Legendre polynomials

The n^{th}-order Legendre polynomial is defined by.

$$\boxed{P_n(\mu) = \frac{1}{2^n n!}\frac{d^n}{d\mu^n}(\mu^2-1)^n}$$

The first few Legendre polynomials are:

$$P_0(\mu) = 1 \ , \quad P_1(\mu) = \mu \ , \quad P_2(\mu) = \frac{1}{2}(3\mu^2-1).$$

$$P_3(\mu) = \frac{1}{2}(5\mu^3-3\mu), \quad P_4(\mu) = \frac{1}{8}(35\mu^4-30\mu^2+3)$$

$$P_5(\mu) = \frac{1}{8}(65\mu^5-70\mu^3+15\mu)$$

$$P_6(\mu) = \frac{1}{16}(231\mu^6-315\mu^4+105\mu^2-5).$$

Numerically, $P_n(\mu)$ is evaluated through recurrencies which we will now develop.

Preliminary results

Define $\gamma_n = \frac{1}{2^n n!}$. Then $\gamma_n = \frac{1}{2n}\gamma_{n-1} = \frac{1}{4n(n-1)}\gamma_{n-2}$.

Also define the following auxilliary polynomials:

$q_{m,n}(\mu) = \frac{d^m}{d\mu^m}(\mu^2-1)^n$	$r_{m,n}(\mu) = \frac{d^m}{d\mu^m}\left[\mu(\mu^2-1)^n\right]$

The relation between these and the Legendre polynomial is:

$$\boxed{P_n(\mu) = \gamma_n \, q_{n,n}(\mu)}$$

The polynomials $q_{m,n}(\mu)$ and $r_{m,n}(\mu)$ are also closely related with each other.

In particular:

$$r_{m,n}(\mu) = \frac{d^m}{d\mu^m}\left[\mu(\mu^2-1)^n\right] = \sum_{k=0}^{m}\binom{m}{k}\frac{d^k\mu}{d\mu^k}\,\frac{d^{m-k}(\mu^2-1)^n}{d\mu^{m-k}} =$$
$$= \sum_{k=0}^{m}\binom{m}{k} q_{m-k,n}(\mu)\frac{d^k\mu}{d\mu^k} =$$
$$= \binom{m}{0}\mu q_{m,n}(\mu) + \binom{m}{1} q_{m-1,n}(\mu) = \mu q_{m,n}(\mu) + m q_{m-1,n}(\mu)$$

therefore: $\boxed{r_{m,n}(\mu) = \mu q_{m,n}(\mu) + m q_{m-1,n}(\mu).}$

Now use this result to attempt to obtain a recursion for $P_n(\mu)$. Begin with $q_{m,n}(\mu)$ and do one of m differentiations and see what comes out:

$$q_{m,n}(\mu) = \frac{d^m}{d\mu^m}(\mu^2-1)^n = \frac{d^{m-1}}{d\mu^{m-1}}\left[\frac{d}{d\mu}(\mu^2-1)^n\right] = \frac{d^{m-1}}{d\mu^{m-1}}\left[n(\mu^2-1)^{n-1}(2\mu)\right] =$$
$$= 2n\frac{d^{m-1}}{d\mu^{m-1}}\left[\mu(\mu^2-1)^{n-1}\right] = 2n\, r_{m-1,n-1}(\mu) =$$
$$= 2n\left[\mu q_{m-1,n-1}(\mu) + (m-1) q_{m-2,n-1}(\mu)\right]$$

therefore: $\boxed{q_{m,n}(\mu) = 2n\left[\mu q_{m-1,n-1}(\mu) + (m-1) q_{m-2,n-1}(\mu)\right]}$

This relation can almost yield a recursion for Legendre polynomials. In particular; for $m=n$

$$P_n(\mu) = \gamma_n q_{n,n}(\mu) = \gamma_n 2n\left[\mu q_{n-1,n-1}(\mu) + (n-1) q_{n-2,n-1}(\mu)\right] =$$
$$= (2n\gamma_n)\mu q_{n-1,n-1}(\mu) + 2n(n-1)\gamma_n q_{n-2,n-1}(\mu) =$$
$$= \mu\gamma_{n-1} q_{n-1,n-1}(\mu) + 2n(n-1)\gamma_n q_{n-2,n-1}(\mu) =$$
$$= \mu P_{n-1}(\mu) + 2n(n-1)\gamma_n q_{n-2,n-1}(\mu).$$

therefore: $\boxed{P_n(\mu) = \mu P_{n-1}(\mu) + 2n(n-1)\gamma_n q_{n-2,n-1}(\mu).}$

Coming this far is quite a challenge. The real magic however is in expressing $q_{n-2,n-1}(\mu)$ in terms of Legendre polynomials.

Evaluating the Legendre polynomial

- The magic trick is to consider $P_{n+1}(\mu) - P_{n-1}(\mu)$ and write it as the $(n-1)$-order derivative of something. That something, by miracle, is $q_{n-1,n}(\mu)$ which is what we want!

$$
\begin{aligned}
P_{n+1}(\mu) - P_{n-1}(\mu) &= \gamma_{n+1}\, q_{n+1,n+1}(\mu) - \gamma_{n-1}\, q_{n-1,n-1}(\mu) = \\
&= \frac{\gamma_{n-1}}{4n(n+1)} \frac{d^{n+1}(\mu^2-1)^{n+1}}{d\mu^{n+1}} - \gamma_{n-1} \frac{d^{n-1}(\mu^2-1)^{n-1}}{d\mu^{n-1}} = \\
&= \gamma_{n-1} \frac{d^{n-1}}{d\mu^{n-1}} \left[\frac{1}{4n(n+1)} \frac{d^2(\mu^2-1)^{n+1}}{d\mu^2} - (\mu^2-1)^{n-1} \right] = \\
&= \gamma_{n-1} \frac{d^{n-1}}{d\mu^{n-1}} \left[\frac{1}{4n\cancel{(n+1)}} \frac{d}{d\mu}\left(\cancel{(n+1)}(\mu^2-1)^n (2\mu) \right) - (\mu^2-1)^{n-1} \right] = \\
&= \gamma_{n-1} \frac{d^{n-1}}{d\mu^{n-1}} \left[\frac{1}{2n} \frac{d}{d\mu}\left(\mu(\mu^2-1)^n \right) - (\mu^2-1)^{n-1} \right] = \\
&= \gamma_{n-1} \frac{d^{n-1}}{d\mu^{n-1}} \left[\frac{1}{2n}\left((\mu^2-1)^n + \mu \cdot n(\mu^2-1)^{n-1}(2\mu) \right) - (\mu^2-1)^{n-1} \right] = \\
&= \gamma_{n-1} \frac{d^{n-1}}{d\mu^{n-1}} \left[\frac{1}{2n}(\mu^2-1)^n + \mu^2(\mu^2-1)^{n-1} - (\mu^2-1)^{n-1} \right] = \\
&= \gamma_{n-1} \frac{d^{n-1}}{d\mu^{n-1}} \left[\frac{1}{2n}(\mu^2-1)^n + (\mu^2-1)(\mu^2-1)^{n-1} \right] = \\
&= \gamma_{n-1} \frac{d^{n-1}}{d\mu^{n-1}} \left[\left(\frac{1}{2n}+1\right)(\mu^2-1)^n \right] = \frac{2n+1}{2n}\gamma_{n-1} \frac{d^{n-1}}{d\mu^{n-1}}\left[(\mu^2-1)^n\right] = \\
&= \frac{2n+1}{2n}\gamma_{n-1}\, q_{n-1,n}(\mu).
\end{aligned}
$$

therefore: $$\boxed{P_{n+1}(\mu) - P_{n-1}(\mu) = \frac{2n+1}{2n}\gamma_{n-1}\, q_{n-1,n}(\mu).}$$

We want to mix this identity with $P_n(\mu) = \mu P_{n-1}(\mu) + 2n(n-1)\gamma_n\, q_{n-2,n-1}(\mu)$. Decrease n by one and solve for $q_{n-2,n-1}(\mu)$:

$$P_n(\mu) - P_{n-2}(\mu) = \frac{2n-1}{2n-2}\gamma_{n-2}\, q_{n-2,n-1}(\mu) \Rightarrow$$

$$\Rightarrow q_{n-2,n-1}(\mu) = \frac{2n-2}{2n-1}\frac{1}{\gamma_{n-2}}\left[P_n(\mu) - P_{n-2}(\mu)\right]$$

and substitute this result in the recurrence for $P_n(\mu)$:

$$P_n(\mu) = \mu P_{n-1}(\mu) + 2n(n-1)\gamma_n\, q_{n-2,n-1}(\mu) =$$

$$= \mu P_{n-1}(\mu) + 2n(n-1)\gamma_n \frac{2n-2}{2n-1}\frac{1}{\gamma_{n-2}}\left[P_n(\mu) - P_{n-2}(\mu)\right] =$$

$$= \mu P_{n-1}(\mu) + 2n(n-1)\frac{2n-2}{2n-1}\frac{1}{4n(n-1)}\left[P_n(\mu) - P_{n-2}(\mu)\right] =$$

$$= \mu P_{n-1}(\mu) + \frac{1}{2}\frac{2n-2}{2n-1}\left[P_n(\mu) - P_{n-2}(\mu)\right] \Rightarrow$$ (multiply by 2 and solve for $P_n(\mu)$).

$$\Rightarrow 2P_n(\mu) - \frac{2n-2}{2n-1}P_n(\mu) = 2\mu P_{n-1}(\mu) - \frac{2n-2}{2n-1}P_{n-2}(\mu).$$

Note that: $2 - \dfrac{2n-2}{2n-1} = \dfrac{2(2n-1)-(2n-2)}{2n-1} = \dfrac{4n-2-2n+2}{2n-1} = \dfrac{2n}{2n-1}$

therefore:

$$P_n(\mu) = \frac{2n-1}{2n}\left[2\mu P_{n-1}(\mu) - \frac{2n-2}{2n-1}P_{n-2}(\mu)\right] =$$

$$= \frac{2n-1}{n}\mu P_{n-1}(\mu) - \frac{n-1}{n}P_{n-2}(\mu).$$

It is customary to rewrite this in terms of $P_{n+1}(\mu)$:

$$\boxed{P_{n+1}(\mu) = \frac{2n+1}{n+1}\mu P_n(\mu) - \frac{n}{n+1}P_{n-1}(\mu).}$$

This recurrence is what we use in practice to evaluate the Legendre polynomial numerically.

- Other properties of Legendre polynomials.

Recall that

$$q_{m,n}(\mu) = 2n\left[\mu q_{m-1,n-1}(\mu) + (m-1)q_{m-2,n-1}(\mu)\right]$$

By yet another "magic" manipulation we can obtain another significant property of Legendre polynomials.
The idea is to take $q_{m,n}(\mu)$, split $(\mu^2-1)^n = (\mu^2-1)(\mu^2-1)^{n-1}$, expand it out and see what happens.

$$q_{m,n}(\mu) = \frac{d^m}{d\mu^m}\left[(\mu^2-1)^n\right] = \frac{d^m}{d\mu^m}\left[(\mu^2-1)(\mu^2-1)^{n-1}\right] =$$

$$= \sum_{k=0}^{m}\binom{m}{k}\frac{d^k(\mu^2-1)}{d\mu^k}\frac{d^{m-k}(\mu^2-1)^{n-1}}{d\mu^{m-k}} =$$

$$= \sum_{k=0}^{m} \binom{m}{k} q_{m-k,n-1}(\mu) \frac{d^k(\mu^2-1)}{d\mu^k} =$$

$$= \binom{m}{0}(\mu^2-1) q_{m,n-1}(\mu) + \binom{m}{1} 2\mu q_{m-1,n-1}(\mu) + \binom{m}{2} 2 q_{m-2,n-1}(\mu) =$$

$$= (\mu^2-1) q_{m,n-1}(\mu) + 2m\mu q_{m-1,n-1}(\mu) + m(m-1) q_{m-2,n-1}(\mu) =$$

$$= (\mu^2-1) q_{m,n-1}(\mu) + m\mu q_{m-1,n-1}(\mu) + m\left[\mu q_{m-1,n-1}(\mu) + (m-1) q_{m-2,n-1}(\mu)\right] =$$

$$= (\mu^2-1) q_{m,n-1}(\mu) + m\mu q_{m-1,n-1}(\mu) + m \frac{1}{2n} q_{m,n}(\mu) \Rightarrow$$

$$\Rightarrow \left(1 - \frac{m}{2n}\right) q_{m,n}(\mu) = (\mu^2-1) q_{m,n-1}(\mu) + m\mu q_{m-1,n-1}(\mu)$$

Note that $q_{m,n-1}(\mu) = \frac{d}{d\mu} q_{m-1,n-1}(\mu)$ and for $m=n$, this is a Legendre polynomial! This is the key to obtaining the relation upon which we base spectral differentiation:

$$(1-\mu^2) q_{m,n-1}(\mu) = m\mu q_{m-1,n-1}(\mu) + \left(\frac{m}{2n} - 1\right) q_{m,n}(\mu) =$$

$$= m\mu q_{m-1,n-1}(\mu) + \frac{m-2n}{2n} q_{m,n}(\mu).$$

To make this more reasonable, increase both m and n by unity:

$$\boxed{(1-\mu^2) q_{m+1,n}(\mu) = (m+1)\mu q_{m,n}(\mu) + \frac{m-2n-1}{2n+2} q_{m+1,n+1}(\mu).}$$

Now we can derive the differentiation recurrence:

$$(1-\mu^2)\frac{dP_n(\mu)}{d\mu} = (1-\mu^2)\frac{d}{d\mu}\left[\gamma_n q_{n,n}(\mu)\right] = \gamma_n (1-\mu^2) q_{n+1,n}(\mu) =$$

$$\cancel{= \gamma_n\left[(n+1)\mu q_{m,n}(\mu) + \frac{n}{\quad}\right.}$$

$$= \gamma_n\left[(n+1)\mu q_{n,n}(\mu) + \frac{n-2n-1}{2n+2} q_{n+1,n+1}(\mu)\right] =$$

$$= (n+1)\mu P_n(\mu) - \frac{n+1}{2n+2}\gamma_n \frac{1}{\gamma_{n+1}} P_{n+1}(\mu) =$$

$$\cancel{= (n+1)\mu P_n(\mu) - \frac{n+1}{2n+2}\frac{\gamma_n}{2(n+1)\gamma_n} P_{n+1}(\mu) =}$$

Note that $\frac{\gamma_n}{\gamma_{n+1}} = \left(\frac{2^n n!}{2^{n+1}(n+1)!}\right)^{-1} = \frac{2^{n+1}(n+1)!}{2^n n!} = 2(n+1)$, therefore

$$(1-\mu^2)\frac{dP_n(\mu)}{d\mu} = (n+1)\mu P_n(\mu) - \frac{1}{2}2(n+1)P_{n+1}(\mu) = (n+1)[\mu P_n(\mu) - P_{n+1}(\mu)]$$

To make this identity useful for numerical use we want to eliminate the $\mu P_n(\mu)$ term. Recall that

$$P_{n+1}(\mu) = \frac{2n+1}{n+1}\mu P_n(\mu) - \frac{n}{n+1}P_{n-1}(\mu) \Rightarrow$$

$$\Rightarrow \mu P_n(\mu) = \frac{n+1}{2n+1}\left[P_{n+1}(\mu) + \frac{n}{n+1}P_{n-1}(\mu)\right] = \frac{n+1}{2n+1}P_{n+1}(\mu) + \frac{n}{2n+1}P_{n-1}(\mu)$$

Substituting, we obtain:

$$\begin{aligned}(1-\mu^2)\frac{dP_n(\mu)}{d\mu} &= (n+1)[\mu P_n(\mu) - P_{n+1}(\mu)] = \\ &= (n+1)\left[\frac{n+1}{2n+1}P_{n+1}(\mu) + \frac{n}{2n+1}P_{n-1}(\mu) - P_{n+1}(\mu)\right] = \\ &= (n+1)\left[\frac{n}{2n+1}P_{n-1}(\mu) + \frac{(n+1)-(2n+1)}{2n+1}P_{n+1}(\mu)\right] = \\ &= (n+1)\left[\frac{n}{2n+1}P_{n-1}(\mu) - \frac{n}{2n+1}P_{n+1}(\mu)\right] = \\ &= \frac{n(n+1)}{2n+1}\left[P_{n-1}(\mu) - P_{n+1}(\mu)\right]\end{aligned}$$

To summarize, the two important properties we derived are:

$$\boxed{\begin{aligned}&\mu P_n(\mu) = \frac{n+1}{2n+1}P_{n+1}(\mu) + \frac{n}{2n+1}P_{n-1}(\mu). \\ &(1-\mu^2)\frac{dP_n(\mu)}{d\mu} = \frac{n(n+1)}{2n+1}\left[P_{n-1}(\mu) - P_{n+1}(\mu)\right]\end{aligned}}$$

We will show their significance later.
We derive now the final important ~~result~~ recursion with Legendre polynomials.
We will show that $P_n(\mu)$ is the eigenfunction of the following operator:
by reusing the result that

$$P_{n+1}(\mu) - P_{n-1}(\mu) = \frac{2n+1}{2n}\gamma_{n-1}q_{n-1,n}(\mu)$$

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_n(\mu)}{d\mu}\right] = \frac{d}{d\mu}\left[\frac{n(n+1)}{2n+1}\left(P_{n-1}(\mu) - P_{n+1}(\mu)\right)\right] =$$

$$= -\frac{n(n+1)}{2n+1}\frac{d}{d\mu}\left[P_{n+1}(\mu) - P_{n-1}(\mu)\right] =$$

$$= -\frac{\cancel{n}(n+1)}{\cancel{2n+1}}\frac{d}{d\mu}\left[\frac{\cancel{2n+1}}{2\cancel{n}}\gamma_{n-1}\, q_{n-1,n}(\mu)\right] =$$

$$= -\frac{n+1}{2}\gamma_{n-1}\frac{d}{d\mu}q_{n-1,n}(\mu) = -\frac{n+1}{2}\gamma_{n-1}\, q_{n,n}(\mu) =$$

$$= -\frac{n+1}{2}\gamma_{n-1}\frac{1}{\gamma_n}P_n(\mu) = -\frac{n+1}{2}\, 2n\, P_n(\mu) = -n(n+1)P_n(\mu).$$

Therefore:

$$\boxed{\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_n(\mu)}{d\mu}\right] = -n(n+1)P_n(\mu).}$$

- <u>Evaluating the derivative of a Legendre polynomial</u>

Consider the recursion for $P_n(\mu)$:

$$P_{n+1}(\mu) = \frac{2n+1}{n+1}\left(\mu P_n(\mu)\right) - \frac{n}{n+1}P_{n-1}(\mu).$$

Differentiating it we get:

$$P'_{n+1}(\mu) = \frac{d}{d\mu}\left[\frac{2n+1}{n+1}\mu P_n(\mu) - \frac{n}{n+1}P_{n-1}(\mu)\right] =$$

$$= \frac{2n+1}{n+1}\frac{d}{d\mu}\left(\mu P_n(\mu)\right) - \frac{n}{n+1}P'_{n-1}(\mu).$$

$$= \frac{2n+1}{n+1}\left(\mu P'_n(\mu) + P_n(\mu)\right) - \frac{n}{n+1}P'_{n-1}(\mu)$$

The recursion for obtaining $P'_n(\mu)$ is:

$$\begin{cases} P'_0(\mu) = 0 \qquad P'_1(\mu) = 1 \\ P'_{n+1}(\mu) = \frac{2n+1}{n+1}\left[\mu P'_n(\mu) + P_n(\mu)\right] - \frac{n}{n+1}P'_{n-1}(\mu). \end{cases}$$

Of course it is necessary to run it parallel with the recursion for $P_n(\mu)$ since we need $P_n(\mu)$.

Computing the roots of Legendre polynomials

All Legendre polynomials $P_n(\mu)$ have n real roots in the open interval $(-1,1)$.

<u>notation</u>: Let $\mu_j^{(n)}$ = the j^{th} root of $P_n(\mu)$.

There is no closed form expression for $\mu_j^{(n)}$, therefore they must be evaluated numerically. We do this iteratively by the Newton iteration.
Let $\mu_{j,k}^{(n)}$ = the k^{th} guess of the j^{th} root of $P_n(\mu)$.
The iteration we use is:

$$\mu_{j,0}^{(n)} = -\cos\left[\left(\frac{4j+3}{2n+1}\right)\frac{\pi}{2}\right] \qquad \mu_{j,k+1}^{(n)} = \mu_{j,k}^{(n)} - \frac{P_n(\mu_{j,k}^{(n)})}{P_n'(\mu_{j,k}^{(n)})}$$

Then $\lim_{k\to+\infty} \mu_{j,k}^{(n)} = \mu_j^{(n)}$.

This converges with double precision accuracy with less than 10 iterations.
One way to decide when to terminate the iterations is:

- Iterate until $P_n(\mu) < 1e-10$. Then iterate 5 more times and stop.

Legendre polynomials and orthogonality.

The Legendre polynomials are orthogonal.
This means that

$$m \neq n \Rightarrow \int_{-1}^{1} P_m(\mu) P_n(\mu) = 0.$$

Proof

Define: $a_{m,n}(\mu) = (1-\mu^2)^{-1}(P_m(\mu) P_n'(\mu) - P_n(\mu) P_m'(\mu))$.
and notice that

$$a_{m,n}(1) = a_{m,n}(-1) = 0$$

because at $\mu = \pm 1 \Rightarrow 1-\mu^2 = 0$.
Now consider

$$\frac{d}{d\mu}\left[(1-\mu^2) P_m(\mu) P_n'(\mu)\right] = (1-\mu^2) P_m'(\mu) P_n'(\mu) + P_m\left[(1-\mu^2) P_n(\mu)\right]' =$$

$$= (1-\mu^2) P_m'(\mu) P_n'(\mu) + P_m(\mu)\left[-n(n+1) P_n(\mu)\right] =$$

$$= (1-\mu^2) P_m'(\mu) P_n'(\mu) - n(n+1) P_m(\mu) P_n(\mu).$$

Similarly:

$$\frac{d}{d\mu}\left[(1-\mu^2) P_n(\mu) P_m'(\mu)\right] = (1-\mu^2) P_m'(\mu) P_n'(\mu) - m(m+1) P_m(\mu) P_n(\mu).$$

Putting it all-together

$$\frac{d a_{m,n}(\mu)}{d\mu} = \frac{d}{d\mu}\left[(1-\mu^2) P_m(\mu) P_n'(\mu)\right] - \frac{d}{d\mu}\left[(1-\mu^2) P_m'(\mu) P_n(\mu)\right] =$$

$$= (1-\mu^2) P_m'(\mu) P_n'(\mu) - n(n+1) P_m(\mu) P_n(\mu) - (1-\mu^2) P_m'(\mu) P_n'(\mu) + m(m+1) P_m(\mu) P_n(\mu) =$$

$$= [m(m+1) - n(n+1)] P_m(\mu) P_n(\mu).$$

Since $m \neq n \Rightarrow m(m+1) - n(n+1) \neq 0$, therefore:

$$P_m(\mu) P_n(\mu) = \frac{1}{m(m+1) - n(n+1)} \frac{d a_{m,n}(\mu)}{d\mu} \Rightarrow$$

$$\Rightarrow \int_{-1}^{1} P_m(\mu) P_n(\mu)\, d\mu = \frac{1}{m(m+1) - n(n+1)} \int_{-1}^{1} \frac{d a_{m,n}(\mu)}{d\mu}\, d\mu =$$

$$= \frac{1}{m(m+1) - n(n+1)}\left[a_{m,n}(1) - a_{m,n}(-1)\right] = 0.$$

Because $a_{m,n}(1) = a_{m,n}(-1) = 0$ □

Now consider the case $m=n$.
Recall that $P_n(\mu) = \gamma_n q_{n,n}(\mu)$ where $\gamma_n = \frac{1}{2^n n!}$ and $q_{m,n}(\mu) = \frac{d^m}{d\mu^m}(\mu^2-1)^n$.

Now consider $q_{m,n}(\mu)$ at $\mu = \pm 1$.
By definition note that

if $m = \text{even} \Rightarrow q_{m,n}(\mu)$ is an even function

$m = \text{odd} \Rightarrow q_{m,n}(\mu)$ is an odd function.

It follows that

$$q_{m,n}(-1) = (-1)^n q_{m,n}(1).$$

If $m<n \Rightarrow \mu^2-1$ is a factor of $q_{m,n}(\mu) \Rightarrow q_{m,n}(1) = 0$
Therefore, combining with the relation for $q_{m,n}(-1)$ we have:

$$\boxed{q_{m,n}(-1) = q_{m,n}(1) = 0 \ , \ \forall m<n}$$

We use this to integrate by parts:

$$\int_{-1}^{1} q_{n+k,n}(\mu)\, q_{n-k,n}(\mu)\, d\mu = \int_{-1}^{1} q_{n+k,n}(\mu)\, q'_{n-k-1,n}(\mu)\, d\mu =$$

$$= \Big[q_{n+k,n}(\mu)\, q_{n-k,n}(\mu) \Big]_{-1}^{1} - \int_{-1}^{1} q'_{n+k,n}(\mu)\, q_{n-k-1}(\mu)\, d\mu =$$

$$= \big(q_{n+k,n}(1)\, q_{n-k}(1) - q_{n+k,n}(-1)\, q_{n-k,n}(-1) \big) - \int_{-1}^{1} q_{n+k+1,n}(\mu)\, q_{n-k-1,n}(\mu)\, d\mu.$$

$$= - \int_{-1}^{1} q_{n+k+1,n}(\mu)\, q_{n-k-1,n}(\mu)\, d\mu \quad \text{, because } q_{n-k,n}(1) = q_{n-k,n}(-1) = 0.$$

Using this result:

$$\int_{-1}^{1} P_n(\mu) P_n(\mu)\, d\mu = \int_{-1}^{1} (\gamma_n q_{n,n}(\mu))^2\, d\mu = \gamma_n^2 \int_{-1}^{1} q_{n,n}(\mu)\, q_{n,n}(\mu)\, d\mu =$$

$$= - \gamma_n^2 \int_{-1}^{1} q_{n+1,n}(\mu)\, q_{n-1,n}(\mu)\, d\mu = (-1)^k \gamma_n^2 \int_{-1}^{1} q_{n+k,n}(\mu)\, q_{n-k,n}(\mu).$$

Note that for $k=n$

$q_{n-k,n}(\mu) = q_{0,n} = (\mu^2-1)^n$

and

$$q_{n+k,n}(\mu) = q_{2n,n}(\mu) = \frac{d^{2n}}{d\mu^{2n}}\left[(\mu^2-1)^n\right] = (2n)!$$

therefore:

$$\int_{-1}^{1} P_n(\mu) P_n(\mu)\, d\mu = (-1)^n \gamma_n^2 \int_{-1}^{1} (2n)!\,(\mu^2-1)^n\, d\mu = \gamma_n^2\,(2n)! \int_{-1}^{1} (1-\mu^2)^n\, d\mu.$$

To evaluate the integral, define:

$$I_{k,n} = \int_{-1}^{1} \mu^k (1-\mu^2)^n\, d\mu.$$

and note that:

$$I_{k,n} = \int_{-1}^{1} \frac{d}{d\mu}\left[\frac{\mu^{k+1}}{k+1}\right](1-\mu^2)^n\, d\mu = -\int_{-1}^{1} \frac{\mu^{k+1}}{k+1}\frac{d}{d\mu}(1-\mu^2)^n\, d\mu =$$

$$= -\int_{-1}^{1} \frac{\mu^{k+1}}{k+1}\, n(1-\mu^2)^{n-1}(-2\mu)\, d\mu = +\frac{2n}{k+1}\int_{-1}^{1} \mu^{k+2}(1-\mu^2)^{n-1}\, d\mu = +\frac{2n}{k+1} I_{k+2,n-1}.$$

If we apply this repeatedly, we obtain:

$$\int_{-1}^{1} P_n(\mu) P_n(\mu)\, d\mu = \gamma_n^2 (2n)! \int_{-1}^{1} (1-\mu^2)^n\, d\mu = \gamma_n^2 (2n)!\, I_{0,n} =$$

$$= \gamma_n^2 (2n)! \left(\frac{2n}{0+1}\right) I_{2,n-1} = \gamma_n^2 (2n)! \left(\frac{2n}{1}\right)\left(\frac{2(n-1)}{3}\right) I_{4,n-2} =$$

$$= \gamma_n^2 (2n)! \left[\prod_{k=0}^{n-1} \frac{2(n-k)}{2k+1}\right] I_{2n,0} =$$

$$= \gamma_n^2 (2n)! \prod_{k=0}^{n-1} 2(n-k) \prod_{k=0}^{n-1} \frac{1}{2k+1} \int_{-1}^{1} \mu^{2n}\, d\mu =$$

$$= \gamma_n^2 (2n)! \prod_{k=1}^{n} 2k \prod_{k=0}^{n-1} \frac{1}{2k+1} \cdot \left[\frac{2}{2n+1}\right] =$$

$$= \gamma_n^2 (2n)!\; 2 \prod_{k=1}^{n} \frac{2k}{2k+1} = 2\frac{(2n)!}{[2^n n!]^2} \prod_{k=1}^{n} \frac{2k}{2k+1} =$$

$$\cancel{= \gamma_n^2 (2n)!}$$

$$= 2 \prod_{k=1}^{n} \frac{2k(2k-1)}{(2k)(2k)} \prod_{k=1}^{n} \frac{2k}{2k+1} = 2\prod_{k=1}^{n} \frac{2k-1}{2k+1} = 2\,\frac{1}{2n+1} =$$

$$= \frac{2}{2n+1}$$

To summorize : $$\boxed{\int_{-1}^{1} P_m(\mu)\, P_n(\mu)\, d\mu = \frac{\delta_{m,n}}{2n+1}}$$

The normalized Legendre polynomial

We define the <u>normalized</u> Legendre polynomial by:

$$\boxed{\tilde{P}_n(\mu) = \left(\frac{2n+1}{2}\right)^{1/2} P_n(\mu).}$$

Then $\tilde{P}_n(\mu)$ are not only orthogonal but also <u>orthonormal</u>:

$$\boxed{\int_{-1}^{1} \tilde{P}_m(\mu)\, \tilde{P}_n(\mu)\, d\mu = \delta_{m,n}}$$

Let $\psi(\mu)$ be a field on asphere with $\mu = \sin\varphi$ where φ=latitude. We assume that $\psi(\mu)$ is indepedent of the longitude λ. Such a field is called <u>axisymmetric</u>. If $\psi(\mu)$ is smooth enough to be a solution to the primitive equations, then it probably has a Taylor expansion around $\mu = 0$ that converges in $(-1,1)$. If that is true, then the Taylor expansion can be rearranged as a Legendre expansion:

$$\boxed{\psi(\mu) = \sum_{n=0}^{+\infty} a_n \tilde{P}_n(\mu)}$$

The orthonormality implies, as we argued in general, that:

$$\boxed{a_n = \int_{-1}^{1} \psi(\mu)\, \tilde{P}_n(\mu)\, d\mu}$$

We prefer to work with normalized Legendre polynomials because for all n, the scale of their typical value has the same order of magnitude. Numerically this has the effect of minimizing round-off error.

Properties of normalized Legendre polynomials.

We want to rewrite the following identities for normalized Legendre polynomials:

1) $P_{n+1}(\mu) = \frac{2n+1}{n+1}\mu P_n(\mu) - \frac{n}{n+1}P_{n-1}(\mu)$.

2) $\mu P_n(\mu) = \frac{n+1}{2n+1}P_{n+1}(\mu) + \frac{n}{2n+1}P_{n-1}(\mu)$.

3) $(1-\mu^2)\frac{dP_n(\mu)}{d\mu} = \frac{n(n+1)}{2n+1}\left[P_{n-1}(\mu) - P_{n+1}(\mu)\right]$

4) $\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_n(\mu)}{d\mu}\right] = -n(n+1)P_n(\mu)$.

~~We wo~~ Equation 4 retains its form:

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\tilde{P}_n(\mu)}{d\mu}\right] = -n(n+1)\tilde{P}_n(\mu)$$

The other equations need to be modified. Define $b_n = \left(\frac{2n+1}{2}\right)^{1/2}$.
Consider equation 2:

$$\mu\tilde{P}_n(\mu) = \mu b_n P_n(\mu) = \frac{n+1}{2n+1} b_n P_{n+1}(\mu) + \frac{n}{2n+1} b_n P_{n-1}(\mu) =$$

$$= \frac{n+1}{2n+1}\frac{b_n}{b_{n+1}}\tilde{P}_{n+1}(\mu) + \frac{n}{2n+1}\frac{b_n}{b_{n-1}}\tilde{P}_{n-1}(\mu)$$

and consider the coefficients seperately:

$$a = \frac{n+1}{2n+1}\frac{b_n}{b_{n+1}} = \frac{n+1}{2n+1}\frac{\left(\frac{2n+1}{2}\right)^{1/2}}{\left(\frac{2n+3}{2}\right)^{1/2}} = \frac{n+1}{2n+1}\left(\frac{2n+1}{2n+3}\right)^{1/2} =$$

$$= \left[\frac{(n+1)^2(2n+1)}{(2n+1)^2(2n+3)}\right]^{1/2} = \left[\frac{(n+1)^2}{(2n+1)(2n+3)}\right]^{1/2}$$

$$b = \frac{n}{2n+1}\frac{b_n}{b_{n-1}} = \frac{n}{2n+1}\left(\frac{2n+1}{2n-1}\right)^{1/2} = \left[\frac{n^2(2n+1)}{(2n+1)^2(2n-1)}\right]^{1/2} =$$

$$= \left[\frac{n^2}{(2n+1)(2n-1)}\right]^{1/2}$$

Note that if we define: $\varepsilon_n = \frac{n}{\sqrt{4n^2-1}}$, then: $a = \varepsilon_{n+1}$, $b = \varepsilon_n$.

It follows that:

$$\boxed{\mu\tilde{P}_n(\mu) = \varepsilon_{n+1}\tilde{P}_{n+1}(\mu) + \varepsilon_n \tilde{P}_{n-1}(\mu).}$$

Which is a very cute result.
Solving for $\tilde{P}_{n+1}(\mu)$ we obtain the substitute for equation 1:

$$\boxed{\tilde{P}_{n+1}(\mu) = \frac{1}{\varepsilon_{n+1}}\mu\tilde{P}_n(\mu) - \frac{\varepsilon_n}{\varepsilon_{n+1}}\tilde{P}_{n-1}(\mu).}$$

To conclude, consider equation 3:

$$(1-\mu^2)\frac{d\tilde{P}_n(\mu)}{d\mu} = b_n(1-\mu^2)\frac{dP_n(\mu)}{d\mu} = b_n\frac{n(n+1)}{2n+1}\left[P_{n-1}(\mu) - P_{n+1}(\mu)\right] =$$

$$= b_n\frac{n(n+1)}{2n+1}\left[\frac{1}{b_{n-1}}\tilde{P}_{n-1}(\mu) - \frac{1}{b_{n+1}}\tilde{P}_{n+1}(\mu)\right] =$$

$$= (n+1)\left[\frac{n}{2n+1}\frac{b_n}{b_{n-1}}\right]\tilde{P}_{n-1}(\mu) - n\left[\frac{n+1}{2n+1}\frac{b_n}{b_{n+1}}\right]\tilde{P}_{n+1}(\mu).$$

Recall that $\frac{n}{2n+1}\frac{b_n}{b_{n-1}} = \varepsilon_n$ and $\frac{n+1}{2n+1}\frac{b_n}{b_{n+1}} = \varepsilon_{n+1}$, therefore:

$$\boxed{(1-\mu^2)\frac{d\tilde{P}_n(\mu)}{d\mu} = (n+1)\varepsilon_n P_{n-1}(\mu) - n\,\varepsilon_{n+1}P_{n+1}(\mu).}$$

Putting it all-together:

1) $\tilde{P}_{n+1}(\mu) = \frac{1}{\varepsilon_{n+1}}\mu\tilde{P}_n(\mu) - \frac{\varepsilon_n}{\varepsilon_{n+1}}\tilde{P}_{n-1}(\mu)$

2) $\mu\tilde{P}_n(\mu) = \varepsilon_{n+1}\tilde{P}_{n+1}(\mu) + \varepsilon_n\tilde{P}_{n-1}(\mu)$

3) $(1-\mu^2)\frac{d\tilde{P}_n(\mu)}{d\mu} = (n+1)\varepsilon_n\tilde{P}_{n-1}(\mu) - n\,\varepsilon_{n+1}\tilde{P}_{n+1}(\mu)$

4) $\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\tilde{P}_n(\mu)}{d\mu}\right] = -n(n+1)\tilde{P}_n(\mu).$

▼ Discrete Legendre transform

Let $\psi(\mu)$ be a field, as before, that has a Legendre expansion. Then:

$$\psi(\mu) = \sum_{n=0}^{+\infty} a_n \tilde{P}_n(\mu).$$

$$a_n = \int_{-1}^{1} \psi(\mu)\, \tilde{P}_n(\mu)\, d\mu.$$

Numerically we retain only a finite number of a_n and we evaluate the integral with Gaussian integration.

● Gaussian-Legendre integration.

We omit the theoretical development and describe how to use this method. The problem is to compute an integral of the form:

$$I = \int_{-1}^{1} f(\mu)\, d\mu.$$

Let N = the resolution of the integration method.
Then we approximate I with the following expression:

$$I_N = \sum_{K=0}^{N-1} A_K f(\mu_K).$$

where $\mu_K = \mu_K^{(N)}$ ⤳ the K'th root of the N-order Legendre polynomial.

$$A_K = \frac{2}{1-(\mu_K^{(N)})^2} \, \frac{1}{[P_N'(\mu_K^{(N)})]^2}$$ ⤳ where P_N' is the un-normalized polynomial.

This method is exact ~~for~~ when $f(\mu)$ is a polynomial of up to ~~Nth~~ $2N-1$ order. It follows that if we use resolution N to evaluate the integral (forward transform) we must use at most ~~resol~~ N terms to represent the expansion of $f(\mu)$ to Legendre polynomials (Backward-transform.

• The discrete transform

Since Gauss-Legendre integration samples at points $\mu_K = \mu_K^{(N)}$ we represent $\psi(\mu)$ in real space by:

$$\boxed{\psi_K = \psi(\mu_K^{(N)}).}$$

The coefficients appearing in the spectral representation of $\psi(\mu)$ we denote as $\tilde{\psi}_n$.
Then the forward and backward ~~Legend~~ discrete Legendre transforms are written as:

$$\boxed{\tilde{\psi}_n = \sum_{K=0}^{N-1} A_K \psi_K \tilde{P}_n(\mu_K^{(N)})}$$

$$\boxed{\psi_K = \sum_{n=0}^{N-1} \tilde{\psi}_n \tilde{P}_n(\mu_K^{(N)}).}$$

If we write ψ and $\tilde{\psi}$ as vectors of ψ_K and $\tilde{\psi}_n$, then these equations are linear transformations that can be written as:

$$\boxed{\tilde{\psi} = F(N)\psi} \qquad \boxed{\psi = B(N)\tilde{\psi}}$$

where $F_{nK} = \tilde{P}_n(\mu_K^{(N)}) A_K$ ⇝ real space ⟶ spectral space
Forward transform

$B_{Kn} = \tilde{P}_n(\mu_K^{(N)}).$ ⇝ spectral space ⟶ real space
Backward transform.

These matrices can be precomputed during initialization.

▼ Operations in spectral space

The following operations can be easily computed in spectral space:

• Polar differentiation

Let $\psi(\mu)$ be a field. We want to compute $f(\mu) = (1-\mu^2)\dfrac{d\psi(\mu)}{d\mu}$.

Suppose that $\psi(\mu) = \sum_{n=0}^{+\infty} a_n \tilde{P}_n(\mu)$

$$f(\mu) = \sum_{n=0}^{+\infty} b_n \tilde{P}_n(\mu).$$

Recall that $(1-\mu^2)\dfrac{d\tilde{P}_n(\mu)}{d\mu} = (n+1)\varepsilon_n \tilde{P}_{n-1}(\mu) - n\varepsilon_{n+1}\tilde{P}_{n+1}(\mu)$

Then,

$$f(\mu) = (1-\mu^2)\frac{d\psi(\mu)}{d\mu} = (1-\mu^2)\frac{d}{d\mu}\sum_{n=0}^{+\infty} a_n \tilde{P}_n(\mu) =$$

$$= \sum_{n=0}^{+\infty} a_n \left[(1-\mu^2)\frac{d\tilde{P}_n(\mu)}{d\mu}\right] =$$

$$= \sum_{n=0}^{+\infty} a_n \left[(n+1)\varepsilon_n \tilde{P}_{n-1}(\mu) - n\varepsilon_{n+1}\tilde{P}_{n+1}(\mu)\right] =$$

$$= \sum_{n=0}^{+\infty} a_n (n+1)\varepsilon_n \tilde{P}_{n-1}(\mu) - \sum_{n=0}^{+\infty} a_n n\,\varepsilon_{n+1}\tilde{P}_{n+1}(\mu).$$

Recall that $\varepsilon_n = \dfrac{n}{\sqrt{4n^2-1}} \Rightarrow \varepsilon_0 = 0$

This means that we can rewrite these summations as:

$$\sum_{n=0}^{+\infty} a_n (n+1)\varepsilon_n \tilde{P}_{n-1}(\mu) = \sum_{n=-1}^{+\infty} a_{n+1}(n+2)\varepsilon_{n+1}\tilde{P}_n(\mu) = \sum_{n=0}^{+\infty} a_{n+1}(n+2)\varepsilon_{n+1}\tilde{P}_n(\mu).$$

$$\sum_{n=0}^{+\infty} a_n n\,\varepsilon_{n+1}\tilde{P}_{n+1}(\mu) = \sum_{n=1}^{+\infty} a_{n-1}(n-1)\varepsilon_n \tilde{P}_n(\mu) = \sum_{n=0}^{+\infty}(n-1)a_{n-1}\varepsilon_n \tilde{P}_n(\mu)$$

therefore:

$$f(\mu) = \sum_{n=0}^{+\infty}\left[(n+2)\varepsilon_{n+1}a_{n+1} - (n-1)\varepsilon_n a_{n-1}\right]\tilde{P}_n(\mu). \Rightarrow$$

$$\Rightarrow \boxed{b_n = (n+2)\varepsilon_{n+1}a_{n+1} - (n-1)\varepsilon_n a_{n-1}}$$

In terms of the discrete Legendre transform:

$$\boxed{\begin{aligned} \tilde{f}_0 &= 2\varepsilon_1\tilde{\psi}_1 \\ \tilde{f}_n &= (n+2)\varepsilon_{n+1}\tilde{\psi}_{n+1} - (n-1)\varepsilon_n\tilde{\psi}_{n-1} \\ \tilde{f}_N &= -(N-1)\varepsilon_N\tilde{\psi}_{N-1} \end{aligned}} \quad \text{for } n \in \{1, 2, \dots, N-1\}.$$

Note that

- μ- multiplication

If we are using a Galerkin spectral method then all fields are represented by default in spectral space. Suppose that we are given $\psi(\mu)$ in terms of $\tilde{\psi}_n$ and we want $f(\mu) = \mu\psi(\mu)$ in terms of $\tilde{f}_K$. It turns out that this computation can also be done entirely in spectral space.

Recall that,

$$\mu \tilde{P}_n(\mu) = \varepsilon_{n+1} \tilde{P}_{n+1}(\mu) + \varepsilon_n \tilde{P}_{n-1}(\mu)$$

Suppose that

$$\psi(\mu) = \sum_{n=0}^{+\infty} a_n \tilde{P}_n(\mu) \quad \text{and} \quad \mu\psi(\mu) = \sum_{n=0}^{+\infty} b_n \tilde{P}_n(\mu).$$

Then,

$$\mu\psi(\mu) = \mu \sum_{n=0}^{+\infty} a_n \tilde{P}_n(\mu) = \sum_{n=0}^{+\infty} a_n \left[\mu \tilde{P}_n(\mu)\right] =$$

$$= \sum_{n=0}^{+\infty} a_n \left[\varepsilon_{n+1} \tilde{P}_{n+1}(\mu) + \varepsilon_n \tilde{P}_{n-1}(\mu)\right] =$$

$$= \sum_{n=0}^{+\infty} a_n \varepsilon_{n+1} \tilde{P}_{n+1}(\mu) + \sum_{n=0}^{+\infty} a_n \varepsilon_n \tilde{P}_{n-1}(\mu) =$$

$$= \sum_{n=1}^{+\infty} a_{n-1} \varepsilon_n \tilde{P}_n(\mu) + \sum_{n=-1}^{+\infty} a_{n+1} \varepsilon_{n+1} \tilde{P}_n(\mu) =$$

$$= \sum_{n=0}^{+\infty} a_{n-1} \varepsilon_n \tilde{P}_n(\mu) + \sum_{n=0}^{+\infty} a_{n+1} \varepsilon_{n+1} \tilde{P}_n(\mu)$$

$$= \sum_{n=0}^{+\infty} \left[a_{n-1} \varepsilon_n + a_{n+1} \varepsilon_{n+1}\right] \tilde{P}_n(\mu) \Rightarrow \boxed{b_n = \varepsilon_n a_{n-1} + \varepsilon_{n+1} a_{n+1}}$$

In terms of the discrete Legendre transform:

$$\boxed{\begin{aligned} \tilde{f}_0 &= \varepsilon_1 \tilde{\psi}_1 \\ \tilde{f}_n &= \varepsilon_n \tilde{\psi}_{n-1} + \varepsilon_{n+1} \tilde{\psi}_{n+1} \\ \tilde{f}_N &= \varepsilon_N \tilde{\psi}_{N-1}. \end{aligned}} \;, \; n \in \{1, 2, \ldots, N-1\}.$$

• Computing the Jacobian

Let $\psi(\mu)$ be a field as before and $f(\mu) = \dfrac{d\psi(\mu)}{d\mu}$

Suppose that $\psi(\mu)$ can be represented by a discrete Legendre expansion with N terms:

$$\psi(\mu) = \sum_{n=0}^{N-1} \tilde{\psi}_n \tilde{P}_n(\mu)$$

Then $\psi(\mu)$ is a polynomial of order $N-1$. Since $f(\mu) = d\psi(\mu)/d\mu$ will be a polynomial of order $N-2$ it can also be resolved by a discrete Legendre expansion with the same amount of terms:
Define a matrix D_{ij} by:

$$\tilde{P}_n'(\mu) = \sum_{j=0}^{N-1} D_{jn} \tilde{P}_j(\mu). \quad , \forall n \in \{0,1,\dots,N-1\}.$$

Note that all $\tilde{P}_n'(\mu)$, $\forall n \in \{0,1,\dots,N-1\}$ have such expansions for the same reason. It follows that:

$$f(\mu) = \frac{d\psi(\mu)}{d\mu} = \frac{d}{d\mu} \sum_{n=0}^{N-1} \tilde{\psi}_n \tilde{P}_n(\mu) = \sum_{n=0}^{N-1} \tilde{\psi}_n \tilde{P}_n'(\mu) =$$

$$= \sum_{n=0}^{N-1} \tilde{\psi}_n \left[\sum_{j=0}^{N-1} D_{jn} \tilde{P}_j(\mu) \right] = \sum_{j=0}^{N-1} \left[\sum_{n=0}^{N-1} D_{jn} \tilde{\psi}_n \right] \tilde{P}_j(\mu).$$

It follows that

$$\tilde{f}_j = \sum_{n=0}^{N-1} D_{jn} \tilde{\psi}_n$$

or: $\tilde{f} = D\tilde{\psi}$.
Therefore, the matrix D is the linear operator which is equivalent to direct differentiation in spectral space.
To compute D transform all $\tilde{P}_n'(\mu)$ to spectral space, $\forall n \in \{0,\dots,N-1\}$ and obtain the <u>columns</u> of D.

Legendre transforms and aliasing.

Suppose that $\psi(\mu)$ has Legendre modes for $n \in B$ such that

$$f(\mu) = \sum_{n\in B} \gamma_n \tilde{P}_n(\mu) \quad \text{with} \quad \gamma_n = \int_{-1}^{1} \psi(\mu)\, \tilde{P}_n(\mu)\, d\mu$$

We would like to derive a result similar to Shannon's sampling theorem.

Problem: Consider a subset of the modes $\Gamma \subseteq B$ that are interesting. How much resolution N is required so that a discrete Legendre transform can correctly resolve γ_n, $\forall n \in \Gamma$?

Solution: The discrete Legendre transform computes:

$$\gamma_n^{(N)} = \sum_{k=0}^{N-1} A_k^{(N)} f(\mu_k^{(N)})\, \tilde{P}_n(\mu_k^{(N)}). \;, \quad \forall n \in \mathbb{N}.$$

Taken for $n \in \mathbb{N}$ this is the Legendre transform of the following function:

$$f_N(\mu) = \sum_{j=0}^{N-1} A_j^{(N)} f(\mu_j^{(N)})\, \delta(\mu - \mu_j^{(N)}). = \sum_{n=0}^{+\infty} \gamma_n^{(N)} \tilde{P}_n(\mu).$$

We verify and proceed:

$$\gamma_n^{(N)} = \int_{-1}^{1} f_N(\mu)\, \tilde{P}_n(\mu)\, d\mu = \int_{-1}^{1} \left[\sum_{j=0}^{N-1} A_j^{(N)} f(\mu_j^{(N)})\, \delta(\mu - \mu_j^{(N)}) \right] \tilde{P}_n(\mu)\, d\mu =$$

$$= \sum_{j=0}^{N-1} A_j^{(N)} f(\mu_j^{(N)}) \left[\int_{-1}^{1} P_n(\mu)\, \delta(\mu - \mu_j^{(N)})\, d\mu \right] =$$

$$= \sum_{j=0}^{N-1} A_j^{(N)} f(\mu_j^{(N)})\, P_n(\mu_j^{(N)}) =$$

$$\cancel{= \sum_{j=0}^{N-1} A_j^{(N)} \left[\sum_{k\in B} \gamma_k \tilde{P}_k(\mu) \right] P_n}$$

$$= \sum_{j=0}^{N-1} A_j^{(N)} \left[\sum_{k\in B} \gamma_k \tilde{P}_k(\mu_j^{(N)}) \right] \tilde{P}_n(\mu_j^{(N)}) =$$

$$= \sum_{j=0}^{N-1} \sum_{k\in B} A_j^{(N)} \gamma_k \tilde{P}_k(\mu_j^{(N)})\, \tilde{P}_n(\mu_j^{(N)}).$$

$$= \sum_{k\in B} \left[\sum_{j=0}^{N-1} A_j^{(N)} \tilde{P}_k(\mu_j^{(N)})\, \tilde{P}_n(\mu_j^{(N)}) \right] \gamma_k. = \sum_{k\in B} I_{nk} \gamma_k.$$

where we define: $I_{nk} = \sum_{j=0}^{N-1} A_j^{(N)} \tilde{P}_k(\mu_j^{(N)}) \tilde{P}_n(\mu_j^{(N)})$

Recall that Gaussian quadrature is exact for polynomials with degree $2N-1$ and since $\deg I_{nk} = n+k$ then

$$I_{nk} = \int_{-1}^{1} \tilde{P}_k(\mu) \tilde{P}_n(\mu)\, d\mu = \delta_{nk}, \quad \forall k+n \leq 2N-1$$

Define
$$\left.\begin{aligned} B_N' &= \{0,1,\dots,N-1\} \cap B \\ B_N &= B - B_N' \end{aligned}\right\} \Rightarrow B_N' \cup B_N = B \wedge B_N' \cap B_N = \emptyset.$$
and split the summation over B_N and B_N' for $n \in \{0,1,\dots,N-1\}$.
Note that
$k \in B_N' \Rightarrow k \leq N-1 \Rightarrow k+n \leq 2N-1 \Rightarrow I_{nk} = \delta_{nk}$, therefore

$$\gamma_n^{(N)} = \sum_{k\in B} I_{nk}\gamma_k = \sum_{k \in B_N'} I_{nk}\gamma_k + \sum_{k\in B_N} I_{nk}\gamma_k =$$
$$= \sum_{k\in B_N'} \delta_{nk}\gamma_k + \sum_{k\in B_N} I_{nk}\gamma_k = \gamma_n + \sum_{k \in B_N} I_{nk}\gamma_k, \quad \forall n \in \{0,1,\dots,N-1\}.$$

This result is the sampling theorem for discrete Legendre transforms.

$$\boxed{\begin{aligned} &\gamma_n^{(N)} = \gamma_n + \sum_{k\in B_N} I_{nk}\gamma_k, \quad \forall n \in \{0,1,\dots,N-1\}. \\ &B_N = B - (B \cap \{0,1,\dots,N-1\}) \end{aligned}}$$

Like the Fourier case there are aliasing terms.
Unlike the Fourier case I_{nk} has no closed form expression when $k \in B_N$.
Now suppose that we only want to resolve the modes $\Gamma \subseteq \{0,\dots,N-1\} \cap B_N'$
That is we want
$$\gamma_n^{(N)} = \gamma_n, \quad \forall n \in \Gamma$$
We want a lower bound on N such that this is true.
We require a stronger condition:
$$I_{nk} = 0, \quad \forall n \in \Gamma, \forall k \in B_N.$$
Since $\Gamma \subseteq B_N'$ and $B_N' \cap B_N = \emptyset \Rightarrow \Gamma \cap B_N = \emptyset \Rightarrow \delta_{nk} = 0, \forall (n,k) \in \Gamma \times B_N$.
It follows that
$$I_{nk} = 0, \ \forall (n,k) \in \Gamma\times B_N \Leftrightarrow I_{nk} = \delta_{nk}, \ \forall (n,k) \in \Gamma \times B_N \Leftrightarrow$$
$$\Leftrightarrow k+n \leq 2N-2, \ \forall (n,k) \in \Gamma \times B_N \Leftrightarrow$$
$$\Leftrightarrow N \geq \frac{k+n+2}{2}, \ \forall (n,k) \in \Gamma \times B_N \Leftrightarrow$$

$$\Longleftrightarrow \boxed{N \geqslant \max_{(n,k)\in\Gamma\times B_N} \frac{k+n+2}{2}}$$

Now consider $B = \{0, 1, \ldots, M\} \Rightarrow B_N = \{N, \ldots, M\}$
$B_N' = \{0, \ldots, N-1\}$.
and $\Gamma = \{0, 1, \ldots, K\}$ with $K \leqslant N-1 \leqslant M$.
In other words we have a function $f(\mu)$ with M modes and we want to resolve correctly $K+1$ first modes.
The resolution required is:

$$N \geqslant \max_{(n,k)\in\Gamma\times B_N} \frac{k+n+2}{2} = \frac{1}{2}\left[\max_{k\in B_N} k + \max_{n\in\Gamma} n + 2\right] =$$

$$= \frac{M+K+2}{2} \Rightarrow \boxed{N \geqslant \frac{M+K+2}{2}}$$

In other words, if we don't want to resolve all the modes then we can lower our resolution, just like in the Fourier case

Products with pseudospectral method

Suppose that
$$\psi(\mu) = \sum_{n=0}^{M-1} a_n \tilde{P}_n(\mu) \quad \text{and} \quad \varphi(\mu) = \sum_{n=0}^{M-1} b_n \tilde{P}_n(\mu).$$

Then their product $f(\mu) = \psi(\mu)\varphi(\mu)$ is equal to:
$$f(\mu) = \psi(\mu)\varphi(\mu) = \left[\sum_{n=0}^{M-1} a_n \tilde{P}_n(\mu)\right]\left[\sum_{k=0}^{M-1} b_k \tilde{P}_k(\mu)\right] =$$
$$= \sum_{n=0}^{M-1}\sum_{k=0}^{M-1} a_n b_k \tilde{P}_n(\mu)\tilde{P}_k(\mu) = \sum_{n=0}^{2M-2} \gamma_n \tilde{P}_n(\mu).$$

There is no easy relation between γ_n and a_n, b_n so a Galerkin approach is not practical. Therefore we prefer the pseudospectral algorithm.
A wasteful approach is to use resolution $2M$ and retain only the first M modes. We will show that we can use resolution lower than $2M$.
For $f(\mu)$: $B = \{0, 1, 2, \dots, 2M-2\}$
and $B_N = \{N, \dots, 2M-2\}$.
Let $\Gamma = \{0, 1, \dots, M-1\}$ be the modes that we want to resolve. Then a lower bound for sufficient resolution, as we already derived, is given by:

$$N \geq \max_{(n,k)\in\Gamma\times B_N} \frac{k+n+2}{2} = \frac{1}{2}\left[\max_{k\in B_N}(k) + \max_{n\in\Gamma}(n) + 2\right] =$$
$$= \frac{1}{2}\left[(2M-2) + (M-1) + 2\right] = \frac{3M-1}{2} = \frac{3}{2}|\Gamma| - \frac{1}{2} \Rightarrow$$
$$\Rightarrow \frac{3}{2}|\Gamma| \leq N + \frac{1}{2} \Rightarrow \boxed{|\Gamma| \leq \frac{2}{3}N + \frac{1}{3}}$$

It follows that if we use resolution N and the pseudospectral method, only the first $(2N+1)/3$ modes will be accurately resolved.

Pseudospectral algorithm

a) Given $\tilde{\psi}_k$ and $\tilde{\varphi}_k$ with resolution N.
b) Set $\tilde{\psi}_k = \tilde{\varphi}_k = 0$, $\forall k > (2N+1)/3$
c) Transform to real space $\tilde{\psi}_k \to \psi_j$ and $\tilde{\varphi}_k \to \varphi_j$, $\forall j \in \{0, \dots, N-1\}$.
d) Compute $f_j = \psi_j \varphi_j$ and $f_j \to \tilde{f}_k$.
e) Set $\tilde{f}_k = 0$, $\forall k > (2N+1)/3$.

Testing an antialiasing implementation.

Testing an implementation of the pseudospectral algorithm is a non-trivial task, because even when it works there maybe some error. The objective is to at least insure that it is not an aliasing error but only a truncation error.
Now we describe a method for cheeking this.
Define
$$f_1(\mu) = (1-\mu^2)^n \quad \text{and} \quad f_2(\mu) = \mu^2(1-\mu^2)^{n-1}$$
and the product:
$$f(\mu) = f_1(\mu) f_2(\mu) = \mu^2(1-\mu^2)^{2n-1}.$$
Note that
$$\deg f_1(\mu) = \deg f_2(\mu) = 2n \Rightarrow \deg f(\mu) = 2n+2n = 4n$$

Then $f_1(\mu), f_2(\mu), f(\mu)$ can be resolved with resolution $4n+1$.
However we only care about the modes of $f_1(\mu)$ and $f_2(\mu)$:
$$\Gamma = \{0,1,\ldots,2n\} \Rightarrow |\Gamma| = 2n+1.$$

It follows that the pseudospectral algorithm requires resolution
$$N \geq \frac{3}{2}|\Gamma| - \frac{1}{2} = \frac{3}{2}(2n+1) - \frac{1}{2} = 3n + \frac{3}{2} - \frac{1}{2} = 3n+1$$

Now we suggest the following verification algorithm:
<u>Algorithm</u>

a) Allocate arrays $a_1(4n+1), a_2(4n+1), a(4n+1)$ with size $4n+1$.
b) Sample f_1, f_2, f in a_1, a_2, a and transform to spectral space. The arrays a_1, a_2, a store the <u>exact</u> values of the modes.
c) Allocate arrays $b_1(3n+1), b_2(3n+1), b(3n+1)$ with size $3n+1$.
d) <u>Resample</u> f_1, f_2 to b_1, b_2. Sanity check: Assert that the nonzero values in b_1, b_2 equal the nonzero values in a_1, a_2. We expect this to be true because f_1, f_2 require only $2n+1$ modes to be fully resolved.
e) Apply the pseudospectral algorithm on b_1, b_2 and obtain b.
f) The first $2n+1$ modes of b must match the first $2n+1$ modes of a.

<u>Note:</u> If you transform a, b to real space and interpolate, they will not agree. It is sufficient only for the first $2n+1$ modes to agree.

Computing the product

Suppose that
$$\psi(\mu)=\sum_{n=0}^{N-1} a_n \tilde{P}_n(\mu) \quad \text{and} \quad \varphi(\mu)=\sum_{n=0}^{N-1} b_n \tilde{P}_n(\mu).$$

Then their product is equal to:
$$\psi(\mu)\varphi(\mu)=\left[\sum_{n=0}^{N-1} a_n \tilde{P}_n(\mu)\right]\cdot\left[\sum_{k=0}^{N-1} b_k \tilde{P}_k(\mu)\right]=$$
$$=\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} a_n b_k \tilde{P}_n(\mu)\tilde{P}_k(\mu).$$

It follows that $\psi(\mu)\varphi(\mu)$ is also a polynomial but of degree $2N-2$. To accurately resolve $\psi(\mu)\varphi(\mu)$ then we need $2N-1$ terms:
$$\psi(\mu)\varphi(\mu)=\sum_{n=0}^{2N-2} c_n \tilde{P}_n(\mu)$$

Since we want to confine our computation to the space spanned by the first N Legendre polynomials we approximate $\psi(\mu)$ and $\varphi(\mu)$ by eliminating the high-order terms. This is ok since the large-scale behaviour is more important than the small scale behaviour. Suppose that we retain the terms $0,1,\ldots,M$. To resolve a product M must be such that

$$M+M\leqslant N-1 \Leftrightarrow M\leqslant \frac{N-1}{2} \Leftrightarrow \boxed{M=\begin{cases}(N-1)/2, & N=\text{odd}\\ N/2-1, & N=\text{even}\end{cases}}$$

Given M we truncate terms $M+1,\ldots,N-1$.
Therefore, the bottom line is:

a) We use $M+1$ numbers for the spectral representation
b) We use $N-1$ numbers for the real representation
c) To multiply two fields, we take them to real space, transform them there and cast them back to spectral space.
d) Although the product needs $N-1$ terms to be resolved we only retain the first M terms.