Porous Media Flow

The governing equations

We consider single-phased incompressible flow in 2d and define:

u = horizontal velocity.
v = vertical velocity.
p = the pressure.

Darcey's law on porous media flow states that

$$\boxed{u = -b(x,y)\frac{\partial p}{\partial x}} \quad \boxed{v = -b(x,y)\frac{\partial p}{\partial y}}$$

where $b(x,y)$ is given by $\boxed{b(x,y) = \dfrac{\kappa(x,y)}{\mu}}$

with $\kappa(x,y)$ = the porosity
μ = the fluid viscosity.

Given that we assume incompressibility we also have:

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

Given the steady velocity field (u,v) we can then compute how it advects a tracer (or contaminant) by:

$$\boxed{\kappa \frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} = \nu_t\left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2}\right).}$$

where ν_t is the diffusion coefficient of the tracer and $0 \leq q(x,y,t) \leq 1$ is the concentration.

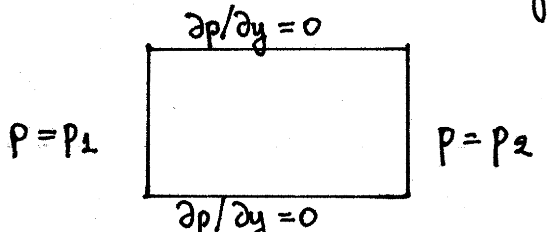
The pressure formulation

Substitute Darcey's law into the continuity equation. We obtain:

$$\boxed{\frac{\partial}{\partial x}\left(b\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(b\frac{\partial p}{\partial y}\right) = 0}$$

We set up the domain Ω to be a square with pressure differences on the left and right edges to drive the flow.

We assume that the other two edges are impenetrable.



So we specify Dirichlet boundary conditions on the left/right edges, which are constant accross the edges.

$p = p_1$ on left edge
$p = p_2$ on right edge.

The nonpenetrability implies that

$$v = 0 \Leftrightarrow -b(x,y)\cdot\frac{\partial p}{\partial y} = 0 \Leftrightarrow \underline{\frac{\partial p}{\partial y} = 0} \quad \text{on upper+lower edges.}$$

This set of equations is complete, and can yield $p(x,y)$.

▼ <u>The streamfunction formulation</u>

Recall that because the flow is incompressible

$$\nabla\cdot\vec{u} = 0 \Leftrightarrow \vec{u} = \nabla\varphi$$

therefore:

$$\nabla\cdot\vec{u} = \nabla\cdot(\nabla\varphi) = \nabla^2\varphi = 0 \Rightarrow \varphi(x,y) \text{ is a harmonic function} \Rightarrow$$
$$\Rightarrow \exists\ \psi(x,y) \text{ also harmonic} : \Phi(x+yi) = \varphi(x,y) + i\psi(x,y) \text{ analytic}$$

► We call $\psi(x,y)$ the <u>streamfunction.</u> of the velocity field.

<u>Physical interpretation:</u>

Consider, $\vec{u}\cdot\nabla\psi = \nabla\varphi\cdot\nabla\psi = \dfrac{\partial\varphi}{\partial x}\dfrac{\partial\psi}{\partial x} + \dfrac{\partial\varphi}{\partial y}\dfrac{\partial\psi}{\partial y} =$

$$= -\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y} + \frac{\partial\varphi}{\partial y}\frac{\partial\varphi}{\partial x} = 0 \Rightarrow \boxed{\vec{u}\perp \nabla\psi(x,y).}$$

It follows that velocity field lines lie on <u>$\psi(x,y) = \text{const.}$</u>

Now we derive the governing equation for $\psi(x,y)$.

From the Cauchy-Riemann relations we have:

$$u = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

If we substitute Dalcey's law:

$$\begin{cases} -b\dfrac{\partial p}{\partial x} = \dfrac{\partial \psi}{\partial y} \\ -b\dfrac{\partial p}{\partial y} = -\dfrac{\partial \psi}{\partial x} \end{cases} \Rightarrow \begin{cases} \dfrac{1}{b}\dfrac{\partial \psi}{\partial y} = -\dfrac{\partial p}{\partial x} \\ \dfrac{1}{b}\dfrac{\partial \psi}{\partial x} = \dfrac{\partial p}{\partial y} \end{cases}.$$

It follows that

$$\frac{\partial}{\partial x}\left(\cancel{b}\frac{1}{b}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{b}\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial p}{\partial x}\right) =$$

$$= \frac{\partial^2 p}{\partial x \partial y} - \frac{\partial^2 p}{\partial x \partial y} = 0.$$

so we obtain:

$$\boxed{\frac{\partial}{\partial x}\left(\frac{1}{b}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{b}\frac{\partial \psi}{\partial y}\right) = 0}$$

Now consider the boundary conditions. On the left and right edges

$$p = p_{1,2} = \text{const} \Rightarrow \frac{\partial p}{\partial y} = 0 \Rightarrow \frac{\partial \psi}{\partial x} = 0 \quad \text{on left/right}$$

of course this must be generalized if p is not constant there along the y direction.
Tohe values p_1, p_2 are not needed to obtain ψ. They are needed however to obtain $p(x,y)$ from $\psi(x,y)$.
On the upper/lower edges

$$\frac{\partial p}{\partial y} = 0 \Rightarrow \frac{\partial \psi}{\partial x} = 0 \quad \text{on upper/lower} \Rightarrow \psi = c_{1,2} \text{ on upper-lower}$$

(because these edges are horizontal).
So we obtain:

$$\frac{\partial \psi}{\partial x} = 0 \quad \text{on } \cancel{\text{left/right upper/l.}} \text{ left/right}$$

$$\psi = c_1 \text{ on } \cancel{\text{up}}\text{lower} \qquad \psi = c_2 \text{ on upper.}$$

$$\boxed{\begin{array}{l} \dfrac{\partial \psi}{\partial x} = 0 \quad \text{on left/right} \\ \psi = c_1 \text{ on lower}, \quad \psi = c_2 \text{ on upper} \end{array}}$$

It is suggested that c_1, c_2 are normalized such that $c_1 = 0$ and $c_2 = 1$

Galerkin formulation

Let H^1 = the space of all functions $w(x,y)$. that are 1st-order differentiable.

Then:

$$\nabla \cdot \left(\frac{1}{b} \nabla \psi\right) = 0 \Leftrightarrow I = \int_{\Omega} \nabla \cdot \left(\frac{1}{b} \nabla \psi\right) w \, dx dy = 0 \quad , \forall \, w \in H^1.$$

Now rewrite the integral as follows:

$$I = \int_{\Omega} \nabla \cdot \left(\frac{1}{b} \nabla \psi\right) w \, dx dy =$$

$$= \int_{\Omega} \frac{\partial}{\partial x}\left(\frac{1}{b} \frac{\partial \psi}{\partial x}\right) w \, dx dy + \int_{\Omega} \frac{\partial}{\partial y}\left(\frac{1}{b} \frac{\partial \psi}{\partial y}\right) w \, dx dy$$

$$= \int_{y_1}^{y_2} \left[\int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\frac{1}{b} \frac{\partial \psi}{\partial x}\right) w \, dx \right] dy + \int_{x_1}^{x_2} \left[\int_{y_1}^{y_2} \frac{\partial}{\partial y}\left(\frac{1}{b} \frac{\partial \psi}{\partial y}\right) w \, dy \right] dx$$

Consider the integrals indepedently:

$$\int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\frac{1}{b} \frac{\partial \psi}{\partial x}\right) w \, dx = \left[\frac{1}{b} \frac{\partial \psi}{\partial x} w \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{1}{b} \frac{\partial \psi}{\partial x} \frac{\partial w}{\partial x} dx =$$

$$= \frac{w(x_2,y)}{b(x_2,y)} \frac{\partial \psi}{\partial x}\bigg|_{x_2} - \frac{w(x_1,y)}{b(x_1,y)} \frac{\partial \psi}{\partial x}\bigg|_{x}^{x_1} - \int_{x_1}^{x_2} \frac{1}{b} \frac{\partial \psi}{\partial x} \frac{\partial w}{\partial x} dx$$

$$= -\int_{x_1}^{x_2} \frac{1}{b} \frac{\partial \psi}{\partial x} \frac{\partial w}{\partial x} dx$$

because $\left.\dfrac{\partial \psi}{\partial x}\right|_{x_1} = \left.\dfrac{\partial \psi}{\partial x}\right|_{x_2} = 0$, from the boundary condition.

Also:

$$\int_{y_1}^{y_2} \frac{\partial}{\partial y}\left(\frac{1}{b}\frac{\partial \psi}{\partial y}\right) w\,dy = \left[\frac{1}{b}\frac{\partial \psi}{\partial y}\, w\right]_{y_1}^{y_2} - \int_{y_1}^{y_2} \frac{1}{b}\frac{\partial \psi}{\partial y}\frac{\partial w}{\partial y}\,dy =$$

$$= \frac{w(x,y_2)}{b(x,y_2)}\frac{\partial \psi}{\partial y}\bigg|_{y_2} - \frac{w(x,y_1)}{b(x,y_1)}\frac{\partial \psi}{\partial y}\bigg|_{y_1} - \int_{y_1}^{y_2} \frac{1}{b}\frac{\partial \psi}{\partial y}\frac{\partial w}{\partial y}\,dy$$

In order to eliminate the "constant" terms we must require that

$$\boxed{w \in \{w \in H^1 \mid w(x,y_2) = w(x,y_1) = 0\} = V}$$

So $\forall w \in V$: $\displaystyle\int_{y_1}^{y_2} \frac{\partial}{\partial y}\left(\frac{1}{b}\frac{\partial \psi}{\partial y}\right) w\,dy = -\int_{y_1}^{y_2} \frac{1}{b}\frac{\partial \psi}{\partial y}\frac{\partial w}{\partial y}\,dy.$

Putting it all together:

$$I = \int_{y_1}^{y_2}\left[-\int_{x_1}^{x_2} \frac{1}{b}\frac{\partial \psi}{\partial x}\frac{\partial w}{\partial x}\,dx\right] dy + \int_{x_1}^{x_2}\left[-\int_{y_1}^{y_2} \frac{1}{b}\frac{\partial \psi}{\partial y}\frac{\partial w}{\partial y}\,dy\right] dx =$$

$$= -\int_{\Omega} \frac{1}{b}\left(\frac{\partial \psi}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial \psi}{\partial y}\frac{\partial w}{\partial y}\right) dx\,dy = -\int_{\Omega} \frac{1}{b}\nabla\psi\cdot\nabla w\,dx\,dy$$

therefore, the Galerkin formulation of the PDE is:

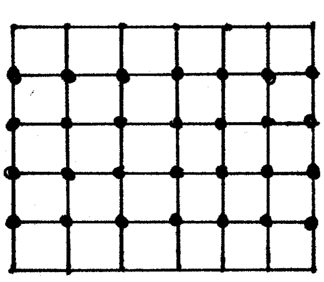
$$\boxed{\int_{\Omega} \frac{1}{b}\nabla\psi\cdot\nabla w\,dx\,dy = 0 \quad , \forall w \in V}$$

▼ Finite element formulation

For convenience, and without loss of generality suppose that

$$x_1 = 0 \quad , \quad x_2 = 1$$
$$y_1 = 0 \quad , \quad y_2 = 1$$

We divide the domain Ω with square elements:



We associate unknowns q_j with the vertices shown in the figure.
We include Neumann boundary points but omit Dirichlet points.

We describe our environment with the following sets:

a) A set of <u>elements</u> E with $\forall A \in E : A \subseteq \Omega$.

b) A set of meshpoints M with $j \in M$.

c) A set of boundary points B with $j \in B$.

M includes the interior vertices and the Neumann boundary vertices.

B includes the Dirichlet boundary vertices.

We also define the following notation:

a) If $j \in M$, then $N(j) \subseteq M$ is the set of all the neighbohrs of j.

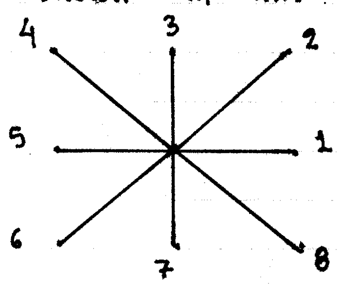
b) If $j \in M$, then $I(j) \subseteq E$ is the set of all elements incident to j.

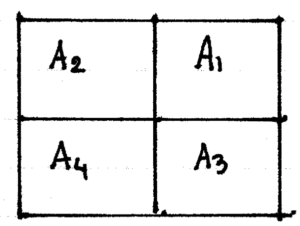
In particular we use the following labelling:

$$N(j) = \{ N_1(j), N_2(j), N_3(j), \ldots, N_8(j) \}.$$

$$E(j) = \{ E_1(j), E_2(j), E_3(j), E_4(j) \}.$$

as shown in this figure:





Finally we use the following notation for the coordinates.

If $j \in M \cup B$ then it has coordinates $(x_j, y_j) \in \Omega$.

If $A \in E$ then we write $A = [b_A^1, b_A^2] \times [b_A^3, b_A^4]$.

With each vertex $j \in M \cup B$ we associate a function $\varphi_j \in V$ with the following desired properties:

a) $\varphi_j(x_j, y_j) = 1$ b) $\forall k \in N(j) : \varphi_j(x_k, y_k) = 0$

c) $\forall (x,y) \in \Omega - \bigcup_{A \in I(j)} A : \varphi(x,y) = 0$.

In other words φ_j is 1 at j, 0 at the neighbohr vertices k and vanishes outside the four incident elements to j.

Then we express our unknown $\psi(x,y)$ as:

$$\boxed{\psi(x,y) = \sum_{j \in G} q_j \varphi_j(x,y) + \sum_{j \in B} q_j \varphi_j(x,y).}$$

$q_j, \forall j \in G$ are unknown. However $q_j, \forall j \in B$ are <u>known</u>

From the three properties of φ_j it follows that ~~$\psi(x_j, y_j$~~

$$\boxed{\psi(x_j, y_j) = q_j}$$

Now we substitute this expression to the Galerkin condition.

- Define:

$$a(f,g) = \int_{\Omega} \frac{1}{\varepsilon(x,y)} \left[\nabla f(x,y) \cdot \nabla g(x,y)\right] dx\,dy.$$

We call this the <u>bilinear form</u> of f and g.
The bilinear form has the following properties:
a) $a(f,g) = a(g,f)$.
b) $a(\lambda_1 f_1 + \lambda_2 f_2, g) = \lambda_1 a(f_1, g) + \lambda_2 a(f_2, g)$

In terms of the bilinear form, the Galerkin condition states that:

$$\boxed{a(\psi, w) = 0 \ , \ \forall w \in V}$$

Note that $w \in V \Leftrightarrow w(x,y) = \sum_{j \in M} \lambda_j \varphi_j(x,y)$. therefore:

$$a(\psi, w) = 0 \ , \ \forall w \in V \Leftrightarrow a(\psi, \sum_{j \in M} \lambda_j \varphi_j) = 0 \ , \ \forall \lambda_j \in \mathbb{R} \Leftrightarrow$$

$$\Leftrightarrow \sum_{k \in M} \lambda_k a(\psi, \varphi_k) = 0 \ , \ \forall \lambda_k \in \mathbb{R} \Leftrightarrow$$

$$\Leftrightarrow \boxed{a(\psi, \varphi_k) = 0 \ , \ \forall k \in M}$$

Now we substitute $\psi(x,y)$:

$$a(\psi, \varphi_k) = a\Big(\sum_{j \in M \cup B} q_j \varphi_j(x,y), \varphi_k\Big) = \sum_{j \in M \cup B} q_j a(\varphi_j, \varphi_k) =$$

$$= \sum_{j \in M} q_j a(\varphi_j, \varphi_k) + \sum_{j \in B} q_j a(\varphi_j, \varphi_k) = 0 \Leftrightarrow$$

$$\Leftrightarrow \boxed{\sum_{j \in M} q_j a(\varphi_j, \varphi_k) = - \sum_{j \in B} q_j a(\varphi_j, \varphi_k).} \ , \ \forall k \in M$$

Note that the right-hand-side is known and we write it as:

$$\beta_k = - \sum_{j \in B} q_j a(\varphi_j, \varphi_k) \ , \ \forall k \in M.$$

Then we have a linear set of equations to solve for q_j.

$$\boxed{\sum_{j\in M} q_j\, a(\varphi_j,\varphi_k) = \beta_k \quad , \forall k\in M}$$

Outstanding issues are:

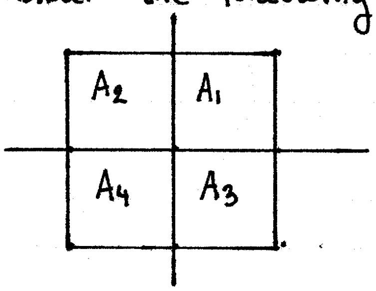
a) Defining $\varphi_j(x,y)$.

b) Computing $a(\varphi_j,\varphi_k)$.

c) Solving the linear system.

▼ <u>Definition of the basis functions.</u>

We say that $f(x,y)$ is <u>bilinear</u> in A if and only if $f(x,y) = a + bx + \gamma y + \delta xy$.
We claim that there is a unique $\varphi_j(x,y)$ with the desired properties if we also require it to be piecewise bilinear.
We now construct this unique $\varphi_j(x,y)$ and use this construction as our definition.
Consider the following 4 regions:



$$A_1 = [0,1]\times[0,1]$$
$$A_2 = [-1,0]\times[0,1]$$
$$A_3 = [0,1]\times[-1,0]$$
$$A_4 = [-1,0]\times[-1,0]$$

We want functions f_1, f_2, f_3, f_4 such that.

$f_1(0,0) = 1 \quad f_1(1,0) = 0 \quad f_1(0,1) = 0 \quad f_1(1,1) = 0 \quad f_1$ bilinear in A_1

$f_2(0,0) = 1 \quad f_2(-1,0) = 0 \quad f_2(0,1) = 0 \quad f_2(-1,1) = 0 \quad f_2$ bilinear in A_2

$f_3(0,0) = 1 \quad f_3(1,0) = 0 \quad f_3(0,-1) = 0 \quad f_3(1,-1) = 0 \quad f_3$ bilinear in A_3

$f_4(0,0) = 1 \quad f_4(-1,0) = 0 \quad f_4(0,-1) = 0 \quad f_4(-1,-1) = 0 \quad f_4$ bilinear in A_4

and also

$$f_1(x,y) = f_2(x,y) = f_3(x,y) = f_4(x,y) = 0 \quad , \forall (x,y) \notin A_1\cup A_2\cup A_3\cup A_4$$

Such functions are unique and they are given by:

$f_1(x,y) = (1-x)(1-y) \quad , \forall (x,y)\in A_1 \qquad f_1(x,y) = 0 \quad , \forall (x,y)\notin A_1$

$f_2(x,y) = (1+x)(1-y) \quad , \forall (x,y)\in A_2 \qquad f_2(x,y) = 0 \quad , \forall (x,y)\notin A_2$

$f_3(x,y) = (1-x)(1+y) \quad , \forall (x,y)\in A_3 \qquad f_3(x,y) = 0 \quad , \forall (x,y)\notin A_3$

$f_4(x,y) = (1+x)(1+y) \quad , \forall (x,y)\in A_4 \qquad f_4(x,y) = 0 \quad , \forall (x,y)\notin A_4$.

Now we define the <u>standard piecewise bilinear</u> function $\tilde{\varphi}_0(x,y)$:

$$\tilde{\varphi}_0(x,y) = \begin{cases} f_1(x,y) & , (x,y)\in A_1 \\ f_2(x,y) & , (x,y)\in A_2 \\ f_3(x,y) & , (x,y)\in A_3 \\ f_4(x,y) & , (x,y)\in A_4 \\ 0 & , \text{otherwise.} \end{cases}$$

Then the function $\varphi_j(x,y)$ associated with $j\in M\cup B$ is defined as a translated + rescaled version of $\tilde{\varphi}_0$:

$$\boxed{\varphi_j(x,y) = \tilde{\varphi}_0((x-x_j)/h, (y-y_j)/h), \quad \forall j\in M\cup B}$$

The uniqueness of $\varphi_j(x,y)$ follows from the uniqueness of f_1, f_2, f_3, f_4.

▼ Computing the bilinear form

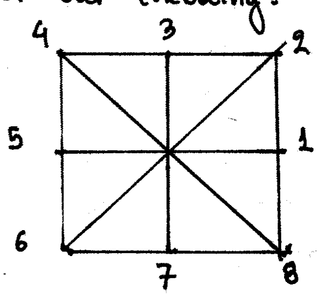
Now we show how to compute $a(\varphi_i,\varphi_j)$.
To do this we need a way to algebraicly encode all the geometric information about $i,j\in M\cup B$.
We encode our sense of "direction" with integers

$$(p_i, q_i)\in\{-1,0,1\}\times\{-1,0,1\}, \quad \forall i\in\{0,1,2,\dots,8\}.$$

such that if

$$j\in M\cup B \text{ and } k = N_i(j)\in M\cup B \Rightarrow (x_k,y_k) = (x_j,y_j) + (p_i,q_i).$$

For our encoding:



$$p_i = (1, 1, 0, -1, -1, -1, 0, 1)$$
$$q_i = (0, 1, 1, 1, 0, -1, -1, -1)$$

We also define displacements of the standard function $\varphi_0(x,y)$ in terms of p_i, q_i:

$$\boxed{\tilde{\varphi}_i(x,y) \doteq \tilde{\varphi}_0(x-p_i, y-q_i), \quad \forall i\in\{1,2,\dots,8\}}$$

~~Now let $j\in M$ be a given vertex and consider the following change of variables:~~

Now let $j \in MUB$ be given and let $\ell = N_K(j)$ be a neighbohr of j for $K \in \{0,1,2,\dots,8\}$ (with $N_0(j) = j$).
We want to compute $a(\varphi_j, \varphi_\ell)$.
We introduce the following change of variables:

$$x' = \frac{1}{h}(x - x_j) \qquad y' = \frac{1}{h}(y - y_j).$$

Note that this change is dependent on j.
Then $dx'dy' = \frac{1}{h^2} dxdy \Rightarrow dxdy = h^2 dx'dy'$ and

$$\left.\begin{aligned} \frac{\partial \varphi_j}{\partial x} &= \frac{\partial \tilde{\varphi}_0}{\partial x'}\frac{\partial x'}{\partial x} = \frac{1}{h}\frac{\partial \tilde{\varphi}_0}{\partial x'} \\ \frac{\partial \varphi_j}{\partial y} &= \frac{\partial \tilde{\varphi}_0}{\partial y'}\frac{\partial y'}{\partial y} = \frac{1}{h}\frac{\partial \tilde{\varphi}_0}{\partial y'} \end{aligned}\right\} \Rightarrow \nabla \varphi_j = \frac{1}{h}\nabla \tilde{\varphi}_0 .$$

and

$$\left.\begin{aligned} \frac{\partial \varphi_\ell}{\partial x} &= \frac{\partial \tilde{\varphi}_K}{\partial x'}\frac{\partial x'}{\partial x} = \frac{1}{h}\frac{\partial \tilde{\varphi}_K}{\partial x'} \\ \frac{\partial \varphi_\ell}{\partial y} &= \frac{\partial \tilde{\varphi}_K}{\partial y'}\frac{\partial y'}{\partial y} = \frac{1}{h}\frac{\partial \tilde{\varphi}_K}{\partial y'} \end{aligned}\right\} \Rightarrow \nabla \varphi_\ell = \frac{1}{h}\nabla \tilde{\varphi}_K$$

Also suppose that $\underline{\Omega}$ is mapped to $\underline{\Omega}_j$.
Finally define: $b_j(x,y) = b((x - x_j)/h, (y - y_j)/h)$.
Then $b_j(x',y') = b(x,y)$

Putting all this together:

$$a(\varphi_j, \varphi_\ell) = \int_{\underline{\Omega}} \frac{1}{b(x,y)}\left[\nabla \varphi_j(x,y) \cdot \nabla \varphi_\ell(x,y)\right] dxdy =$$

$$= \int_{\underline{\Omega}_j^1} \frac{1}{b_j(x,y)}\left[\frac{1}{h}\nabla \tilde{\varphi}_0 \cdot \frac{1}{h}\nabla \tilde{\varphi}_K\right] h^2 dxdy =$$

$$= \int_{\underline{\Omega}_j} \frac{1}{b_j(x,y)}\left[\nabla \tilde{\varphi}_0 \cdot \nabla \tilde{\varphi}_K\right] dxdy$$

Note that if j is an interior vertex then $\underline{\Omega}_j$ includes A_1, A_2, A_3, A_4
If j lies on the left boundary then $\underline{\Omega}_j$ includes only A_1, A_3
If j lies on the right boundary then $\underline{\Omega}_j$ includes only A_2, A_4.

In general we define the vertex-element incidence:

$$I_{ij} = \begin{cases} 1, & A_i \subseteq \Omega_j \\ 0, & \text{otherwise} \end{cases}, \quad \forall j \in MUB, \ \forall i \in \{1,2,3,4\}$$

we also define the following form:

$$\boxed{a(b_j, \tilde{\varphi}_0, \tilde{\varphi}_k ; A_i) = \int_{A_i} \frac{1}{b_j(x,y)} \left[\nabla\tilde{\varphi}_0 \cdot \nabla\tilde{\varphi}_k \right] dx\,dy}$$

Then we have:

$$\begin{aligned} a(\varphi_j, \varphi_\ell) &= \int_{\Omega_j} \frac{1}{b_j(x,y)} \left[\nabla\tilde{\varphi}_0 \cdot \nabla\tilde{\varphi}_k \right] dx\,dy = \\ &= \sum_{i=1}^{4} I_{ij} \int_{A_i} \frac{1}{b_j(x,y)} \left[\nabla\tilde{\varphi}_0 \cdot \nabla\tilde{\varphi}_k \right] dx\,dy = \\ &= \sum_{i=1}^{4} I_{ij}\, a(b_j, \tilde{\varphi}_0, \tilde{\varphi}_k ; A_i). \end{aligned}$$

To summarize, we've shown that:

Thm: If $j \in MUB$ and $\ell = N_k(j) \in N(j)$ then

$$\boxed{a(\varphi_j, \varphi_\ell) = \sum_{i=1}^{4} I_{ij}\, a(b_j, \tilde{\varphi}_0, \tilde{\varphi}_k ; A_i)}$$

▼ Computing the a.

Now discuss the problem of computing $a(b_j, \tilde{\varphi}_0, \tilde{\varphi}_k, A_i)$. Consider first the ~~arbitrar~~ general case with

$$\begin{aligned} b(x,y) &= \text{arbitrary}. \\ f(x,y) &= (a_1 + b_1 x)(\gamma_1 + \delta_1 y). \\ g(x,y) &= (a_2 + b_2 x)(\gamma_2 + \delta_2 y). \end{aligned}$$

Then:

$$\frac{\partial f}{\partial x} \frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left[(a_1 + b_1 x)(\gamma_1 + \delta_1 y) \right] \frac{\partial}{\partial x}\left[(a_2 + b_2 x)(\gamma_2 + \delta_2 y) \right] =$$

$$= b_1(\gamma_1+\delta_1 y)\, b_2(\gamma_2+\delta_2 y) = b_1 b_2 (\gamma_1+\delta_1 y)(\gamma_2+\delta_2 y).$$

$$\frac{\partial f}{\partial y}\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left[(a_1+b_1 x)(\gamma_1+\delta_1 y)\right]\frac{\partial}{\partial y}\left[(a_2+b_2 x)(\gamma_2+\delta_2 y)\right] =$$

$$= \delta_1(a_1+b_1 x)\,\delta_2(a_2+b_2 x) = \delta_1\delta_2 (a_1+b_1 x)(a_2+b_2 x).$$

therefore:

$$a(b,f,g;A_i) = \int_{A_i} \frac{1}{b(x,y)}\left[\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right] dx\,dy =$$

$$= \int_{A_i} \frac{1}{b(x,y)}\left[\delta_1\delta_2 (a_1+b_1 x)(a_2+b_2 x) + b_1 b_2 (\gamma_1+\delta_1 y)(\gamma_2+\delta_2 y)\right] dx\,dy.$$

This integral can be evaluated numerically. The evaluating routine then requires the following input:

a) The function $b(x,y)$.

b) The parameters $(a_1, b_1, \gamma_1, \delta_1)$ and $(a_2, b_2, \gamma_2, \delta_2)$.

γ) $i \in \{1,2,3,4\}$.

To evaluate $a(\cancel{\varphi_0} b_j, \tilde{\varphi}_0, \tilde{\varphi}_k; A_i)$ we need <u>incidence matrices</u> $a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij}$ such that

$$\boxed{\tilde{\varphi}_j(x,y) = (a_{ij}+b_{ij}x)(\gamma_{ij}+\delta_{ij}y)\ ,\ \forall (x,y)\in A_i\ ,\ \forall j\in\{0,1,2,\dots,8\}}$$

These are problem indepedent and can be inlined to the code. Now we describe a way for generating them:

<u>Generating the incidences</u>

The $\tilde{\varphi}_j(x,y)$ functions are displaced versions of $\tilde{\varphi}_0(x,y)$ in the (p_j, q_j) direction. The incidences $a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij}$ describe what part of $\tilde{\varphi}_j$ shows up on region A_i. Note that it may be that it vanishes at region A_i.

We define an artificial "nowhere" region $i=0$. Everything vanishes in the nowhere region:

$$a_{0j} = b_{0j} = \gamma_{0j} = \delta_{0j}.$$

This is notational convenience for the following definition:

<u>Def</u>: Let $i \in \{1,2,3,4\}$ index a region A_i. and $j \in \{0,1,2,\dots,8\}$ be a direction. Then we define

$$m(i,j) = k \in \{0,1,2,3,4\} \quad \text{such that}$$

a) For $k \neq 0$, $\forall (x,y) \in A_i : (x-p_j, y-q_j) \in A_k$

b) For $k = 0$, $\forall (x,y) \in A_i : (x-p_j, y-q_j) \notin A_1 \cup A_2 \cup A_3 \cup A_4$

Finally we also define:

$$m(0,j) = 0. \qquad \square$$

<u>interpretation</u>: When you evaluate $\tilde{\varphi}_j$ at a point $(x,y) \in A_i$, you are actually evaluating ~~$\varphi_0(x,y)$~~ $\tilde{\varphi}_0$ at a point in A_k with $k = m(i,j)$.

Given $m(i,j)$ we can compute the incidences as follows:

For $j=0$ the incidences are known by definition:

$$a_{i0} = (1,1,1,1) \qquad \gamma_{i0} = (1,1,1,1)$$
$$b_{i0} = (-1,1,-1,1) \qquad \delta_{i0} = (-1,-1,1,1)$$

For $j>0$, let $(x,y) \in A_i$ be given and let $k = m(i,j)$. Then:

$$\tilde{\varphi}_j(x,y) = \tilde{\varphi}_0(x-p_j, y-q_j) = f_k(x-p_j, y-q_j) =$$
$$= [a_{k0} + b_{k0}(x-p_j)][\gamma_{k0} + \delta_{k0}(y-q_j)] =$$
$$= [(a_{k0} - b_{k0} p_j) + b_{k0} x][(\gamma_{k0} - \delta_{k0} q_j) + \delta_{k0} y]$$
$$= (a_{ij} + b_{ij} x)(\gamma_{ij} + \delta_{ij} y) \Rightarrow$$

$$\Rightarrow \boxed{\begin{array}{l} a_{ij} = a_{k0} - b_{k0} p_j \\ b_{ij} = b_{k0} \\ \gamma_{ij} = \gamma_{k0} - \delta_{k0} q_j \\ \delta_{ij} = \delta_{k0} \qquad , \ k = m(i,j) \end{array}}$$

To compute $m(i,j)$ introduce the following base-2 decompositions:

$i = i_1 + 2 i_2 + 1$	$(i_1, i_2) \in \{0,1\} \times \{0,1\}$
$m(i,j) = k = k_1 + 2k_2 + 1$	$(k_1, k_2) \in \{0,1\} \times \{0,1\}$

Note that (i_1, i_2) work as coordinates:

$2 = (1,0)$	$1 = (0,0)$
$4 = (1,1)$	$3 = (0,1)$

It follows that if $m(i,j) \neq 0$ then $(k_1, k_2) = (i_1, i_2) + (p_j, q_j)$.

Conversely if $(k_1,k_2)=(i_1,i_2)+(p_j,q_j)\in\{0,1\}\times\{0,1\}\Rightarrow m(i,j)=k$.
It follows that $m(i,j)$ can be evaluated with the following algorithm:

<u>Algorithm</u>: Let $i\in\{1,2,3,4\}$ and $j\in\{0,1,2,3,4,5,6,7,8\}$.

Decompose $i=i_1+2i_2+1$ with $(i_1,i_2)\in\{0,1\}\times\{0,1\}$.
Let $k_1=i_1+p_j$ and $k_2=i_2+q_j$.
If $(k_1,k_2)\in\{0,1\}\times\{0,1\}$ then
  $k=k_1+2k_2+1$
else
  $k=0$
endif
$m(i,j)=k$ $\square$

Note that there is a redundancy in this approach in two ways.
One stems from the following proposition:

<u>Prop</u>: If $\boxed{(p_k,q_k)\notin A_i\Rightarrow a(b_j,\tilde{\varphi}_0,\tilde{\varphi}_k;A_i)=0}$

<u>Proof</u>

▶ We claim that $(p_k,q_k)\notin A_i\Rightarrow m(i,k)=0$
It follows that
$\forall(x,y)\in A_i:(x-p_k,y-q_k)\notin A_1\cup A_2\cup A_3\cup A_4\Rightarrow$

$$\Rightarrow\tilde{\varphi}_0(x-p_k,y-p_k)=0\Rightarrow\tilde{\varphi}_k(x,y)=0,\ \forall(x,y)\in A_i\Rightarrow$$

$$\Rightarrow\nabla\tilde{\varphi}_k(x,y)=0,\forall(x,y)\in A_i\Rightarrow a(b_j,\tilde{\varphi}_0,\tilde{\varphi}_k;A_i)=0$$

To show the claim, suppose that
$m(i,k)\neq0\Rightarrow\exists\,l\in\{1,2,3,4\}:\forall[(x,y)\in A_i\Leftrightarrow(x-p_k,y-q_k)\in A_l]$ (1)

But because $(0,0)\in A_l,\forall l\in\{1,2,3,4\}$ it follows that (1) is satisfied when $(x-p_k,y-q_k)=(0,0)\Leftrightarrow(x,y)=(p_k,q_k)$.
Therefore:
 $(p_k,q_k)=(x,y)\in A_i$
which is a contradiction to our hypothesis.
Therefore $m(i,k)\neq0$ $\square$.

Solving for the solution

Recall that we have expressed the solution $\psi(x,y)$ as

$$\boxed{\psi(x,y) = \sum_{j\in M} q_j \varphi_j(x,y) + \sum_{j\in B} q_j \varphi_j(x,y).}$$

The Galerkin condition yielded.

$$\sum_{j\in M} a(\varphi_j,\varphi_k) q_j = \beta_k \ , \ \forall k\in M$$

$$\text{with } \beta_k = -\sum_{j\in B} q_j\, a(\varphi_j,\varphi_k). \ , \ \forall k\in M.$$

Note that:

$$j \notin N(k) \Rightarrow a(\varphi_j,\varphi_k) = 0.$$

It follows we can rewrite our equation for q_j as:

$$a(\varphi_k,\varphi_k) q_k + \sum_{j\in M\cap N(k)} a(\varphi_j,\varphi_k) q_j = \beta_k \Leftrightarrow$$

$$\Leftrightarrow \boxed{q_k = \frac{\beta_k}{a(\varphi_k,\varphi_k)} - \sum_{j\in M\cap N(k)} \frac{a(\varphi_j,\varphi_k)}{a(\varphi_k,\varphi_k)} q_j} \ , \ \forall k\in M$$

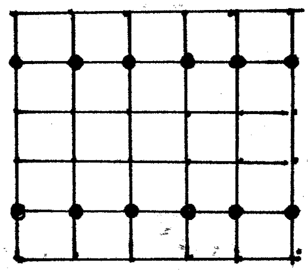
This expression can be used in a Gauss-Seidel iteration to relax q_k. Because we are summing over $M\cap N(k)$, if we are at a point in the Neumann boundary or the Dirichlet boundary, some of the neighbours do not lie in M. These neighbours we simply drop them from the summation, with the other neighbours unaffected!
Once the q_k are given: $\psi(x_k,y_k) = q_k$.

Initialization of the algorithm.

First we need to evaluate $\beta_k, \forall k\in M$. In general β_k is nonzero only at meshpoints in M which are neighbours to points in B. So the algorithm here is:
a) Initially set $\beta_k = 0, \forall k\in M$.
b) Consider points $k\in M$ such that $B\cap N(k) \neq \varnothing$ and then evaluate β_k noting that:

$$\beta_K = -\sum_{j\in B} q_j\, a(\varphi_j,\varphi_K) = -\sum_{j\in B\cap N(K)} q_j\, a(\varphi_j,\varphi_K).$$

These points are the ones shown in this figure:



The iterations propagate the infaluence of these points to the entire mesh.

Then for each vertex K we associate a vector of 9 numbers containing $a(\varphi_j,\varphi_K)$, $\forall j\in N(K)\cup\{K\}$. The ordering is based on (p_i,q_i). If a there is no $j\in N(K)$ for a given direction we simply drop the term and set the vector entry equal to 0.

► Verification : One way to confirm our computations of $a(\varphi_i,\varphi_j)$ is to assert the following theorem:

Thm : Let $K\in M$ be an interior vertex such that $B\cap N(K)=\emptyset$.
Then:

$$\boxed{\sum_{j\in M\cap N(K)} \frac{a(\varphi_j,\varphi_K)}{a(\varphi_K,\varphi_K)} + 1 = 0}$$

Proof

Note that $\psi(x,y)=1,\ \forall (x,y)\in\Omega$ is a solution to $\nabla\cdot\left(\frac{1}{\varepsilon}\nabla\psi\right)=0$ and it satisfies the Galerkin condition.

Also note that if we set $q_j=1,\ \forall j\in M\cup B$ then the finite element expansion is exactly constant:

$$\psi(x,y) = \sum_{j\in M\cup B} \varphi_j(x,y) = 1\ ,\ \forall (x,y)\in\Omega$$

The proof is tedious but straight forward.
This means then that

$$\sum_{j\in M} a(\varphi_j,\varphi_K) = -\sum_{j\in B} a(\varphi_j,\varphi_K).$$