▼ Spherical harmonic expansion

Let $\psi(\lambda,\mu)$ be a field on a sphere where
λ = the longitude
$\mu = \sin\varphi$ with φ = the latitude
Then $\psi(\lambda,\mu)$ is periodic in λ and has appropriate boundary condition at the poles $\mu = \pm 1$. Such functions have a spherical harmonic expansion:

$$\boxed{\psi(\lambda,\mu) = \sum_{m=0}^{+\infty} \sum_{n=|m|}^{+\infty} \tilde{\psi}_{m,n} Y_{m,n}(\lambda,\mu)}$$

where $Y_{m,n}(\lambda,\mu)$ = the spherical harmonic function of order (m,n) and it is defined as:

$$\boxed{Y_{m,n}(\lambda,\mu) = P_{m,n}(\mu)\, e^{2\pi i(m\lambda)}}$$

$$\boxed{P_{m,n}(\mu) = \left[\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!}\right]^{1/2} (1-\mu^2)^{m/2} \frac{d^m P_n(\mu)}{d\mu^m}}$$

$P_{m,n}(\mu)$ is the associated Legendre polynomial of order (m,n).
$P_n(\mu)$ is the Legendre polynomial of order n.
Then $\tilde{\psi}_{m,n}$ is the representation of $\psi(\lambda,\mu)$ in spherical harmonic space or, shortly, spectral space.
Note that:

$$\psi(\lambda,\mu) = \sum_{m=0}^{+\infty} \sum_{n=|m|}^{+\infty} \tilde{\psi}_{m,n} Y_{m,n}(\lambda,\mu) =$$

$$= \sum_{m=0}^{+\infty} \sum_{n=|m|}^{+\infty} \tilde{\psi}_{m,n} P_{m,n}(\mu)\, e^{2\pi i(m\lambda)} =$$

$$= \sum_{m=0}^{+\infty} \left[\sum_{n=|m|}^{+\infty} \tilde{\psi}_{m,n} P_{m,n}(\mu)\right] e^{2\pi i(m\lambda)}$$

Then the expansion can be evaluated faster in two steps:

$$\boxed{\begin{aligned} \psi_m(\mu) &= \sum_{n=|m|}^{+\infty} \tilde{\psi}_{m,n} P_{m,n}(\mu) \\ \psi(\lambda,\mu) &= \sum_{m=0}^{+\infty} \psi_m(\mu)\, e^{2\pi i(m\lambda)} \end{aligned}} \qquad (1)$$

For fixed m, the associated Legendre polynomials are orthogonal with orthogonality condition:

$$\int_{-1}^{1} P_{m,n}(\mu)\, P_{m,s}(\mu)\, d\mu = \delta_{ns}.$$

so the first step can be inverted. The second step is a Fourier transform. Together, we can obtain $\tilde{\psi}_{m,n}$ from $\psi(\lambda,\mu)$ like this:

$$\boxed{\begin{aligned} \tilde{\psi}_{m,n} &= \int_{-1}^{1} \psi_m(\mu)\, P_{m,n}(\mu)\, d\mu \\ \psi_m(\mu) &= \int_{-\pi}^{\pi} \psi(\lambda,\mu)\, e^{2\pi i(m\lambda)}\, d\lambda \end{aligned}}$$

▼ Evaluating the associated Legendre polynomial.

The problem of evaluating $P_{m,n}(\mu)$ is very non-trivial. For the purposes of theoretical development we define:

$$Q_{m,n}(\mu) = \frac{d^m P_n(\mu)}{d\mu^m} \quad \text{and} \quad A_{m,n} = \left[\frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}\right]^{1/2}$$

Then $P_{m,n}(\mu)$ is given by:

$$\boxed{P_{m,n}(\mu) = A_{m,n}\,(1-\mu^2)^{m/2}\, Q_{m,n}(\mu).}$$

Numerically $A_{m,n}$ grows factorially and $Q_{m,n}(\mu)$ vanishes factorially so we do not compute $Q_{m,n}(\mu)$ but $P_{m,n}(\mu)$ directly. To derive the relations for ~~$Q_{m,n}$~~ $P_{m,n}(\mu)$ we need relations for $Q_{m,n}(\mu)$.
We also make use of the following results for Legendre polynomials:

a) $\begin{cases} P_0(\mu) = 1, \quad P_1(\mu) = \mu \\ P_{n+1}(\mu) = \dfrac{2n+1}{n+1}\,\mu P_n(\mu) - \dfrac{n}{n+1} P_{n-1}(\mu). \end{cases}$

b) $(1-\mu^2)\dfrac{dP_n}{d\mu} = \dfrac{(n+1)n}{2n+1}\left(P_{n-1}(\mu) - P_{n+1}(\mu)\right)$

c) $\dfrac{d}{d\mu}\left[(1-\mu^2)\dfrac{dP_n}{d\mu}\right] = -n(n+1)\,P_n(\mu)$

To deal with m-order differentiation, we use the Leibnitz identity:

$$\frac{d^m}{d\mu^m}\left[f(\mu)g(\mu)\right] = \sum_{k=0}^{m} \binom{m}{k} f^{(k)}(\mu)\, g^{(m-k)}(\mu)$$

where $\binom{m}{k} = \frac{m!}{k!(m-k)!}$

● <u>Evaluating the $Q_{m,n}(\mu)$</u>

$$Q_{m,n+1}(\mu) = \frac{d^m P_{n+1}(\mu)}{d\mu^m} = \frac{d^m}{d\mu^m}\left[\frac{2n+1}{n+1}\mu P_n(\mu) - \frac{n}{n+1}P_{n-1}(\mu)\right] =$$

$$= \frac{2n+1}{n+1}\frac{d^m}{d\mu^m}\left[\mu P_n(\mu)\right] - \frac{n}{n+1}\frac{d^m}{d\mu^m}P_{n-1}(\mu) =$$

$$= \frac{2n+1}{n+1}\left[\sum_{k=0}^{m}\binom{m}{k}\left(\frac{d^k}{d\mu^k}\mu\right)\frac{d^{m-k}P_n(\mu)}{d\mu}\right] - \frac{n}{n+1}Q_{m,n-1}(\mu) =$$

$$= \frac{2n+1}{n+1}\left[\binom{m}{0}\mu Q_{m,n}(\mu) + \binom{m}{1}Q_{m-1,n}(\mu)\right] - \frac{n}{n+1}Q_{m,n-1}(\mu) =$$

$$= \frac{2n+1}{n+1}\left[\mu Q_{m,n}(\mu) + m Q_{m-1,n}(\mu)\right] - \frac{n}{n+1}Q_{m,n-1}(\mu).$$

This result will be useful later ~~it~~ in this form:

$$\boxed{Q_{m,n+1}(\mu) = \frac{2n+1}{n+1}\left[\mu Q_{m,n}(\mu) + m Q_{m-1,n}(\mu)\right] - \frac{n}{n+1}Q_{m,n-1}(\mu)} \quad (1)$$

To obtain a recurrence for $P_{m,n}(\mu)$ however we want to eliminate the $Q_{m-1,n}(\mu)$ term. To do this, consider:

$$\frac{d^m}{d\mu^m}\left[(1-\mu^2)\frac{dP_n(\mu)}{d\mu}\right] = \frac{d^{m-1}}{d\mu^{m-1}}\left\{\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_n(\mu)}{d\mu}\right]\right\} =$$

$$= \frac{d^{m-1}}{d\mu^{m-1}}\left[-n(n+1)P_n(\mu)\right] = -n(n+1)Q_{m-1,n}(\mu).$$

and alternatively:

$$\frac{d^m}{d\mu^m}\left[(1-\mu^2)\frac{dP_n(\mu)}{d\mu}\right] = \frac{d^m}{d\mu^m}\left[\frac{(n+1)n}{2n+1}\left(P_{n-1}(\mu) - P_{n+1}(\mu)\right)\right] =$$

$$= \frac{(n+1)n}{2n+1}\left[Q_{m,n-1}(\mu) - Q_{m,n+1}\right]$$

Putting these two relations together we get:

$$-n(n+1)\, Q_{m-1,n}(\mu) = \frac{n(n+1)}{2n+1}\Big[Q_{m,n-1}(\mu) - Q_{m,n+1}(\mu)\Big] \Rightarrow$$

$$\Rightarrow \boxed{Q_{m-1,n}(\mu) = \frac{1}{2n+1}\Big[Q_{m,n+1}(\mu) - Q_{m,n-1}(\mu)\Big]}$$

Substitute this to the recurrence:

$$Q_{m,n+1}(\mu) = \frac{2n+1}{n+1}\Big[\mu\, Q_{m,n}(\mu) + m\, Q_{m-1,n}(\mu)\Big] - \frac{n}{n+1} Q_{m,n-1}(\mu) =$$

$$= \frac{2n+1}{n+1}\Big[\mu\, Q_{m,n}(\mu) + \frac{m}{2n+1}\Big(Q_{m,n+1}(\mu) - Q_{m,n-1}(\mu)\Big)\Big] - \frac{n}{n+1} Q_{m,n-1}(\mu) =$$

$$= \frac{2n+1}{n+1}\mu\, Q_{m,n}(\mu) + \frac{m}{n+1} Q_{m,n+1} - \frac{m}{n+1} Q_{m,n-1} - \frac{n}{n+1} Q_{m,n-1}(\mu) =$$

$$= \frac{2n+1}{n+1}\mu\, Q_{m,n}(\mu) + \frac{m}{n+1} Q_{m,n+1} - \frac{m+n}{n+1} Q_{m,n-1}(\mu) \Rightarrow$$

$$\Rightarrow \Big(1 - \frac{m}{n+1}\Big) Q_{m,n+1}(\mu) = \frac{2n+1}{n+1}\mu\, Q_{m,n}(\mu) - \frac{m+n}{n+1} Q_{m,n-1}(\mu) \Rightarrow$$

$$\Rightarrow Q_{m,n+1}(\mu) = \frac{2n+1}{n-m+1}\mu\, Q_{m,n}(\mu) - \frac{n+m}{n-m+1} Q_{m,n-1}(\mu).$$

This is a nicer recurrence beacause all the Q polynomials have the same order m, and that simplifies manipulations later:

$$\boxed{Q_{m,n+1}(\mu) = \frac{2n+1}{n-m+1}\mu\, Q_{m,n}(\mu) - \frac{n+m}{n-m+1} Q_{m,n-1}(\mu).}$$

For $m=0$ this recurrence reduces to the one for Legendre polynomials. For verification, you ~~d~~ can check that it ~~satf~~ satisfies:

$$Q_{1,2}(\mu) = 3\mu \qquad Q_{1,3}(\mu) = \frac{1}{2}(15\mu^2 - 3) \qquad Q_{1,4}(\mu) = \frac{1}{2}(35\mu^3 - 15\mu).$$

To bootstrap the recursion we need an initial condition. From the definition we know that

$$\boxed{\begin{aligned} Q_{m,n}(\mu) &= 0 &&, \forall n < m \\ Q_{n,n}(\mu) &= a_n &&, \forall n \in \{0,1,2,\dots\} \end{aligned}}$$

where a_n is a constant, to be determined

To do that we need to use the original recurrence because it connects $Q_{m,n}(\mu), \forall n$ with $Q_{m-1,n}(\mu), \forall n$:

$$a_{n+1} = Q_{n+1,n+1}(\mu) = \frac{2n+1}{n+1}\left[\mu\, Q_{n+1,n}(\mu) + (n+1)\, Q_{n,n}(\mu)\right] - \frac{n}{n+1} Q_{n+1,n-1}(\mu) =$$

$$= \frac{2n+1}{n+1}\left[\mu \cdot 0 + (n+1) a_n\right] - \frac{n}{n+1}\cdot 0 = (2n+1) a_n \Rightarrow a_n = (2n-1) a_{n-1}.$$

Recall that $a_0 = Q_{0,0}(\mu) = P_0(\mu) = 1 \Rightarrow a_n = \prod_{k=1}^{n} (2k-1)$

Therefore putting it all-together, the recursion for computing $Q_{m,n}(\mu)$ is:

$$\begin{cases} Q_{n,n-1}(\mu) = 0 \qquad Q_{n,n}(\mu) = \prod_{k=1}^{n}(2k-1), & \forall n \in \{1,2,\dots\} \\ Q_{m,n+1}(\mu) = \dfrac{2n+1}{n-m+1}\,\mu\, Q_{m,n}(\mu) - \dfrac{n+m}{n-m+1} Q_{m,n-1}(\mu), & \forall m \in \{1,2,\dots\},\ \forall n \geq m. \end{cases}$$

- <u>Recurrence for associated Legendre polynomials.</u>

Recall that

$$P_{m,n}(\mu) = A_{m,n}\,(1-\mu^2)^{m/2}\, Q_{m,n}(\mu).$$

where $A_{m,n}$ has be defined as:

$$A_{m,n} = \left[\frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}\right]^{1/2}$$

Multiply both sides of the recurrence by $(1-\mu^2)^{m/2}$ and substitute in the associated Legendre polynomial

$$\frac{P_{m,n+1}(\mu)}{A_{m,n+1}} = \frac{2n+1}{n-m+1}\,\mu\,\frac{P_{m,n}(\mu)}{A_{m,n}} - \frac{n+m}{n-m+1}\,\frac{P_{m,n-1}(\mu)}{A_{m,n-1}} \Rightarrow$$

$$\Rightarrow P_{m,n+1}(\mu) = \frac{2n+1}{n-m+1}\,\frac{A_{m,n+1}}{A_{m,n}}\,\mu\, P_{m,n}(\mu) - \frac{n+m}{n-m+1}\,\frac{A_{m,n+1}}{A_{m,n-1}}\, P_{m,n-1}(\mu).$$

An interesting result can be obtained if we solve this relation for $\mu P_{m,n}(\mu)$:

$$\mu P_{m,n}(\mu) = \frac{n-m+1}{2n+1}\,\frac{A_{m,n}}{A_{m,n+1}}\left[P_{m,n+1} + \frac{n+m}{n-m+1}\,\frac{A_{m,n+1}}{A_{m,n-1}}\, P_{m,n-1}(\mu)\right] =$$

$$= \frac{n-m+1}{2n+1}\,\frac{A_{m,n}}{A_{m,n+1}}\, P_{m,n+1}(\mu) + \frac{n+m}{2n+1}\,\frac{A_{m,n}}{A_{m,n-1}}\, P_{m,n-1}(\mu)$$

Now define:

$$a = \frac{n-m+1}{2n+1}\,\frac{A_{m,n}}{A_{m,n+1}} \qquad \text{and} \qquad b = \frac{n+m}{2n+1}\,\frac{A_{m,n}}{A_{m,n-1}}.$$

and substitute the definition for $A_{m,n}$:

$$a = \frac{n-m+1}{2n+1}\,\frac{A_{m,n}}{A_{m,n+1}} = \frac{n-m+1}{2n+1}\,\frac{\left[\frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}\right]^{1/2}}{\left[\frac{2n+3}{2}\,\frac{(n-m+1)!}{(n+m+1)!}\right]^{1/2}} =$$

$$= \frac{n-m+1}{2n+1}\left[\frac{2n+1}{2n+3}\,\frac{(n-m)!}{(n-m+1)!}\,\frac{(n+m+1)!}{(n+m)!}\right]^{1/2} =$$

$$= \frac{n-m+1}{2n+1}\left[\frac{2n+1}{2n+3}\,\frac{n+m+1}{n-m+1}\right]^{1/2} = \left[\frac{(n-m+1)^2}{(2n+1)^2}\,\frac{2n+1}{2n+3}\,\frac{n+m+1}{n-m+1}\right]^{1/2} =$$

$$= \left[\frac{(n-m+1)(n+m+1)}{(2n+1)(2n+3)}\right]^{1/2} = \left[\frac{(n+1)^2-m^2}{(2(n+1)-1)(2(n+1)+1)}\right]^{1/2} =$$

$$= \left[\frac{(n+1)^2-m^2}{4(n+1)^2-1}\right]^{1/2}.$$

$$b = \frac{n+m}{2n+1}\,\frac{A_{m,n}}{A_{m,n-1}} = \frac{n+m}{2n+1}\,\frac{\left[\frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}\right]^{1/2}}{\left[\frac{2n-1}{2}\,\frac{(n-m-1)!}{(n+m-1)!}\right]^{1/2}} =$$

$$= \frac{n+m}{2n+1}\left[\frac{2n+1}{2n-1}\,\frac{(n-m)!}{(n-m-1)!}\,\frac{(n+m-1)!}{(n+m)!}\right]^{1/2} =$$

$$= \frac{n+m}{2n+1}\left[\frac{2n+1}{2n-1}\,\frac{n-m}{n+m}\right]^{1/2} = \left[\frac{(n+m)^2}{(2n+1)^2}\,\frac{2n+1}{2n-1}\,\frac{n-m}{n+m}\right]^{1/2} =$$

$$= \left[\frac{(n+m)(n-m)}{(2n+1)(2n-1)}\right]^{1/2} = \left[\frac{n^2-m^2}{4n^2-1}\right]^{1/2}.$$

If you compare a and b, you will notice that they are related as follows:

Define $\varepsilon_{m,n}$ by $\quad \varepsilon_{m,n} = \left[\dfrac{n^2-m^2}{4n^2-1}\right]^{1/2}.$

Then:

$$\frac{n-m+1}{2n+1}\,\frac{A_{m,n}}{A_{m,n+1}} = \varepsilon_{m,n+1} \qquad \text{and} \qquad \frac{n+m}{2n+1}\,\frac{A_{m,n}}{A_{m,n-1}} = \varepsilon_{m,n}$$

and we obtain the following recurrence:

$$\boxed{\mu P_{m,n}(\mu) = \varepsilon_{m,n+1} P_{m,n+1}(\mu) + \varepsilon_{m,n} P_{m,n-1}.}$$

That itself is a useful recurrence. For the purposes of ~~solving~~ evaluating $P_{m,n}(\mu)$ we want to solve it for $P_{m,n+1}(\mu)$:

$$\mu P_{m,n}(\mu) = \varepsilon_{m,n+1} P_{m,n+1}(\mu) + \varepsilon_{m,n} P_{m,n-1}(\mu) \Rightarrow$$

$$\Rightarrow \varepsilon_{m,n+1} P_{m,n+1}(\mu) = \mu P_{m,n}(\mu) - \varepsilon_{m,n} P_{m,n-1}(\mu) \Rightarrow$$

$$\Rightarrow \boxed{P_{m,n+1}(\mu) = \frac{1}{\varepsilon_{m,n+1}} \mu P_{m,n}(\mu) - \frac{\varepsilon_{m,n}}{\varepsilon_{m,n+1}} P_{m,n-1}(\mu).}$$

This is the recurrence that we use in practice.

● <u>Initial condition for associated Legendre polynomials.</u>

Similarly,

$$P_{n,n-1}(\mu) = 0 \quad \text{and} \quad P_{n,n}(\mu) = A_n (1-\mu^2)^{n/2}$$

where A_n is a constant given by:

$$A_n = A_{n,n} Q_{n,n}(\mu) = A_{n,n} a_n = \left[\frac{2n+1}{2} \frac{(n-n)!}{(n+n)!}\right]^{1/2} \prod_{k=1}^{n} (2k-1) =$$

$$= \left(\frac{2n+1}{2}\right)^{1/2} \frac{1}{[(2n)!]^{1/2}} \prod_{k=1}^{n} (2k-1) = \left(\frac{2n+1}{2}\right)^{1/2} \left[\prod_{k=1}^{n} \frac{1}{2k(2k-1)}\right]^{1/2} \prod_{k=1}^{n} (2k-1) =$$

$$= \left(\frac{2n+1}{2}\right)^{1/2} \left[\prod_{k=1}^{n} \frac{(2k-1)^2}{2k(2k-1)}\right]^{1/2} = \left(\frac{2n+1}{2}\right)^{1/2} \prod_{k=1}^{n} \left[\frac{2k-1}{2k}\right]^{1/2}$$

$$= \left(\frac{2n+1}{2}\right)^{1/2} \exp\left[\frac{1}{2} \sum_{k=1}^{n} \ln\left(\frac{2k-1}{2k}\right)\right]$$

Numerically A_n is well-behaved:

$A_1 \cong 0.86$	$A_{1000} \cong 4.22$
$A_{10} \cong 1.36$	$A_{10000} \cong 7.51$
$A_{100} \cong 2.37$	$A_{100000} \cong 13.35$

To summarize, we have:

$$
\begin{cases}
P_{n,n-1}(\mu) = 0 \,, \qquad P_{n,n}(\mu) = A_n (1-\mu^2)^{n/2} \\
P_{m,n+1}(\mu) = \dfrac{1}{\varepsilon_{m,n+1}} \, \mu P_{m,n}(\mu) - \dfrac{\varepsilon_{m,n}}{\varepsilon_{m,n+1}} \, P_{m,n-1}(\mu).
\end{cases}
$$

with,

$$A_n = \left(\frac{2n+1}{2}\right)^{1/2} \exp\left[\frac{1}{2} \sum_{k=1}^{n} \ln\left(\frac{2k-1}{2k}\right)\right]$$

$$\varepsilon_{m,n} = \left(\frac{n^2-m^2}{4n^2-1}\right)^{1/2}$$

The derivative $P'_{m,n}(\mu)$ can be evaluated by differentiating this reccurence :

$$
\begin{cases}
P'_{n,n-1}(\mu) = 0 \,, \qquad P'_{n,n}(\mu) = n A_n \mu (1-\mu^2)^{(n-1)/2} \\
P'_{m,n+1}(\mu) = \dfrac{1}{\varepsilon_{m,n+1}} \left(\mu P'_{m,n}(\mu) + P_{m,n}(\mu)\right) - \dfrac{\varepsilon_{m,n}}{\varepsilon_{m,n+1}} \, P_{m,n-1}(\mu)
\end{cases}
$$

▼ Properties of associated Legendre polynomials

Recall that $P_{m,n}(\mu) = A_{m,n}\,(1-\mu^2)^{m/2}\, Q_{m,n}(\mu)$.
with

$$A_{m,n} = \left[\frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}\right]^{1/2} \quad \text{and} \quad Q_{m,n}(\mu) = \frac{d^m P_n(\mu)}{d\mu}$$

We have already shown the following Lemmas:

a) $Q_{m,n+1}(\mu) = \frac{2n+1}{n+1}\left[\mu\, Q_{m,n} + m\, Q_{m-1,n}(\mu)\right] - \frac{n}{n+1}\, Q_{m,n-1}(\mu)$.

b) $Q_{m-1,n}(\mu) = \frac{1}{2n+1}\left[Q_{m,n+1}(\mu) - Q_{m,n-1}(\mu)\right]$

c) $Q_{m,n+1}(\mu) = \frac{2n+1}{n-m+1}\,\mu\, Q_{m,n}(\mu) - \frac{n+m}{n-m+1}\, Q_{m,n-1}(\mu)$

We have also shown that

a) $\frac{n-m+1}{2n+1}\,\frac{A_{m,n}}{A_{m,n+1}} = \varepsilon_{m,n+1}$

b) $\frac{n+m}{2n+1}\,\frac{A_{m,n}}{A_{m,n-1}} = \varepsilon_{m,n}$ where $\varepsilon_{m,n} = \left(\frac{n^2-m^2}{4n^2-1}\right)^{1/2}$

Finally we have already derived two identities:

$$\boxed{\mu P_{m,n}(\mu) = \varepsilon_{m,n+1} P_{m,n+1}(\mu) + \varepsilon_{m,n} P_{m,n-1}(\mu).}$$

$$\boxed{P_{m,n+1}(\mu) = \frac{1}{\varepsilon_{m,n+1}}\,\mu\, P_{m,n}(\mu) - \frac{\varepsilon_{m,n}}{\varepsilon_{m,n+1}}\, P_{m,n-1}(\mu)}$$

Now we derive a few more:

● Polar differentiation

For Legendre polynomials we have shown that

$$(1-\mu^2)\frac{d\tilde{P}_n(\mu)}{d\mu} = (n+1)\,\varepsilon_n \tilde{P}_{n-1}(\mu) - n\,\varepsilon_{n+1}\tilde{P}_{n+1}(\mu).$$

with $\varepsilon_n = \varepsilon_{0,n} = \frac{n}{\sqrt{4n^2-1}}$

It turns out that this generalizes to:

$$(1-\mu^2)\frac{dP_{m,n}(\mu)}{d\mu} = (n+1)\varepsilon_{m,n}P_{m,n-1}(\mu) - n\varepsilon_{m,n+1}P_{m,n+1}(\mu).$$

To show this, we can reuse some of the proof of the result for the Legendre polynomial.
Recall that $P_n(\mu) = \gamma_n q_{n,n}(\mu)$ with

$$\gamma_n = \frac{1}{2^n n!} \quad \text{and} \quad q_{m,n} = \frac{d^m}{d\mu^m}(\mu^2-1)^n.$$

The proof for the Legendre polynomial was based on the following identity:

$$\boxed{(1-\mu^2)q_{m+1,n}(\mu) = (m+1)\mu q_{m,n}(\mu) + \frac{m-2n-1}{2n+2}q_{m+1,n+1}(\mu).}$$

The problem for the associated Legendre polynomial can be also put in terms of $q_{m,n}(\mu)$. Note that:

$$Q_{m,n}(\mu) = \frac{d^m P_n(\mu)}{d\mu^m} = \frac{d^m}{d\mu^m}\left[\gamma_n q_{n,n}(\mu)\right] =$$
$$= \gamma_n \frac{d^m}{d\mu^m} q_{n,n}(\mu) = \gamma_n q_{m+n,n}(\mu).$$

therefore: $\boxed{Q_{m,n}(\mu) = \gamma_n q_{m+n,n}(\mu).}$

Given these, we first reduce the problem down to $Q_{m,n}(\mu)$:

$$\cancel{(1-\mu^2)\frac{dP_{m,n}(\mu)}{d\mu} = (1-\mu^2)\frac{d}{d\mu}\left[A_{m,n}(1-\mu^2)^{m/2}Q_{m,n}(\mu)\right] =}$$
$$\cancel{= (1-\mu^2)A_{m,n}Q_{m,n}(\mu)\frac{m}{2}(1-\mu^2)^{(m-2)/2}(-2\mu) + A_{m,n}(1-\mu^2)^{m/2}}$$

$$(1-\mu^2)\frac{dP_{m,n}(\mu)}{d\mu} = (1-\mu^2)\frac{d}{d\mu}\left[A_{m,n}(1-\mu^2)^{m/2}Q_{m,n}(\mu)\right] =$$
$$= (1-\mu^2)A_{m,n}\left[\frac{m}{2}(-2\mu)(1-\mu^2)^{(m-2)/2}Q_{m,n}(\mu) + (1-\mu^2)^{m/2}\frac{d}{d\mu}Q_{m,n}(\mu)\right] =$$
$$= -m\mu\left[A_{m,n}(1-\mu^2)^{m/2}Q_{m,n}(\mu)\right] + A_{m,n}(1-\mu^2)^{m/2}\left[(1-\mu^2)\frac{d}{d\mu}Q_{m,n}(\mu)\right] =$$
$$= -m\mu P_{m,n}(\mu) + A_{m,n}(1-\mu^2)^{m/2}\left[(1-\mu^2)\frac{d}{d\mu}Q_{m,n}(\mu)\right]$$

We already have an identity to eliminate $\mu P_{m,n}(\mu)$. The problem now reduces to evaluating:

$$(1-\mu^2)\frac{d}{d\mu}Q_{m,n}(\mu) = (1-\mu^2)Q_{m+1,n}(\mu) = (1-\mu^2)\gamma_n q_{m+n+1,n}(\mu).$$

To eliminate $(1-\mu^2)$ we apply the lemma for $q_{m,n}(\mu)$ with $m \to m+n$ and we obtain:

$$(1-\mu^2)\frac{dQ_{m,n}(\mu)}{d\mu} = \gamma_n\left[(m+n+1)\mu q_{m+n,n}(\mu) + \frac{(m+n)-2n-1}{2n+2}q_{m+n+1,n+1}(\mu)\right] =$$

$$= (m+n+1)\mu\left[\gamma_n q_{m+n,n}(\mu)\right] + \frac{m-n-1}{2n+2}\frac{\gamma_n}{\gamma_{n+1}}\left[\gamma_{n+1}q_{m+n+1,n+1}(\mu)\right] =$$

$$= (m+n+1)\mu Q_{m,n}(\mu) + \frac{m-n-1}{2n+2}\frac{\gamma_n}{\gamma_{n+1}}Q_{m,n+1}(\mu).$$

Recall that $\frac{\gamma_n}{\gamma_{n+1}} = 2(n+1) = 2n+2.$

Therefore: $$\boxed{(1-\mu^2)\frac{dQ_{m,n}(\mu)}{d\mu} = (m+n+1)\mu Q_{m,n}(\mu) + (m-n-1)Q_{m,n+1}(\mu).}$$

Now, back to the original problem:

$$(1-\mu^2)\frac{dP_{m,n}(\mu)}{d\mu} = -m\mu P_{m,n}(\mu) + A_{m,n}(1-\mu^2)^{m/2}\left[(1-\mu^2)\frac{d}{d\mu}Q_{m,n}(\mu)\right] =$$

$$= -m\mu P_{m,n}(\mu) + A_{m,n}(1-\mu^2)^{m/2}\left[(m+n+1)\mu Q_{m,n}(\mu) + (m-n-1)Q_{m,n+1}(\mu)\right] =$$

$$= -m\mu P_{m,n}(\mu) + (m+n+1)\mu P_{m,n}(\mu) + (m-n-1)\frac{A_{m,n}}{A_{m,n+1}}\left[A_{m,n+1}(1-\mu^2)^{m/2}Q_{m,n+1}(\mu)\right] =$$

$$= (n+1)\mu P_{m,n}(\mu) + (m-n-1)\frac{A_{m,n}}{A_{m,n+1}}P_{m,n+1}(\mu)$$

Recall that

$$\frac{n-m+1}{2n+1}\frac{A_{m,n}}{A_{m,n+1}} = \varepsilon_{m,n+1} \Rightarrow (m-n-1)\frac{A_{m,n}}{A_{m,n+1}} = -(2n+1)\varepsilon_{m,n+1}$$

and $\mu P_{m,n}(\mu) = \varepsilon_{m,n+1}P_{m,n+1}(\mu) + \varepsilon_{m,n}P_{m,n-1}(\mu)$

Substituting these two we get:

$$(1-\mu^2)\frac{dP_{m,n}(\mu)}{d\mu} = (n+1)\left[\varepsilon_{m,n+1}P_{m,n+1}(\mu) + \varepsilon_{m,n}P_{m,n-1}(\mu)\right] - (2n+1)\varepsilon_{m,n+1}P_{m,n+1}(\mu) =$$

$$= (n+1)\varepsilon_{m,n}P_{m,n-1}(\mu) + \left[(n+1)-(2n+1)\right]\varepsilon_{m,n+1}P_{m,n+1}(\mu) =$$

$$= (n+1)\varepsilon_{m,n}P_{m,n-1}(\mu) - n\varepsilon_{m,n+1}P_{m,n+1}(\mu)$$

- <u>Quasi-eigenfunction result</u>

With ordinary Legendre polynomials

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_n(\mu)}{d\mu}\right] = -n(n+1)P_n(\mu) \quad \text{or} \quad \frac{d}{d\mu}\left[(1-\mu^2)\frac{dQ_{0,n}(\mu)}{d\mu}\right] = -n(n+1)Q_{0,n}(\mu).$$

With associated Legendre polynomials this generalizes to:

$$\boxed{\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_{m,n}(\mu)}{d\mu}\right] = \left(-n(n+1)+\frac{m^2}{1-\mu^2}\right)P_{m,n}(\mu).}$$

To show this, we begin by generalizing the result for $Q_{0,n}(\mu)$ to $Q_{m,n}(\mu)$. Consider:

$$\frac{d^m}{d\mu^m}\left[(1-\mu^2)\frac{dQ_{0,n}(\mu)}{d\mu}\right] = \sum_{k=0}^{m}\binom{m}{k}\left[\frac{d^k}{d\mu^k}(1-\mu^2)\right]\left[\frac{d^{m-k}}{d\mu^{m-k}}\left(\frac{dQ_{0,n}}{d\mu}\right)\right] =$$

$$= \binom{m}{0}(1-\mu^2)Q_{m+1,n}(\mu) + \binom{m}{1}(-2\mu)Q_{m,n}(\mu) + \binom{m}{2}(-2)Q_{m-1,n}(\mu) =$$

$$= (1-\mu^2)Q_{m+1,n}(\mu) - 2m\mu Q_{m,n}(\mu) - m(m-1)Q_{m-1,n}(\mu) =$$

$$= (1-\mu^2)\frac{dQ_{m,n}(\mu)}{d\mu} - m\left[2\mu Q_{m,n}(\mu) + (m-1)Q_{m-1,n}(\mu)\right] \Rightarrow$$

$$\Rightarrow (1-\mu^2)\frac{dQ_{m,n}(\mu)}{d\mu} = \frac{d^m}{d\mu^m}\left[(1-\mu^2)\frac{dQ_{0,n}(\mu)}{d\mu}\right] + m\left[2\mu Q_{m,n}(\mu) + (m-1)Q_{m-1,n}(\mu)\right]$$

Now differentiate both sides wrt μ, substitute the relation for $Q_{0,n}(\mu)$ and simplify:

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dQ_{m,n}(\mu)}{d\mu}\right] = \frac{d^{m+1}}{d\mu^{m+1}}\left[(1-\mu^2)\frac{dQ_{0,n}(\mu)}{d\mu}\right] + m\frac{d}{d\mu}\left[2\mu Q_{m,n}(\mu) + (m-1)Q_{m-1,n}(\mu)\right] =$$

$$= \frac{d^m}{d\mu^m}\left[-n(n+1)Q_{0,n}(\mu)\right] + m\left[2\mu Q_{m+1,n}(\mu) + 2Q_{m,n}(\mu) + (m-1)Q_{m,n}(\mu)\right] =$$

$$= -n(n+1)Q_{m,n}(\mu) + m\left[2\mu Q_{m+1,n}(\mu) + (m+1)Q_{m,n}(\mu)\right] =$$

$$= \left[m(m+1) - n(n+1)\right]Q_{m,n}(\mu) + m\left[2\mu Q_{m+1,n}(\mu)\right]$$

We obtain then the following generalization:

$$\boxed{\frac{d}{d\mu}\left[(1-\mu^2)\frac{dQ_{m,n}(\mu)}{d\mu}\right]=[m(m+1)-n(n+1)]Q_{m,n}(\mu)+2m\mu\, Q_{m+1,n}(\mu).}$$

Now define $\boxed{S_{m,n}(\mu)=A_{m,n}\frac{d}{d\mu}\left[(1-\mu^2)^{m/2}(1-\mu^2)\frac{dQ_{m,n}(\mu)}{d\mu}\right]}$

We wish to derive a result for $S_{m,n}(\mu)$ as well before carrying on with the derivation.

$$S_{m,n}(\mu)=A_{m,n}(1-\mu^2)^{m/2}\frac{d}{d\mu}\left[(1-\mu^2)\frac{dQ_{m,n}(\mu)}{d\mu}\right]+$$
$$+A_{m,n}(1-\mu^2)\frac{dQ_{m,n}(\mu)}{d\mu}\frac{d}{d\mu}\left[(1-\mu^2)^{m/2}\right]=S^{(1)}_{m,n}(\mu)+S^{(2)}_{m,n}(\mu).$$

where $S^{(1)}_{m,n}(\mu)$ = the first term
$S^{(2)}_{m,n}(\mu)$ = the second term.

The first term can be simplified by the previous result:

$$S^{(1)}_{m,n}(\mu)=A_{m,n}(1-\mu^2)^{m/2}\left\{[m(m+1)-n(n+1)]Q_{m,n}(\mu)+2m\mu\, Q_{m+1,n}(\mu)\right\}=$$
$$=[m(m+1)-n(n+1)]P_{m,n}(\mu)+(2m\mu)A_{m,n}(1-\mu^2)^{m/2}Q_{m+1,n}(\mu).$$

For the second term, note that

$$\frac{d}{d\mu}\left[(1-\mu^2)^{m/2}\right]=\frac{m}{2}(1-\mu^2)^{m/2-1}(-2\mu)=-m\mu(1-\mu^2)^{m/2-1}\Rightarrow$$
$$\Rightarrow S^{(2)}_{m,n}(\mu)=A_{m,n}(1-\mu^2)\frac{dQ_{m,n}(\mu)}{d\mu}\left[-m\mu(1-\mu^2)^{m/2-1}\right]=$$
$$=-m\mu A_{m,n}(1-\mu^2)^{m/2}Q_{m+1,n}(\mu)$$

Putting them both together we obtain:

$$\boxed{S_{m,n}(\mu)=[m(m+1)-n(n+1)]P_{m,n}(\mu)+m\mu A_{m,n}(1-\mu^2)^{m/2}Q_{m+1,n}(\mu).}$$

Now consider the LHS of the identity that we want to prove:

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_{m,n}(\mu)}{d\mu}\right] = \frac{d}{d\mu}\left[(1-\mu^2)\frac{d}{d\mu}\left[A_{m,n}(1-\mu^2)^{m/2}Q_{m,n}(\mu)\right]\right] =$$

$$= \frac{d}{d\mu}\left[A_{m,n}(1-\mu^2)^{m/2}(1-\mu^2)\frac{dQ_{m,n}}{d\mu} + A_{m,n}(1-\mu^2)\,Q_{m,n}(\mu)\frac{d}{d\mu}\left[(1-\mu^2)^{m/2}\right]\right]$$

$$= S_{m,n}(\mu) + A_{m,n}\frac{d}{d\mu}\left[(1-\mu^2)\,Q_{m,n}(\mu)(-m\mu)(1-\mu^2)^{m/2-1}\right] =$$

$$= S_{m,n}(\mu) + A_{m,n}\frac{d}{d\mu}\left[-m\mu\,(1-\mu^2)^{m/2}Q_{m,n}(\mu)\right] =$$

$$= S_{m,n}(\mu) - m\frac{d}{d\mu}\left[\mu\,P_{m,n}(\mu)\right] =$$

$$= S_{m,n}(\mu) - m\left[P_{m,n}(\mu) + \mu\frac{d}{d\mu}P_{m,n}(\mu)\right] =$$

$$= \left[m(m\cancel{+1}) - n(n+1)\right]P_{m,n}(\mu) + m\mu A_{m,n}(1-\mu^2)^{m/2}Q_{m+1,n} - m\cancel{P_{m,n}(\mu)} - m\mu\frac{dP_{m,n}(\mu)}{d\mu} =$$

$$= \left[m^2 - n(n+1)\right]P_{m,n}(\mu) + m\mu A_{m,n}(1-\mu^2)^{m/2}Q_{m+1,n} - m\mu\frac{dP_{m,n}(\mu)}{d\mu}$$

Already we see that we have obtained part of the RHS.
Consider the last term:

$$\frac{dP_{m,n}(\mu)}{d\mu} = \frac{d}{d\mu}\left[A_{m,n}(1-\mu^2)^{m/2}Q_{m,n}(\mu)\right] =$$

$$= A_{m,n}(1-\mu^2)^{m/2}Q_{m+1,n}(\mu) + A_{m,n}Q_{m,n}(\mu)\frac{d}{d\mu}\left[(1-\mu^2)^{m/2}\right]$$

$$= A_{m,n}(1-\mu^2)^{m/2}Q_{m+1,n}(\mu) + A_{m,n}Q_{m,n}(\mu)\left[-m\mu(1-\mu^2)^{m/2-1}\right] \Rightarrow$$

$$\Rightarrow -m\mu\frac{dP_{m,n}(\mu)}{d\mu} = -m\mu A_{m,n}(1-\mu^2)^{m/2}Q_{m+1,n}(\mu) + m^2\mu^2 A_{m,n}Q_{m,n}(\mu)(1-\mu^2)^{m/2-1}$$

Part of the last term cancels the middle term and we obtain:

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_{m,n}(\mu)}{d\mu}\right] = \left[m^2 - n(n+1)\right]P_{m,n}(\mu) + (m\mu)^2 A_{m,n}(1-\mu^2)^{m/2-1}Q_{m,n}(\mu)$$

$$= \left[m^2 - n(n+1)\right]P_{m,n}(\mu) - m^2(1-\mu^2)A_{m,n}(1-\mu^2)^{m/2-1}Q_{m,n}(\mu)$$

$$+ m^2 A_{m,n}(1-\mu^2)^{m/2-1}Q_{m,n}(\mu) =$$

$$= \left[m^2 - n(n+1)\right] P_{m,n}(\mu) - m^2 P_{m,n}(\mu) + m^2 \frac{P_{m,n}(\mu)}{1-\mu^2} =$$

$$= \left[-n(n+1) + \frac{m^2}{1-\mu^2}\right] P_{m,n}(\mu).$$

We obtain then that

$$\boxed{\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_{m,n}(\mu)}{d\mu}\right] = \left[-n(n+1) + \frac{m^2}{1-\mu^2}\right] P_{m,n}(\mu)}$$

▼ The associated Legendre transform

The associated Legendre transform is an intermediate step to the more general spherical harmonic transform.
From the orthogonality condition

$$\int_{-1}^{1} P_{m,n}(\mu)\, P_{m,k}(\mu)\, d\mu = \delta_{nk}$$

it follows that if a function $f(\mu)$ can be expressed by an m-order associated Legendre expansion

$$f(\mu) = \sum_{n=m}^{+\infty} \tilde{f}_n P_{m,n}(\mu).$$

then $\tilde{f}_n$ is related with $f(\mu)$ with

$$\tilde{f}_n = \int_{-1}^{1} f(\mu)\, P_{m,n}(\mu)\, d\mu.$$

For the purposes of these transforms we <u>extend</u> the definition of $P_{m,n}(\mu)$ to negative $m<0$:

$$\boxed{P_{-m,n}(\mu) = (-1)^m P_{m,n}(\mu)}$$

Because $\varepsilon_{-m,n} = \varepsilon_{m,n}$ all the properties follow through.
Also, the orthogonality condition is true for $m<0$:

$$\int_{-1}^{1} P_{-m,n}(\mu)\, P_{-m,k}(\mu)\, d\mu = (-1)^m (-1)^m \int_{-1}^{1} P_{m,n}(\mu)\, P_{m,k}(\mu)\, d\mu = (-1)^{2m}\delta_{nk} = \delta_{nk}$$

We may then define the transforms in general by:

$$\boxed{f(\mu) = \sum_{n=|m|}^{+\infty} \tilde{f}_n P_{m,n}(\mu) \quad\Big|\quad \tilde{f}_n = \int_{-1}^{1} f(\mu)\, P_{m,n}(\mu)\, d\mu}$$

backward transform. forward transform

Note that because $P_{m,n}(\mu)$ in general is not a polynomial, it is not obvious if we can use Gauss-Legendre quadrature to discretize it. We discuss this next.

▼ Discrete orthogonality condition.

Recall that $P_{m,n}(\mu) = A_{m,n}(1-\mu^2)^{m/2} Q_{m,n}(\mu)$. for $m \geq 0$
Then,

$$P_{m,n}(\mu) P_{m,k}(\mu) = \left[A_{m,n}(1-\mu^2)^{m/2} Q_{m,n}(\mu)\right]\left[A_{m,k}(1-\mu^2)^{m/2} Q_{m,k}(\mu)\right] =$$

$$= (A_{m,n} A_{m,k})(1-\mu^2)^m Q_{m,n}(\mu) Q_{m,k}(\mu).$$

so the product $\underline{\text{is}}$ a polynomial. To compute the degree, note first that

$$\deg Q_{m,n}(\mu) = \deg\left[\gamma_n q_{m+n,n}(\mu)\right] = \deg\left[\frac{d^{m+n}}{d\mu^{m+n}}(1-\mu^2)^n\right] =$$

$$= \deg(1-\mu^2)^n - (m+n) = 2n-(m+n) = n-m$$

therefore:

$$\deg P_{m,n}(\mu) P_{m,k}(\mu) = \deg(1-\mu^2)^m + \deg Q_{m,n}(\mu) + \deg Q_{m,k}(\mu) =$$

$$= 2m + (n-m) + (k-m) = n+k.$$

$$\longrightarrow \boxed{\deg P_{m,n}(\mu) P_{m,k}(\mu) = n+k}$$

Note that the degree is independent of m.
Recall that N-resolution Gauss-Legendre integration is exact when the integrand is a polynomial up to degree $2N-1$.
Therefore:

$$\boxed{\sum_{j=0}^{N-1} A_j^{(N)} P_{m,n}(\mu_j^{(N)}) P_{m,k}(\mu_j^{(N)}) = \delta_{nk} \quad , \forall n,k \in \mathbb{N} : n \geq |m| \wedge k \geq |m| \quad , \forall N \in \mathbb{N} : n+k \leq 2N-1.}$$

This can be also verified numerically. Now we use this result to produce the discrete transforms.

▼ Discrete associated Legendre transforms

Suppose that for a given $M > |m|$ we want to consider functions that can be expanded as:

$$f(\mu) = \sum_{n=|m|}^{M} \tilde{f}_n P_{m,n}(\mu).$$

We would like to resolve the inverse with Gauss – Legendre quadrature. Note that:

$$\deg f(\mu)\, P_{m,n}(\mu) = \deg\left[\left(\sum_{k=|m|}^{M} \tilde{f}_k P_{m,k}(\mu) \right) P_{m,n}(\mu) \right] =$$

$$= \max_{k=|m|}^{M} \deg P_{m,k}(\mu) P_{m,n}(\mu) = \max_{k=|m|}^{M} (k+n) = M+n.$$

To resolve $\tilde{f}_n = \int_{-1}^{1} f(\mu) P_{m,n}(\mu)\, d\mu \;, \forall n \in \{|m|, \ldots, M\}$ we need

$$\deg f(\mu) P_{m,n}(\mu) \leqslant 2N-1 \;, \forall n \in \{|m|, \ldots, M\} \Leftrightarrow$$

$$\Leftrightarrow M+n \leqslant 2N-1 \,, \forall n \in \{|m|, \ldots, M\} \Leftrightarrow M+M \leqslant 2N-1 \Leftrightarrow N \geqslant M + \frac{1}{2} \Leftrightarrow$$

$\Leftrightarrow \underline{N \geqslant M+1}$, therefore

$$\boxed{N \geqslant M+1 \Rightarrow \tilde{f}_n = \sum_{j=0}^{N-1} A_j^{(N)} f(\mu_j^{(N)}) P_{m,n}(\mu_j^{(N)}) \;, \forall n \in \{|m|, \ldots, M\}.}$$

Given a resolution $N \geqslant M+1$, ~~then~~ we sample $f_j = f(\mu_j^{(N)})$. Then the forward and backward transforms are:

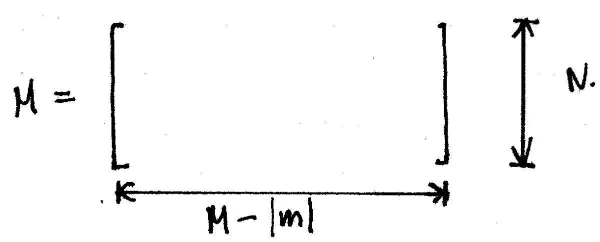
$f_j = \sum_{n=\lvert m\rvert}^{M} \tilde{f}_n P_{m,n}(\mu_j^{(N)})$	$\tilde{f}_n = \sum_{j=0}^{N-1} A_j^{(N)} \cancel{f(\mu)} f_j P_{m,n}(\mu_j^{(N)})$

with $n \in \{|m|, \ldots, M\}$ and $j \in \{0, \ldots, N-1\}$.

For implementation purposes, we may represent both transformations with a matrix

$$M_{jn} = P_{m,n}(\mu_j^{(N)}) \;, \forall (j,n) \in \{0, \ldots, N-1\} \times \{|m|, \ldots, M\}.$$

It is important to note that this is not a square matrix:



If we want to write:

$$\begin{aligned} f_j &= \sum_{n=|m|}^{M} B_{jn} \tilde{f}_n \\ \tilde{f}_n &= \sum_{j=0}^{N-1} f_j F_{nj} \end{aligned} \qquad \text{then} \qquad \begin{aligned} B_{jn} &= M_{jn} \\ F_{nj} &= A_j M_{jn}. \end{aligned}$$

Precomputing the matrix M is sufficient for doing transforms in both directions.

The spherical harmonic function

We define the (m,n)-order spherical harmonic function, where $(m,n) \in \mathbb{Z} \times \mathbb{N}$ by:

$$\boxed{Y_{m,n}(\lambda,\mu) = \begin{cases} P_{m,n}(\mu)\, e^{im\lambda} & \text{for } m \geq 0 \\ (-1)^m P_{-m,n}(\mu)\, e^{im\lambda}, & \text{for } m < 0 \end{cases}}$$

It is common to extend the definition of associated Legendre polynomials by

$$\boxed{P_{-m,n}(\mu) = (-1)^m P_{m,n}(\mu) \quad , \forall m \geq 0}$$

Since $\varepsilon_{m,n} = \varepsilon_{-m,n}$ all the properties of associated Legendre polynomials that we derived so far carry over to the extended definition. The extended definition allows us to write:

$$\boxed{Y_{m,n}(\lambda,\mu) = P_{m,n}(\mu)\, e^{im\lambda} \quad , \forall (m,n) \in \mathbb{Z} \times \mathbb{N}}$$

Properties of the spherical harmonics.

Two properties are direct corollaries of properties of associated Legendre polynomials:

1) $\mu Y_{m,n}(\lambda,\mu) = \varepsilon_{m,n+1} Y_{m,n+1}(\lambda,\mu) + \varepsilon_{m,n} Y_{m,n-1}(\lambda,\mu).$

2) $(1-\mu^2)\dfrac{\partial Y_{m,n}(\lambda,\mu)}{\partial \mu} = (n+1)\varepsilon_{m,n} Y_{m,n-1}(\lambda,\mu) - n\varepsilon_{m,n+1} Y_{m,n+1}(\lambda,\mu).$

where $\boxed{\varepsilon_{m,n} = \left(\dfrac{n^2-m^2}{4n^2-1}\right)^{1/2}}$

Another property is a direct corollary from the Fourier part:

$$\boxed{\frac{\partial^k Y_{m,n}(\lambda,\mu)}{\partial \lambda^k} = (im)^k\, Y_{m,n}(\lambda,\mu).}$$

A new result is that $Y_{m,n}(\lambda,\mu)$ is an eigenfunction to the spherical Laplacian which is defined by:

$$\boxed{\nabla^2\psi = \frac{1}{1-\mu^2}\frac{\partial^2\psi}{\partial\lambda^2} + \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu}\right]}$$

To show this:

$$\nabla^2 Y_{m,n}(\lambda,\mu) = \nabla^2\left[P_{m,n}(\mu)e^{im\lambda}\right] =$$

$$= \frac{1}{1-\mu^2}\frac{\partial^2}{\partial\lambda^2}\left[P_{m,n}(\mu)e^{im\lambda}\right] + \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial}{\partial\mu}\left(P_{m,n}(\mu)e^{im\lambda}\right)\right] =$$

$$= \frac{(im)^2}{1-\mu^2}Y_{m,n}(\lambda,\mu) + \left[-n(n+1) + \frac{m^2}{1-\mu^2}\right]Y_{m,n}(\lambda,\mu) =$$

$$= \left[-\frac{m^2}{1-\mu^2} - n(n+1) + \frac{m^2}{1-\mu^2}\right]Y_{m,n}(\lambda,\mu) = -n(n+1)Y_{m,n}(\lambda,\mu).$$

therefore:

$$\boxed{\nabla^2 Y_{m,n}(\lambda,\mu) = -n(n+1)Y_{m,n}(\lambda,\mu).}$$

It is this property that makes spherical harmonics nice for solving problems on a sphere.

▼ Spherical harmonic expansion

Let $\psi(\lambda,\mu)$ be a field on a sphere where

λ = the longitude

$\mu = \sin\varphi$ with φ = the latitude

Then $\psi(\lambda,\mu)$ is periodic in λ and has a fixed value for all λ at $\mu=\pm 1$. Such functions have a spherical harmonic expansion:

$$\boxed{\psi(\lambda,\mu) = \sum_{m=-\infty}^{+\infty}\sum_{n=|m|}^{+\infty}\tilde{\psi}_{m,n}Y_{m,n}(\lambda,\mu)}$$

$\tilde{\psi}_{m,n}$ = representation of $\psi(\lambda,\mu)$ in spherical harmonic space

Note that the expansion can be decomposed as follows:

$$\psi(\lambda,\mu) = \sum_{m\in\mathbb{Z}}\sum_{n=|m|}^{+\infty}\tilde{\psi}_{m,n}Y_{m,n}(\lambda,\mu) = \sum_{m\in\mathbb{Z}}\sum_{n=|m|}^{+\infty}\tilde{\psi}_{m,n}P_{m,n}(\mu)e^{im\lambda} =$$

$$= \sum_{m\in\mathbb{Z}}\left[\sum_{n=|m|}^{+\infty}\tilde{\psi}_{m,n}P_{m,n}(\mu)\right]e^{im\lambda}$$

therefore:

$$\boxed{\begin{aligned}\psi_m(\mu) &= \sum_{n=|m|}^{+\infty} \tilde{\psi}_{m,n} P_{m,n}(\mu).\\ \psi(\lambda,\mu) &= \sum_{m=-\infty}^{+\infty} \psi_m(\mu)\, e^{im\lambda}\end{aligned}}$$

Both steps can be inverted so the whole can be inverted:

$$\boxed{\begin{aligned}\tilde{\psi}_{m,n} &= \int_{-1}^{1} \psi_m(\mu) P_{m,n}(\mu)\, d\mu\\ \psi_m(\mu) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(\lambda,\mu)\, e^{-im\lambda}\, d\lambda\end{aligned}}$$

▼ The discrete spherical harmonic transform

The idea is to truncate the spherical harmonic expansion to a finite set of modes. In general:

$$\boxed{\psi(\lambda,\mu) = \sum_{(m,n)\in T} c_{m,n} Y_{m,n}(\lambda,\mu).}$$

We call T the truncation set of the expansion.
Also note the change in notation. We use $c_{m,n}$ for the mode coefficients and reserve $\tilde{\psi}_{m,n}$ for something else.

● Truncation and sampling issues.

The most general interesting way to define T is to set:

$$\boxed{\begin{aligned}S &= \{-M_1,\ldots,0,\ldots,M_2\}.\\ T &= \{(m,n)\in\mathbb{Z}^2 \mid -M_1 \leq m \leq M_2 \land |m| \leq n \leq N(m)\}\end{aligned}}$$

where M_1, M_2 are arbitrary and $N(m)$ is a function of $m \in S$.
At this point we will ignore concerns related to aliasing and the pseudospectral algorithm.

Define $M = |S| = M_1 + M_2 + 1$
Two common choices for $N(m)$ are:
a) Triangular truncation: $N(m) = M-1$
b) Rhomboidal truncation: $N(m) = |m| + M - 1$

However we will keep the analysis general for arbitrary $N(m)$.
Now we discuss discretizing $\psi(\lambda,\mu)$ and $\psi_m(\mu)$.
We discretize $\psi(\lambda,\mu)$ by:

$$\boxed{\psi_{j,k} = \psi\left(-\pi+2\pi\frac{j}{M},\ \mu_k^{(N)}\right)}$$

where $\mu_k^{(N)}$ = the k'th root of the order-N Legendre polynomial.
N = a constant related to $N(m)$.
We discretize $c_m(\mu)$ in a similar manner:

$$\boxed{c^m{}_k = c_m(\mu_k^{(N)})}$$

In terms of all this notation the truncated expansion can be written as:

$$\boxed{\begin{aligned} c^m{}_k &= \sum_{n=|m|}^{N(m)} c_{m,n} P_{m,n}(\mu) \\ \psi_{j,k} &= \sum_{m=-M_1}^{M_2} c^m{}_k \exp\left[im\left(-\pi+2\pi\frac{j}{M}\right)\right] \end{aligned}}$$

Note that the second step is a DFT but not exactly in FFTPACK form. So we now discuss how to handle this complication.

- ~~Forward~~ Backword DSHT (discrete spherical harmonic transform).

Suppose that $M_1 = M_2$ or $M_1 = M_2 - 1$ and $M = M_1 + M_2 + 1$.
Define $\ell = \begin{cases} M/2 & , M = \text{even} \\ (M-1)/2 & , M = \text{odd} \end{cases}$
and define $\tilde{\psi}_{m,n}$ in terms of $c_{m,n}$:

$$\boxed{\tilde{\psi}_{m,n} = \begin{cases} (-1)^m c_{m,n} & , m \in \{0,1,\ldots,\ell\} \\ (-1)^{m-M} c_{m-M,n} & , m \in \{\ell+1,\ldots,M-1\} \end{cases}}$$

This definition reorders the Fourier wavenumbers in such a manner so that they match the ones computed by FFTPACK DFT. An alternative statement is to use the wavenumber permutation:

$$\boxed{w(m) = \begin{cases} m & , m \in \{0,1,\ldots,\ell\} \\ m-M & ; m \in \{\ell+1,\ldots,M-1\}. \end{cases}}$$

and rewrite: $\boxed{\tilde{\psi}_{m,n} = (-1)^{w(m)} \; c_{w(m),n}}$

If we do a forward associated Legendre transform on $\tilde{\psi}_{m,n}$ over n of order $w(m)$ we obtain $\psi_m(\mu)$:

$$\boxed{\psi_m(\mu) = \sum_{n=|w(m)|}^{N(m)} \tilde{\psi}_{m,n} \, P_{w(m),n}(\mu)}$$

We sample $\psi_m(\mu)$ as we do $c_m(\mu)$:

$$\boxed{\psi^m{}_K = \psi_m(\mu_K^{(N)})}$$

Given this definitions we perform the forward DSHT as follows:

$$\boxed{\begin{aligned} \psi^m{}_K &= \sum_{n=|w(m)|}^{N(w(m))} \tilde{\psi}_{m,n} \, P_{w(m),n}(\mu_K^{(N)}) \\ \psi_{j,K} &= \sum_{m=0}^{M-1} \psi^m{}_K \exp\left[2\pi i \frac{jm}{M}\right] \end{aligned}}$$

The resulting $\tilde{\psi}_{m,n}$ is related to the $Y_{w(m),n}(\lambda,\mu)$ mode.

- ~~Backward~~ Forward DSHT

The backward DSHT can be obtained by inverting the steps in the forward DSHT:

$$\cancel{\psi^m{}_K = \frac{1}{M} \sum_{K\,m=0}^{M-1} v}$$

$$\boxed{\begin{aligned} \psi^m{}_K &= \frac{1}{M} \sum_{j=0}^{M-1} \psi_{j,K} \exp\left[-2\pi i \frac{jm}{M}\right] \\ \tilde{\psi}_{m,n} &= \sum_{K=0}^{N-1} A_K^{(N)} \, \psi^m{}_K \, P_{w(m),n}(\mu_K^{(N)}) \end{aligned}}$$

The first step is backward DFT. In the second step we use Gaussian quadrature to ~~ap~~ compute the integral

$$\tilde{\psi}_{m,n} = \int_{-1}^{1} \psi_m(\mu) \, P_{w(m),n}(\mu) \, d\mu$$

However recall that $P_{m,n}(\mu) = A_{m,n}\,(1-\mu^2)^{m/2}\,Q_{m,n}(\mu)$ is not a polynomial for m=odd so the use of Gaussian quadrature needs to be justified.

- Define for abbreviation purposes the following sets:

$$A(m) = \{k\in\mathbb{Z} \mid |w(m)| \leq k \leq N(w(m))\}\ ,\ \forall m \in \{0,1,\dots,M-1\}.$$
$$B = \{(m,n)\in\mathbb{Z}^2 \mid 0\leq m\leq M-1 \wedge n\in A(m)\}.$$

Then recall that

$$\psi_m(\mu) = \sum_{k\in B} \tilde{\psi}_{m,k}\, P_{w(m),k}(\mu)$$

therefore,

$$\psi_m(\mu)\, P_{w(m),n}(\mu) = P_{w(m),n}(\mu) \sum_{k\in B} \tilde{\psi}_{m,k}\, P_{w(m),k}(\mu) =$$
$$= \sum_{k\in B} \tilde{\psi}_{m,k}\, P_{w(m),k}(\mu)\, P_{w(m),n}(\mu) =$$
$$= \sum_{k\in B} \tilde{\psi}_{m,k} \left[A_{w(m),n}(1-\mu^2)^{w(m)/2} Q_{w(m),n}(\mu)\right]\left[A_{w(m),k}(1-\mu^2)^{w(m)/2} Q_{w(m),k}(\mu)\right] =$$
$$= \sum_{k\in B} \tilde{\psi}_{m,k} \left(A_{w(m),n} A_{w(m),k}\right)(1-\mu^2)^{w(m)}\, Q_{w(m),n}(\mu)\, Q_{w(m),k}(\mu).$$

Note that all the terms in this summation are polynomials so given sufficient resolution N, the integrand $\psi_m(\mu)\, P_{w(m),n}(\mu)$ can be computed with Gaussian quadrature.

The lower bound for N is:

$$2N-2 \geq \max_{(m,n)\in B} \deg\left[\psi_m(\mu)\, P_{w(m),n}(\mu)\right]$$

which we will now compute:

Note that

$$\deg Q_{m,n}(\mu) = \deg\left[\gamma_n\, q_{m+n,n}(\mu)\right] = \deg\left[\frac{d^{m+n}}{d\mu^{m+n}}(1-\mu^2)^n\right] =$$
$$= \deg(1-\mu^2)^n - (m+n) = 2n-(m+n) = n-m \Rightarrow$$

$$\Rightarrow \deg \psi_m(\mu)\, P_{w(m),n}(\mu) = \deg\left[\sum_{k\in B} \tilde{\psi}_{m,k}\left(A_{w(m),n} A_{w(m),k}\right)(1-\mu^2)^{w(m)} Q_{w(m),n}(\mu)\, Q_{w(m),k}(\mu)\right] =$$
$$= \max_{k\in B}\left[\deg(1-\mu^2)^{w(m)} + \deg Q_{w(m),n}(\mu) + \deg Q_{w(m),k}(\mu)\right] =$$
$$= \max_{k\in B}\left[2w(m) + (n-w(m)) + (k-w(m))\right] =$$
$$= \max_{k\in B}(k+n) = N(w(m)) + n \Rightarrow$$

$$\Rightarrow \max_{(m,n)\in B} \deg\left[\psi_m(\mu) P_{w(m),n}(\mu)\right] = \max_{(m,n)\in B}\left[N(w(m))+n\right]$$

$$= \max_{m=0}^{M-1} N(w(m)) + \max_{m=0}^{M-1} \max_{n\in A(m)} n =$$

$$= \max_{m\in S} N(m) + \max_{m=0}^{M-1} N(w(m)) = \max_{m\in S} N(m) + \max_{m\in S} N(m) =$$

$$= 2\max_{m\in S} N(m).$$

where recall that $S=\{-M_1,\dots,0,\dots,M_2\}$ where M_1, M_2 are chosen such that

$$\begin{cases} M_1+M_2+1 = M \\ M_1 = M_2 \ \vee \ M_1 = M_2 - 1 \end{cases}$$

It follows that

$$2N-2 \geqslant \max_{(m,n)\in B} \deg\left[\psi_m(\mu) P_{w(m),n}(\mu)\right] = 2\max_{m\in S} N(m) \Leftrightarrow$$

$$\Leftrightarrow \boxed{N \geqslant \max_{m\in S} N(m)+1}$$

This argument shows that we can indeed use Gaussian quadrature in the backward DSHT and provides a relation between N and $N(m)$.

Consider specific truncations:

a) ~~Nm~~ For triangular truncation

$$N(m) = M-1 \Rightarrow N \geqslant \max_{m\in S} N(m)+1 = (M-1)+1 = M$$

b) For rhomboidal truncation

$$N(m) = |m|+M-1 \Rightarrow N \geqslant \max_{m\in S}\left[|m|+M-1\right] = M_1+M-1$$

A relation between M_1 and M is obtained as follows:

i) If $M=\text{odd} \Leftrightarrow M_1+M_2=\text{even} \Leftrightarrow M_2=M_1 \Leftrightarrow 2M_1+1=M \Leftrightarrow M_1=(M-1)/2$.

ii) If $M=\text{even} \Leftrightarrow M_1+M_2=\text{odd} \Leftrightarrow M_1=M_2-1 \Leftrightarrow M_2=M_1+1 \Leftrightarrow$

$$\Leftrightarrow M_1+(M_1+1)+1 = M \Leftrightarrow 2(M_1+1) = M \Leftrightarrow M_1 = M/2-1$$

therefore: $$M_1 = \begin{cases} (M-1)/2, & M=\text{odd} \\ M/2-1, & M=\text{even}. \end{cases}$$

The real DSHT

If $\psi(\lambda,\mu) \in \mathbb{R}$ then the complex representation $\tilde{\psi}_{m,n}$ inherits redundancy from the Fourier step.
To resolve the redundancy define

$$a_{2m,n} = \mathrm{Re}[\tilde{\psi}_{m,n}] \quad \text{and} \quad a_{2m+1,n} = \mathrm{Im}[\tilde{\psi}_{m,n}]$$

such that

$$\tilde{\psi}_{m,n} = a_{2m,n} + i a_{2m+1,n}$$

Then it is sufficient to use:

$$a_{0,n}, a_{2,n}, \dots, a_{M,n}$$

We prefer to denote this sequence as:

$$\hat{\psi}_{0,n}, \hat{\psi}_{1,n}, \dots, \hat{\psi}_{M-1,n}$$

Recall that the Fourier ~~wavenulmber~~ wavenumber that corresponds to $\psi_{m,n}$ is given by:

$$w(m) = \begin{cases} m & , m \in \{0,1,\dots,\ell\} \\ m-M & , m \in \{\ell+1,\dots,M-1\} \end{cases} \quad \text{where } \ell = \begin{cases} M/2 & , M = \text{even} \\ (M-1)/2 & , M = \text{odd} \end{cases}$$

Now let $a(m)$ be the wavenumber corresponding to $\hat{\psi}_{m,n}$.
We now show how to compute $a(m)$.
Note that:

wavenumber of $a_{2m,n}$ = wavenumber of $\tilde{\psi}_{m,n}$ $= w(m)$.
wavenumber of $a_{2m+1,n}$ = wavenumber of $\tilde{\psi}_{m,n}$ $= w(m)$

The trick is to match m in $\hat{\psi}_{m,n}$ with the integer $K(m)$ in the corresponding $\tilde{\psi}_{K(m),n}$. Then $a(m) = w(K(m))$.

▶ Define $K(m) = a \Leftrightarrow \hat{\psi}_{m,n} = \mathrm{Re}[\tilde{\psi}_{a(m),n}] \vee \hat{\psi}_{m,n} = \mathrm{Im}[\tilde{\psi}_{a,n}]$
Then note that the correspondances work as follows:

$m = 0 \Rightarrow K(m) = 0$
$m+1 = 2a \Rightarrow K(m) = a$
$m+1 = 2a+1 \Rightarrow K(m) = a$, for $a \neq 0$

~~Sep Now solve for~~
To obtain $K(m)$ in closed form:

$$K(m)=a \Leftrightarrow \begin{cases} m+1=2a \\ m=\text{odd}\end{cases} \vee \begin{cases} m+1=2a+1 \\ m=\text{even}\end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a=(m+1)/2 \\ m=\text{odd}\end{cases} \vee \begin{cases} a=m/2 \\ m=\text{even}\end{cases} \Leftrightarrow a=\begin{cases}(m+1)/2 &, m=\text{odd}\\ m/2 &, m=\text{even}.\end{cases}$$

It follows that

$$K(m)=\begin{cases} 0 & \text{if } m=0 \\ (m+1)/2 & \text{if } m=\text{odd} \\ m/2 & \text{if } m=\text{even and } m\neq 0.\end{cases}$$

Note that this is a monotonically increasing function.
Also note that

$$K(M-1)=\begin{cases}(M-1+1)/2 &, \text{if } M-1=\text{odd}\\ (M-1)/2 &, \text{if } M-1=\text{even}\end{cases}$$

$$=\begin{cases} M/2 &, \text{if } M=\text{even}\\ (M-1)/2 &, \text{if } M=\text{odd}\end{cases} = \ell \Rightarrow$$

$$\Rightarrow \forall m\in\{0,1,\ldots,M-1\}: K(m)\leqslant \ell \Rightarrow a(m)=w(K(m))=K(m) \;\; (!!) \Rightarrow$$

$$\Rightarrow \boxed{a(m)=\begin{cases} 0 &, \text{if } m=0\\ (m+1)/2 &, \text{if } m=\text{odd}\\ m/2 &, \text{if } m=\text{even}\end{cases}}$$

Typical values of $a(m)$ are:

	$\hat{\psi}_{0,n}$	$\hat{\psi}_{1,n}$	$\hat{\psi}_{2,n}$	$\hat{\psi}_{3,n}$	$\hat{\psi}_{4,n}$	$\hat{\psi}_{5,n}$	$\hat{\psi}_{6,n}$
m	0	1	2	3	4	5	6
$a(m)$	0	1	1	2	2	3	3
	$\tilde{\psi}_{0,n}$	$\tilde{\psi}_{1,n}$		$\tilde{\psi}_{2,n}$		$\tilde{\psi}_{3,n}$	

In the figure we also show the correspondance between $\tilde{\psi}_{m,n}$ and $\hat{\psi}_{m,n}$ and how $a(m)$ encodes this correspondance.
Now recall that the forward and backward DSHT are:

Forward DSHT	Backward DSHT
$\psi^m{}_K=\frac{1}{M}\sum_{j=0}^{M-1}\psi_{j,K}\exp\left[-2\pi i\frac{jm}{M}\right]$	$\psi^m{}_K=\sum_{n=\lvert w(m)\rvert}^{N(w(m))}\tilde{\psi}_{m,n}P_{w(m),n}(\mu_K^{(N)})$
$\tilde{\psi}_{m,n}=\sum_{K=0}^{N-1}A_K^{(N)}\psi^m{}_K P_{w(m),n}(\mu_K^{(N)})$	$\psi_{j,K}=\sum_{m=0}^{M-1}\psi^m{}_K\exp\left[2\pi i\frac{jm}{M}\right]$

To obtain the real (forward and backward) DSHTs we substitute the DFT steps with real DFTs and we track the wavenumbers with $a(m)$ instead of $w(m)$:

$\hat{\psi}^m{}_k = \text{Forward RDFT } \psi_{j,k}$	$\hat{\psi}^m{}_k = \sum_{n=\lvert a(m)\rvert}^{N(a(m))} \tilde{\psi}_{m,n} P_{a(m),n}(\mu_k^{(N)})$
$\hat{\psi}_{m,n} = \sum_{k=0}^{N-1} A_k^{(N)} \hat{\psi}^m{}_k P_{a(m),n}(\mu_k^{(N)})$	$\psi_{j,k} = \text{Backward RDFT } \hat{\psi}^m{}_k$

Forward real DSHT   Backward real DSHT

Remarks.

a) The real DFT is complicated to express but it is understood that it is available as a subroutine

b) Both steps are implemented as vector $\longrightarrow$ vector transforms.
In the DFT step k loops over $k \in \{0,1,\dots,N-1\}$
In the associated Legendre step m loops over $m \in \{0,1,\dots,M-1\}$ and $a(m)$ is the order of the associated transform used.

c) Ignoring aliasing concerns we store $\psi_{j,k}$ and $\hat{\psi}_{m,n}$ in an array with size (M,N).
The indices j,k map directly to the ~~area~~ array indices.
The same is true for the m index.
However the n index is mapped to $n-\lvert a(m)\rvert$

d) The constraint $N \geq \max_{m\in S} N(m)+1$ guarantees that an array with dimensions (M,N) has sufficient space to store both the real and spectral representation of the field

e) These are the transforms used in practice