tau > 0$$

Then the corresponding sequence of spectra $\langle E_n(\omega) \mid n \in \mathbb{N} \rangle$ also converges uniformly and there is a $\lambda \in \mathbb{R}$ such that

$$\boxed{\lim_{n \in \mathbb{N}} E_n(\omega) = \lambda \delta(\omega) + E_0(\omega)}$$

<u>Proof</u>

From the relation $\quad S_2^n(\tau) = \int_0^{+\infty} 8 E_n(\omega) \sin^2(\omega\tau/2)\, d\omega$

we see that $\lim_{n \in \mathbb{N}} E_n(\omega)$ diverges $\rightarrow \lim_{n \in \mathbb{N}} S_2^n(\tau)$ diverges.

By contrapositive we conclude that $\lim_{n \in \mathbb{N}} E_n(\omega) = E(\omega)$ converges. Define $f(\omega) = E(\omega) - E_0(\omega)$. To find $f(\omega)$ we argue as follows: By theorem, p. 64 we have that

$$S_2(\tau) = \int_0^{+\infty} E(\omega) \sin^2(\omega\tau/2)\, d\omega \quad , \forall \tau > 0$$

so

$$8\int_0^{+\infty} f(\omega) \sin^2(\omega\tau/2)\, d\omega = 8\int_0^{+\infty} [E(\omega) - E_0(\omega)] \sin^2(\omega\tau/2)\, d\omega =$$

$$= 8\int_0^{+\infty} E(\omega) \sin^2(\omega\tau/2)\, d\omega - 8\int_0^{+\infty} E_0(\omega) \sin^2(\omega\tau/2)\, d\omega =$$

$$= S_2(\tau) - S_2(\tau) = 0 \quad , \forall \tau > 0 \Rightarrow$$

$$\Rightarrow \int_0^{+\infty} f(\omega)\sin^2(\omega\tau/2)\,d\omega = 0 \quad , \forall \tau > 0 \qquad (1)$$

We claim that this implies that $f(\omega) = \lambda\delta(\omega)$.
To bring this out we recall from the theory of generalized functions that every generalized function is definable as the transform of a non-generalized function. Conversely, the transform of an arbitrary non-generalized function defines a generalized function, when it diverges. In the context of the spectrum, and $f(\omega)$ in particular, the relevant transform is the <u>cosine transform</u>. Let $g(t)$ be a non-generalized function such that

$$f(\omega) = \int_0^{+\infty} g(t)\cos(\omega t)\,dt$$

Then we evaluate the integral in (1) as follows:

$$I = \int_0^{+\infty} f(\omega)\sin^2(\omega\tau/2)\,d\omega = \int_0^{+\infty}\left[\int_0^{+\infty} g(t)\cos(\omega t)\,dt\right]\sin^2(\omega\tau/2)\,d\omega =$$
$$= \int_0^{+\infty} dt\; g(t)\left\{\int_0^{+\infty}\cos(\omega t)\sin^2(\omega\tau/2)\,d\omega\right\}$$
$$= \int_0^{+\infty} dt\; g(t)\left\{\int_0^{+\infty}\cos(\omega t)\,\frac{1-\cos(\omega\tau)}{2}\,d\omega\right\} =$$
$$= \int_0^{+\infty} dt\; g(t)\left\{\frac{1}{2}\int_0^{+\infty}\cos(\omega t)\,d\omega - \frac{1}{2}\int_0^{+\infty}\cos(\omega\tau)\cos(\omega t)\,d\omega\right\} =$$
$$= \frac{\pi}{2}\int_0^{+\infty} dt\; g(t)\left[\delta(t) - \delta(t-\tau)\right] = \frac{\pi}{2}\left(g(0) - g(\tau)\right)$$

It follows from (1) that

$$g(0) - g(\tau) = 0 \,,\ \forall \tau > 0 \Rightarrow g(\tau) = c = \text{constant} \,,\ \forall \tau \geq 0. \Rightarrow$$

$$\Rightarrow f(\omega) = \int_0^{+\infty} c\cdot\cos(\omega t)\,dt = c\pi\delta(\omega) = \lambda\delta(\omega). \Rightarrow$$

$$\Rightarrow \underline{E(\omega) = \lambda\delta(\omega)} \qquad \square$$

The significance of the previous theorem is the following: there is no unique $E(\omega)$ corresponding to an infinite variance $S_2(\tau)$. We obtain $\delta$-peak terms of various magnitudes in $E(\omega)$ depending on what sequence $\langle S_2^n(\tau) \mid n \in \mathbb{N} \rangle$ we've used to approach $S_2(\tau)$. Note that we have already placed a constraint on $S_2^n(\tau)$ by requiring that each member of the sequence has finite variance. That means that the limit of $S_2^n(\tau)$ as $\tau \to +\infty$ has to be finite. Without that constraint even higher derivatives of the $\delta$-function would appear in $E(\omega)$. Then $E(\omega)$ would have the form:

$$E(\omega) = E_0(\omega) + \lambda_0 \delta(\omega) + \lambda_1 \delta''(\omega) + \lambda_2 \delta''''(\omega) + \cdots$$

which would be very unphysical.

The meaning of a $\lambda\delta(\omega)$ term in the spectrum was discussed in comment 4, page 62-63. In light of that discussion we see that if we add a second constraint on the sequence $\langle S_2^n(\tau) \mid n \in \mathbb{N} \rangle$ and require that the corresponding correlation functions satisfy

$$\cancel{\lim_{\tau \to +\infty} \Gamma_n(\tau) = 0} \quad \lim_{\tau \to +\infty} \lim_{n \in \mathbb{N}} \Gamma_n(\tau) = 0$$

then we obtain a unique spectrum $E(\omega)$ that does not have a $\lambda\delta(\omega)$ term. This requirement, in physical terms, states that the behaviour of $\varphi(t)$ at large scales is uncorrelated.

## Power-law spectra and structure functions.

The following result is the precursor of a more general statement that has crucial importance in K41 theory.

<u>Theorem</u>: Let $\varphi$ be a locally stationary fluctuation with structure function $S_2$ and spectrum $E$. Then:

$$\boxed{E(\omega) = a\omega^{-n} \Rightarrow S_2(\tau) = b\tau^{n-1}}$$

where $1<n<3$, $\lambda \geq 0$ and $a, b$ are related by:

$$\boxed{b = \frac{8a}{2^{n-1}} \int_0^{+\infty} x^{-n} \sin^2 x \, dx}$$

<u>Proof</u>:

$$S_2(\tau) = 8\int_0^{+\infty} E(\omega)\sin^2(\omega\tau/2)\,d\omega = 8a\int_0^{+\infty} \omega^{-n}\sin^2(\omega\tau/2)\,d\omega$$

Substitute $x = \omega\tau/2 \Rightarrow dx = (\tau/2)d\omega \Rightarrow d\omega = (\tau/2)^{-1}dx$

Also $\omega = (\tau/2)^{-1}x$ and $\omega = 0 \Rightarrow x = 0$

$\omega = +\infty \Rightarrow x = +\infty$

It follows that:

$$S_2(\tau) = 8a\int_0^{+\infty} [(\tau/2)^{-1}x]^{-n}(\sin^2 x)(\tau/2)^{-1}dx =$$

$$= 8a(\tau/2)^{n-1}\int_0^{+\infty} x^{-n}\sin^2 x\,dx = b\tau^{n-1} \qquad \square$$

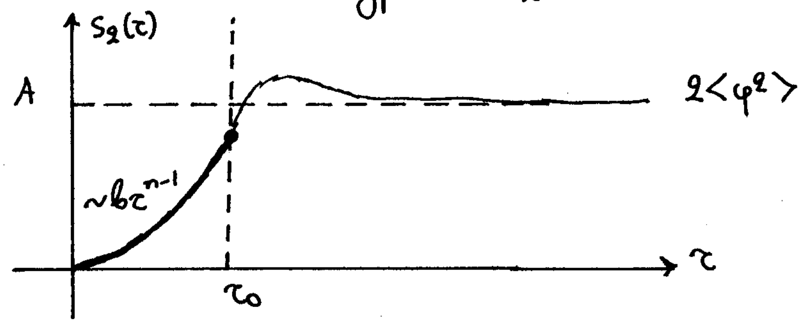
The interesting aspect is the converse, as implied by our results in pages 66-67. Suppose that we know that $S_2(\tau)$ takes the following form:

$$S_2(\tau) = b\tau^{n-1}\mathcal{F}(\tau/\tau_0) + A\,(1-\mathcal{F}(\tau/\tau_0)).$$

where $\mathcal{F}$ is a function such that

$$\mathcal{F}(x) \cong 1 \quad \text{when} \quad x \ll 1$$
$$\mathcal{F}(x) \cong 0 \quad \text{when} \quad x \gg 1$$

Then for $\tau \ll \tau_0$, $S_2(\tau)$ follows a power-law and for $\tau \gg \tau_0$ it has converged to the variance $A = 2\langle\varphi^2\rangle$.
A sketch of a typical $S_2(\tau)$ is shown bellow:




For every $\tau_0$ there corresponds a spectrum $E(\omega;\tau_0)$. What can we say about $E$ in the limit $\tau_0 \to \infty$?
This is an application of theorem, p.66 with $E_0(\omega) = a\omega^{-n}$.
It follows that

$$\lim_{\tau_0 \to +\infty} E(\omega;\tau_0) = a\omega^{-n}$$

Of cource, it is easy to see that the limit $\tau_0 \to +\infty$ corresponds to infinite variance, which is non-physical; $S_2(\tau)$ grows without bound which pushes the $2\langle\varphi^2\rangle$ horizontal line to infinity. Therefore we must ask what happens when $\tau_0$ is finite. Then $E(\omega)$ follows a power-law <u>approximately</u> only within an <u>inertial range</u> $\omega_0 < \omega < \omega_1$ where $\omega_0 \sim \tau_0^{-1}$.

A generalization of this result is the basis for deriving the inertial range energy spectrum in the K41 theory.

## Homogeneity and isotropy in general

Now we turn to functions of the form
$$\varphi : \mathbb{R}^n \times \underline{\Omega} \longrightarrow \mathbb{R}^m$$
where $n=2$ or $n=3$ depending on whether we deal with a 2d or 3d problem. The value of $m$ depends on the nature of $\varphi$. In particular

a) $m=1$, when $\varphi$ is a scalar
b) $m=n$, when $\varphi$ is a vector
c) $m=n^2$, when $\varphi$ is a tensor

We will consider these ~~spatial~~ special cases later. For now we develop our definitions in a general setting.
The increment function $\delta\varphi$ is defined consistently with p. 48 by:
$$\delta\varphi(x,\Delta x,\omega) = \varphi(x,\omega) - \varphi(x-\Delta x,\omega).$$
We interpret $x \in \mathbb{R}^n$ as a spatial variable.
We begin with the meaning of scalar, vector and tensor.

### The rotation group and its representations

Let ${}^{n\times n}\mathbb{R}$ be the set of all $n\times n$ real matrices.
This is a specialization of the standard notation ${}^{A}B$ for the set of all functions $f: A \rightarrow B$.
We introduce the following subgroups of $({}^{n\times n}\mathbb{R}, \cdot)$ where "$\cdot$" is matrix multiplication.

Definition:

a) The group of orthogonal matrices $O(n)$ is defined by:
$$O(n) = \{A \in {}^{n\times n}\mathbb{R} \mid A^T A = A A^T = I\}$$
b) The rotation group $SO(n)$ is defined by
$$SO(n) = \{A \in O(n) \mid \det A = 1\}$$

It is easy to verify that these are indeed groups by proving the statement

$$A \in G \wedge B \in G \Rightarrow AB^{-1} \in G$$

for $G = O(n)$ and $G = SO(n)$.

We define the action of $O(n)$ and $SO(n)$ on $\mathbb{R}^n$ by the usual matrix-vector multiplication.

When we apply $A$ on a vector $x \in \mathbb{R}^n$ to obtain

$$y = Ax$$

Then $y_i$ is the projection of $x$ on the axis defined by the $i^{th}$ row of $A$. Thus $A$ defines a new set of axises and thus a new coordinate frame.

If $A \in O(n)$, then the new set of axis remains orthonormal, so the group $O(n)$ represents <u>rotations</u> and <u>reflections</u>

If $A \in SO(n)$, then no axis is reflected, and $SO(n)$ restricts $O(n)$ only to <u>rotations</u>.

To bring out this interpretation we argue as follows:

<u>Lemma</u>: $\boxed{A \in O(n) \Rightarrow \det A = 1 \vee \det A = -1}$

<u>Proof</u>

$A \in O(n) \Rightarrow \underline{AA^T = I}$ (1)

Then

$$[\det A]^2 = (\det A)(\det A) = (\det A)(\det A^T) =$$
$$= \det(AA^T) = \det I = 1 \Rightarrow$$
$$\Rightarrow \det A = 1 \vee \det A = -1 \qquad \square$$

We may used this result to motivatge <u>parity</u>:

<u>Definition</u>: Let $A \in O(n)$ be a transformation. We define the parity of $A$ by

$$\boxed{\varepsilon(A) = \det A}$$

Clearly when $\varepsilon(A)=1$, then $A$ is a rotation. We expect then that the meaning of $\varepsilon(A)=-1$ is that it involves a reflection. We would like to make that more precise. First we define reflection groups:

<u>Definition</u>: Let $1 \leq k \leq n$ be an integer:

a) The <u>reflection matrix</u> $P_n(k) = [p_{ij}]$ is defined by:

$$\boxed{p_{ij} = \delta_{ij}(1 - 2\delta_{ik})}$$

b) The <u>reflection group</u> $\mathcal{P}_k(n)$ is the subgroup of ${}^{n\times n}\mathbb{R}$ that has the following elements:

$$\boxed{\mathcal{P}_k(n) = \{I_n, P_n(k)\}}$$

The multiplication table of $\mathcal{P}_k(n)$ can be shown to be:

| $\mathcal{P}_k(n)$ | $I_n$ | $P_n(k)$ |
|---|---|---|
| $I_n$ | $I_n$ | $P_n(k)$. |
| $P_n(k)$ | $P_n(k)$ | $I_n$ |

The group $\mathcal{P}_k(n)$ acts on an $n$-dimensional geometry by reflecting the $k$-axis, and leaving everything else the same.

<u>example</u>: For $n=3$:

$$P_3(1) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_3(2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_3(3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$\boxed{\mathcal{P}_1(3)} \qquad \boxed{\mathcal{P}_2(3)} \qquad \boxed{\mathcal{P}_3(3)}$$

Second, we define a method for composing groups:

<u>Definition</u>: Let $G_1, G_2$ be subgroups of a common group $G$. The product set $G_1G_2$ is given by:

$$\boxed{G_1G_2 = \{g_1g_2 \mid g_1 \in G_1 \land g_2 \in G_2\}}$$

In general $G_1G_2$ is not a group. However iff $G_1G_2 = G_2G_1$ then it is a group too!

<u>Proposition</u>: $\boxed{G_1G_2 \text{ is a group} \Leftrightarrow G_1G_2 = G_2G_1}$

<u>Proof</u>

<u>($\Rightarrow$)</u>: Assume $G_1G_2$ is a group.
We will show that $(\forall a \in G)(a \in G_1G_2 \Leftrightarrow a \in G_2G_1)$.
Let $a \in G$ be given.
$a \in G_1G_2 \Rightarrow (\exists a_1 \in G_1)(\exists a_2 \in G_2)(a = a_1a_2)$
It follows that

$$\left.\begin{array}{l} a^{-1} = (a_1a_2)^{-1} = a_2^{-1}a_1^{-1} \\ a_2^{-1} \in G_2 \land a_1^{-1} \in G_1 \end{array}\right\} \Rightarrow a^{-1} \in G_2G_1 \Rightarrow a \in G_2G_1$$

So $a \in G_1G_2 \Rightarrow a \in G_2G_1$, $\forall a \in G$.
The converse can be shown similarly, so $G_1G_2 = G_2G_1$.

<u>($\Leftarrow$)</u>: Assume $G_1G_2 = G_2G_1$
Let $a, b \in G_1G_2$ be given. We will show that $ab^{-1} \in G_1G_2$.
$a \in G_1G_2 \Rightarrow (\exists a_1 \in G_1)(\exists a_2 \in G_2)(a = a_1a_2)$
$b \in G_1G_2 \Rightarrow (\exists b_1 \in G_1)(\exists b_2 \in G_2)(b = b_1b_2)$
It follows that

$$ab^{-1} = (a_1a_2)(b_1b_2)^{-1} = (a_1a_2)(b_2^{-1}b_1^{-1}) = a_1(a_2b_2^{-1})b_1^{-1} \quad (1)$$

Define $\gamma = (a_2b_2^{-1})b_1^{-1}$. It suffices to show that $\gamma \in G_1G_2$

$$\left.\begin{array}{l} \left.\begin{array}{l} a_2 \in G_2 \\ b_2^{-1} \in G_2 \end{array}\right\} \Rightarrow a_2b_2^{-1} \in G_2 \\ \qquad\qquad b_1^{-1} \in G_1 \end{array}\right\} \Rightarrow \gamma \in G_2G_1 = G_1G_2 \Rightarrow \gamma \in G_1G_2 \Rightarrow$$

$\Rightarrow (\exists \gamma_1 \in G_1)(\exists \gamma_2 \in G_2)(\gamma = \gamma_1 \gamma_2)$.

It follows that

$ab^{-1} = a_1 (a_2 b_2^{-1}) b_1^{-1} = a_1 \gamma = a_1 \gamma_1 \gamma_2 = (a_1 \gamma_1)\gamma_2 \in G_1 G_2$

because $a_1 \gamma_1 \in G_1$ and $\gamma_2 \in G_2$.

Since

$(\forall a, b \in G_1 G_2)(ab^{-1} \in G_1 G_2) \Rightarrow G_1 G_2$ is a group. $\square$

The main result, that "$O(n)$ is rotations and reflections" is the following theorem:

<u>Theorem</u>: $\boxed{O(n) = SO(n) \mathcal{P}_k(n) = \mathcal{P}_k(n) SO(n), \ \forall k \in \{1, 2, \dots, n\}}$

<u>Proof</u>

It is easy to see that $SO(n) \subset O(n)$ and $\mathcal{P}_k(n) \subseteq O(n)$. The latter follows from computing and showing that

$$P_n(k) P_n^T(k) = P_n^T(k) P_n(k) = I$$

which is trivial, since these are diagonal matrices with 1 or $-1$ on the diagonal.

It follows that $SO(n)\mathcal{P}_k(n) \subseteq O(n)$ so it suffices to show that $O(n) \subseteq SO(n)\mathcal{P}_k(n)$.

Let $A \in O(n)$ be given. Consider two cases:

a) If $\det A = 1 \Rightarrow \left.\begin{array}{l} A \in SO(n) \\ A = AI_n \end{array}\right\} \Rightarrow A \in SO(n)\mathcal{P}_k(n)$

b) If $\det A = -1$ then define $B = AP_n(k)$. It follows that

$\left.\begin{array}{l} \det B = \det(AP_n(k)) = (\det A)(\det P_n(k)) = (-1)(-1) = +1 \Rightarrow \\ B^{-1} = (AP_n(k))^{-1} = P_n^{-1}(k) A^{-1} = P_n^T(k) A^T = (AP_n(k))^T = B^T \end{array}\right\} \Rightarrow$

$\Rightarrow \underline{B \in SO(n)}.$

Since $A = AI_n = AP_n^2(k) = (AP_n(k))P_n(k) = BP_n(k) \Rightarrow$

$\Rightarrow A \in SO(n)\mathcal{P}_k(n)$.

In both cases we have $A \in SO(n)\mathcal{P}_k(n)$ so $O(n) \subseteq SO(n)\mathcal{P}_k(n)$.

It follows that $\underline{O(n) = SO(n)\,P_K(n)}$ (1)
To complete the proof note that (1) implies that $SO(n)\,P_K(n)$ is a group $\Rightarrow$ $\underline{SO(n)\,P_K(n) = P_K(n)\,SO(n)}$

Let $x \in \mathbb{R}^n$. Then it is easy to see that $SO(n)x$ is an $n$-sphere with radius $r = \|x\|$. Thus we may define an isomorphism between $SO(n)$ and any coordinate representation of an $n$-sphere. Recall that a point on an $n$-sphere with radius $r$ can be fixed by $n-1$ additional angles $\vec{\varphi} \in [0,\pi]^{n-2} \times [0, 2\pi)$, and its cartesian coordinates are given by:

$$\boxed{\begin{aligned} &x_1 = r\cos\varphi_1 \quad , \quad x_n = r\prod_{i=1}^{n-1} \sin\varphi_i \\ &x_K = r\cos\varphi_K \prod_{i=1}^{K-1} \sin\varphi_i \quad , \quad \forall K \in \{2,\dots,n-1\} \end{aligned}}$$

It can be shown that the isomorphism between the sphere and $SO(n)$ can be written as a product of matrix exponentials

$$\boxed{A = \prod_{i=1}^{n-1} \exp\left[i\varphi_i E_i\right]} \quad , \quad \forall A \in SO(n)$$

where $E_i$ are generators that correspond to the successive rotations. We may define a measure $d\underline{\Omega}$ on $SO(n)$ as the "surface area" of the corresponding patch on the sphere. It can be shown then that

$$\boxed{d\underline{\Omega} = \prod_{K=1}^{n-1} (\sin\varphi_K)^{n-1-K}\, d\varphi_K} \quad \text{(solid angle measure).}$$

The spherical $\to$ cartesian coordinate transformation are an isomorphism for $\mathbb{R}^n$: $\mathbb{R}^n \cong [0, +\infty) \times SO(n)$ The volume measure of $\mathbb{R}^n$, in spherical representation is given by

$$\boxed{dV = r^{n-1}\, dr\, d\underline{\Omega}}$$

## Physical quantities and functions

The mathematical representation of a physical quantity, in general, depends on the frame of reference, namely the coordinate axis that we use. If we associate our prefered frame with $I_n$, then every other frame (that is orthonormal), can be represented by $A \in O(n)$. If $\det A = -1$, then $A$ and the prefered frame have opposite parity (i.e. left-handed $\longleftrightarrow$ right-handed). It follows that $SO(n)$ represents all the frames with the same parity.
We can distinguish the physical quantity itself from its mathematical representation by treating it as a function

$$\varphi : O(n) \longrightarrow \mathbb{R}^m$$

where $m$ is the number of components of $\varphi$.

Definition : A function $\varphi : O(n) \longrightarrow \mathbb{R}^m$ is a physical quantity iff there is a homomorphism* $\mathcal{M}_\varphi : O(n) \longrightarrow \mathbb{R}^{m\times m}$ such that $\varphi$ is given by:

$$\boxed{\varphi(A) = \mathcal{M}_\varphi(A)\,\varphi(I_n)}$$

a) The above equation is called the transformation law of $\varphi$.
b) The group $\mathrm{Im}(\varphi) = \mathcal{M}[O(n)]$, which is the image of $O(n)$ under the homomorphism, is called the transformation group of $\varphi$.

Physical quantities can be classified according to $m$ and the transformation group $\mathrm{Im}(\varphi)$. The transformation law can be generalized to change between two arbitrary frames $A, B \in O(n)$ as follows:

(*) The term homomorphism is abused. The correct term is representation and the required properties are:

$$\mathcal{M}(AB) = \mathcal{M}(A)\mathcal{M}(B) \quad , \quad \mathcal{M}(I_n) = I_m$$

<u>Proposition</u> : Let $\varphi$ be a physical quantity. Then :

$$\boxed{\varphi(B) = m_\varphi(BA^{-1})\,\varphi(A)\ ,\ \forall A,B \in O(n)}$$

<u>Proof</u>

$m$ is a homomorphism $\Rightarrow (\forall A,B \in O(n))(m_\varphi(AB) = m_\varphi(A)m_\varphi(B))$.
It follows that

$$\begin{aligned}\varphi(B) &= m_\varphi(B)\varphi(I_n) = m_\varphi(BI_n)\varphi(I_n) = \\ &= m_\varphi(BA^{-1}A)\varphi(I_n) = \\ &= m_\varphi(BA^{-1})m_\varphi(A)\varphi(I_n) = \\ &= m_\varphi(BA^{-1})\varphi(A) \qquad \square\end{aligned}$$

Physical quantities are classified as follows:

1) A <u>scalar</u> is $\varphi : O(n) \longrightarrow \mathbb{R}$ such that
$$\varphi(Ax) = \varphi(x)\ ,\ \forall A \in O(n)$$
2) A <u>pseudoscalar</u> is $\varphi : O(n) \longrightarrow \mathbb{R}$ such that
$$\varphi(Ax) = \varepsilon(A)\varphi(x)\ ,\ \forall A \in O(n)$$
3) A <u>vector</u> (n-dim) is $\varphi : O(n) \longrightarrow \mathbb{R}^n$ such that
$$\varphi(Ax) = A\varphi(x)$$
4) A <u>pseudovector</u> (n-dim) is $\varphi : O(n) \longrightarrow \mathbb{R}^n$ such that
$$\varphi(Ax) = \varepsilon(A)A\varphi(x)$$
$\hookrightarrow$ We denote the components of a vector or a pseudovector as $\varphi_i$ with $i \in \{1,2,\ldots,n\}$

Tensors have very complicated transformation rules, so they are defined indirectly as follows:

5) A <u>tensor</u> is $\varphi : O(n) \longrightarrow \mathbb{R}^{n^2}$ such that it transforms under the same law as the product of two vectors:
$$\varphi_{ij}(x) = a_i(x)\,b_j(x).$$
6) <u>Higher-rank tensors</u> are defined similarly:
$$\varphi_{ijk}(x) = a_i(x)\,b_j\,\gamma_k(x)$$
$$\varphi_{ijkl}(x) = a_i\,b_j\,\gamma_k\,\delta_l \qquad , \text{etc.}$$

It is possible to generalize the definition of physical quantities to obtain <u>physical fields</u>:

<u>Definition</u>: A function $\varphi: \mathbb{R}^n \times O(n) \longrightarrow \mathbb{R}^m$ is a <u>physical field</u> iff there is a homomorphism $m_\varphi : O(n) \longrightarrow \mathbb{R}^{m\times m}$ such that $\varphi$ satisfies:

$$\boxed{\varphi(x,A) = m_\varphi(A)\,\varphi(x,I_n),\ \forall x \in \mathbb{R}^n, \forall A \in O(n)}$$

A direct corollary of the proposition, p.78, is that for any two frames of reference $A,B \in O(n)$:

$$\boxed{\varphi(x,B) = m_\varphi(BA^{-1})\,\varphi(x,A)\ ,\ \forall\, x \in \mathbb{R}^n}$$

Physical fields can be classified as scalars, pseudoscalars, etc. according to $m_\varphi$, same as in p.78. When we fix a frame then we do not show $A \in O(n)$ explicitly. Then we should think of $\varphi(x)$ as a <u>functional</u> that maps $x \in \mathbb{R}^n$ into a physical quantity. $y: O(n) \longrightarrow \mathbb{R}^m$.

## <u>Invariance theorems</u>

We review without proofs the following well known results:

<u>Theorem</u>: Let $\varphi_1, \varphi_2$ be two physical fields with transformation law $m$. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be two real numbers. Then the field

$$\varphi = \lambda_1 \varphi_1 + \lambda_2 \varphi_2$$

also has transformation law $m$.

For the sake of brevity we write:

$$(\text{number}) \times (\text{field}) + (\text{number}) \times (\text{field}) = (\text{field})$$

We state the rest of the invariance theorems in the same abbreviated notation:

(scalar)(scalar) = (scalar)
(scalar)(pseudoscalar) = (pseudoscalar).
(pseudoscalar)(pseudoscalar) = (scalar).
(scalar)(anything) = (anything) (anything = s, ps, v, pv).
(vector) • (vector) = (scalar)
(vector) x (vector) = (pseudovector) (in 3d only)
(vector) • (pseudovector) = (pseudoscalar)
(pseudovector) • (pseudovector) = (scalar).

Every vector defines a unique scalar, its norm.
To deal with tensors, we have to be careful with the definitions, and their development.

Definition: Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ be two vectors with components $x_i$ and $y_j$. Then the tensor product $z = x \otimes y \in \mathbb{R}^{mn}$ has components:

$$\boxed{z_K \equiv z_{ij} \equiv x_i y_j}$$

where $K = i + jm$ is one possible labeling of the components.

Proposition: Let $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $A \in {}^{m\times m}\mathbb{R}$, $B \in {}^{n\times n}\mathbb{R}$.
Then there exists a matrix $\Gamma \in {}^{mn\times mn}\mathbb{R}$ which we denote as $\Gamma = A \oplus B$ such that

$$\boxed{[Ax]\otimes[By] = (A\oplus B)(x\otimes y)}$$

Proof

Let $\gamma_{ij}$ be the $ij$ component of $[Ax]\otimes[By]$.
Then:

$$\gamma_{ij} = (Ax)_i (By)_j = \left[ \sum_{k=1}^{m} a_{ki} x_i \right] \left[ \sum_{l=1}^{n} b_{lj} y_j \right] =$$
$$= \sum_{k=1}^{m} \sum_{l=1}^{n} a_{ki} b_{lj} (x_i y_j)$$

However $z_{ij} = x_i y_j$ are the components of $x \otimes y$ $\square$

We also introduce the notation:

$$\oplus^2 A = A \oplus A$$
$$\oplus^3 A = A \oplus A \oplus A \qquad \left( \underline{\oplus^0 A \equiv 1 \in \mathbb{R}} \ , \ \underline{\oplus^1 A \equiv A} \right)$$

etc. With this notation we may give the general definition of tensors:

<u>Definition</u>: A <u>tensor</u> of rank $m$ in $n$-dimensions is a physical quantity $\varphi : O(n) \longrightarrow \mathbb{R}^{n^m}$ with transformation law

$$\boxed{\varphi(A) = [\oplus^n A]\, \varphi(I_n)} \ , \ \forall A \in O(n)$$

<u>Definition</u>: A <u>pseudotensor</u> of rank $m$ in $n$-dimensions is a physical quantity $\varphi : O(n) \longrightarrow \mathbb{R}^{n^m}$ with transformation law

$$\boxed{\varphi(A) = \varepsilon(A) [\oplus^n A]\, \varphi(I_n)} \ , \ \forall A \in O(n)$$

In the above we made implicit use of the following associativity property:

$$\underline{(A \oplus B) \oplus \Gamma = A \oplus (B \oplus \Gamma)} \ , \ \forall A, B, \Gamma$$

To state the invariance theorem for tensors we must introduce a generalization of the dot-product; <u>contraction</u>.
Because defining contraction formally in a general manner requires a lot of formalism we will instead list contractions for 2nd, 3rd rank tensors:

We employ Einstein's notation: repeated indices indicate summation. Repeated Greek indices do not.

| | Transformation law $m(A)$ |
|---|---|
| $\gamma_j = a_{ji} b_i$ | $\oplus A$ |
| $\lambda = a_{ii} \equiv tr(a)$ | $1$ |
| $\lambda = a_{ij} b_{ij} \equiv a : b$ | $1$ |
| $\gamma_{ij} = a_{ijk} b_k$ | $\oplus^2 A$ |
| $\gamma_i = a_{ijk} b_{jk}$ | $\oplus A$ |
| $\lambda = a_{ijk} b_{ijk} = a : b$ | $1$ |
| $\gamma_k = a_{iik}$ | $A$ |
| $\gamma_j = a_{iji}$ | $A$ |
| $\gamma_i = a_{ijj}$ | $A$ |
| $\lambda = a_{iii} = tr(a)$ | $1$ |

In all of the above we assume that the transformation laws for "a" and "b" are one of: $1, A, \oplus^2 A, \oplus^3 A$ depending on the number of indices.

## Isotropic fields

An isotropic field is one that "looks the same" when we rotate or reflect our frame of reference. A formal definition is as follows:

<u>Definition</u>: A physical field is <u>isotropic</u> $\varphi : \mathbb{R}^n \times O(n) \longrightarrow \mathbb{R}^m$ iff $\varphi$ satisfies:

$$\boxed{\varphi(Ax, B) = \varphi(x, AB) = m_\varphi(A)\varphi(x, B)}, \forall x \in \mathbb{R}^n, \forall A, B \in O(n)$$

It can be shown that if we know that a physical field is isotropic then its representation takes a very specific form:

Let $\vec{r} \in \mathbb{R}^n$ and $r = \|\vec{r}\|$ which is a scalar.
If $\varphi(\vec{r})$ is isotropic and it is also a ~~scalar~~

1) scalar $\longrightarrow$ $\varphi(\vec{r}) = A(r)$
2) vector $\longrightarrow$ $\varphi_i(\vec{r}) = A(r)\, r_i$
3) tensor $\longrightarrow$ $\varphi_{ij}(\vec{r}) = A(r)\, r_i r_j + B(r)\,\delta_{ij}$

$$\varphi_{ijk}(\vec{r}) = A(r)\, r_i r_j r_k + \\ + B(r)\, r_i \delta_{jk} + C(r)\, r_j \delta_{ki} + D(r)\, r_k \delta_{ij}$$

## Random fields and their symmetries

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space with
$\Omega$ = sample space
$\mathcal{B}$ = set of events
$\mu$ = probability measure

<u>Definition</u>: A <u>random field</u> is a function of the form
$$\varphi : \mathbb{R}^n \times \Omega \times O(n) \longrightarrow \mathbb{R}^m$$
such that there is a representation $M_\varphi : O(n) \longrightarrow {}^{m\times m}\mathbb{R}$ for which
$$\boxed{\varphi(x, \omega, A) = M_\varphi(A)\, \varphi(x, \omega, I_n)}$$

From now on we assume that the frame of reference is fixed, but remain aware of the transformation law $M_\varphi$ with which $\varphi$ is associated.

### Statistical symmetries of random fields

When we study fields we define the increment as follows:
$$\delta\varphi(x, \Delta x, \omega, A) = \varphi(x+\Delta x, \omega, A) - \varphi(x, \omega, A)$$
Recall the definition of equivalence in law (p. 48).

<u>Definition</u>: Let $\varphi$ be a random field with transformation law $M_\varphi$. Then:

| | | |
|---|---|---|
| $\varphi$ homogeneous | $\leftrightarrow$ | $\forall y \in \mathbb{R}^n : \varphi(x+y) \simeq \varphi(x)$ |
| $\varphi$ locally homogeneous | $\leftrightarrow$ | $\forall x_0 \in \mathbb{R}^n : \delta\varphi(x+x_0, \Delta x) \simeq \delta\varphi(x, \Delta x)$ |
| $\varphi$ isotropic | $\leftrightarrow$ | $\forall A \in O(n) : \varphi(Ax) \simeq M_\varphi(A)\, \varphi(x)$ |
| $\varphi$ locally isotropic | $\leftrightarrow$ | $\forall A \in O(n) : \delta\varphi(x, A\Delta x) \simeq M_\varphi(A)\, \delta\varphi(x, \Delta x)$ |

The following propositions relate these symmetries:

**Proposition:** $\boxed{\varphi \text{ homogeneous} \Rightarrow \varphi \text{ locally homogeneous}}$

~~Proposition~~

Since $\varphi$ homogeneous let $g(x_0) : \underline{\Omega} \to \underline{\Omega}$ be the corresponding measure preserving and complete transformation. It follows that

$$\begin{aligned}
\delta\varphi(x+x_0, \Delta x, \vartheta) &= \varphi(x+x_0+\Delta x, \vartheta) - \varphi(x+x_0, \vartheta) = \\
&= \varphi(x+\Delta x, g(x_0)\vartheta) - \varphi(x, g(x_0)\vartheta) = \\
&= \delta\varphi(x, \Delta x, g(x_0)\vartheta) \Rightarrow
\end{aligned}$$

$\Rightarrow \delta\varphi(x+x_0, \Delta x) \simeq \delta\varphi(x, \Delta x) \Rightarrow \varphi$ locally homogeneous.

**Proposition:** $\boxed{\left.\begin{array}{l}\varphi \text{ locally homogeneous} \\ \varphi \text{ isotropic}\end{array}\right\} \Rightarrow \varphi \text{ locally isotropic}}$

**Proof**

Since $\varphi$ is locally homogeneous let $g(x_0) : \underline{\Omega} \to \underline{\Omega}$ be the corresponding transformation such that

$$\delta\varphi(x+x_0, \Delta x, \vartheta) = \delta\varphi(x, \Delta x, g(x_0)\vartheta)$$

Since $\varphi$ is isotropic let ~~g~~ $h(A) : \underline{\Omega} \to \underline{\Omega}$ be the corresponding transformation such that

$$\cancel{\delta\varphi(x, A\Delta x, \vartheta) = m_\varphi(A)\delta\varphi}$$

$$\varphi(Ax, \vartheta) = m_\varphi(A)\, \varphi(x, h(A)\vartheta)$$

It follows that:

$$\delta\varphi(x, A\Delta x, \vartheta) = \delta\varphi(0, A\Delta x, g^{-1}(x)\vartheta) =$$
$$= \varphi(A\Delta x, g^{-1}(x)\vartheta) - \varphi(0, g^{-1}(x)\vartheta) =$$
$$= m_\varphi(A)\left[\varphi(\Delta x, h(A)g^{-1}(x)\vartheta) - \varphi(0, h(A)g^{-1}(x)\vartheta)\right] =$$
$$= m_\varphi(A)\,\delta\varphi(0, \Delta x, h(A)g^{-1}(x)\vartheta) =$$
$$= m_\varphi(A)\,\delta\varphi(x, \Delta x, g(x)h(A)g^{-1}(x)\vartheta) \qquad (1)$$

Define $f(A) \equiv g(x)h(A)g^{-1}(x)$ where $x \in \mathbb{R}^n$ is assumed to be fixed. We want to show that $f$ is measure preserving and complete. First note that $f, g, h$ all have unique inverses so it suffices to show that $\mu(f(A)E) = \mu(E)$ and $f(A)\underline{o} = \underline{o}$ mod $\mu$. Both relations follow from easy computations:

$$\mu(f(A)E) = \mu(g(x)h(A)g^{-1}(x)E) =$$
$$= \mu(h(A)g^{-1}(x)E) = \mu(g^{-1}(x)E) =$$
$$= \mu(E), \quad \forall E \in \mathcal{B}, \quad \forall A \in O(n). \qquad (2)$$

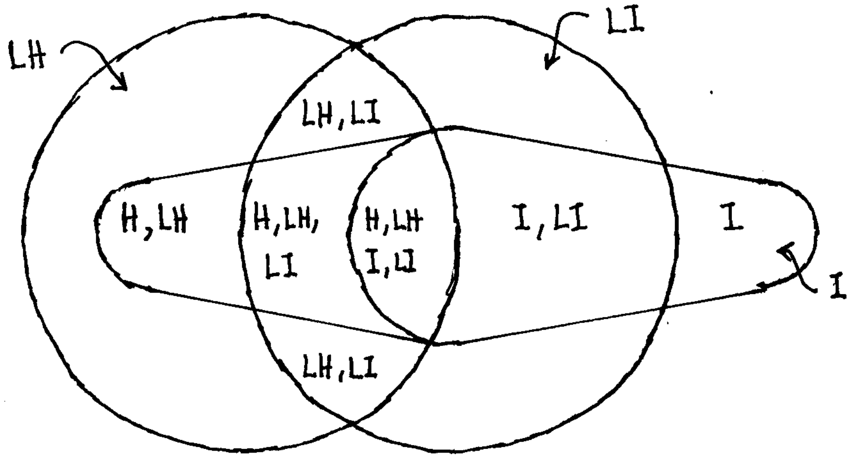
and

$$f(A)\underline{o} = g(x)h(A)g^{-1}(x)\underline{o} = g(x)h(A)\underline{o}$$
$$= g(x)\underline{o} = \underline{o} \ \text{mod}\,\mu \ \text{(for each step)}, \ \forall A \in O(n) \qquad (3)$$

From (1), (2), (3) we obtain:

$$\delta\varphi(x, A\Delta x) \simeq m_\varphi(A)\,\delta\varphi(x, \Delta x) \Rightarrow \varphi \text{ locally isotropic.} \qquad \square$$

It is easy to see, from the above proof, that "$\varphi$ isotropic" alone cannot conclude that "$\varphi$ local isotropic"; local homogeneity is required. It is however possible to have local isotropy without local homogeneity and vice versa. The following diagram shows these logical relationships:




where

H = homogeneous  LH = locally homogeneous

I = isotropic  LI = locally isotropic

There are 8 possibilities that correspond to the 8 regions shown above. We list them in the following truth table:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| H | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 |
| LH | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| I | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| LI | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 |