Ordinary Differential Equations

Introduction

Def: An ordinary differential equation (ODE) is an equation that contains one or more derivatives of the unknown function. A function that satisfies the equation is called a solution of the ODE.

The most general form of an ODE is:

$$F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0$$

where $F: \mathbb{R}^{n+1} \to \mathbb{R}$

Classification of ODEs

a) By order ⇝ this is the order of the highest derivative.

b) Linear / Nonlinear ⇝ depending on whether the function F is linear or not.

c) Scalar / System ⇝ is F a scalar function or a ~~system~~ vector function.

Techniques for solving ODEs.

a) Exact analytic methods: obtain exact solution in closed form.

b) Approximate methods: obtain an approximate solution in closed form.

i) Local methods: obtain approximate solution which is good in a neighborhood of special points.

ii) Global methods: obtain approximate solution on entire range of ODE.

c) Numerical methods: obtain an approximate discretized solution with the use of a computer.

Types of ODE problems.

a) Initial value problems → These are problems of the form:

$$F(x, y(x), y'(x), \dots, y^{(n-1)}(x), y^{(n)}(x)) = 0$$
$$y(x_0) = a_0, \; y'(x_0) = a_1, \dots, y^{(n-1)}(x_0) = a_{n-1}$$

where $y, y', \dots, y^{(n-1)}$ are specified at one point.

b) <u>Boundary value problems.</u> $\longrightarrow$ These are problems of the form

$$F(x, y(x), y'(x), \dots, y^{(n-1)}(x), y^{(n)}(x)) = 0$$
$$y^{(k_1)}(x_1) = a_1,\ y^{(k_2)}(x_2) = a_2, \dots, y^{(k_n)}(x_n) = a_n$$

where n quantities are specified on more than one point.

<u>Def</u> : Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a vector function. We say that f is <u>uniformly Lipschitz continuous</u> $\Leftrightarrow$

$$\exists L > 0, \forall u, v \in \mathbb{R}^n : \|f(u) - f(v)\| \leqslant L \|u - v\|.$$

<u>Prop</u> : Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be given. If

f differentiable wrt $u_1, u_2, \dots, u_n$
$\partial f / \partial u_1, \partial f / \partial u_2, \dots, \partial f / \partial u_n$ bounded in $\mathbb{R}^n$ $\Bigg\} \Rightarrow$ f uniformly Lipschitz continuous.

<u>Thm</u> : Consider the problem $\begin{cases} y'(t) = f(t, y(t)) \\ y(0) = y_0 . \end{cases}$

If f continuous wrt to t
f uniformly Lipschitz continuous wrt to y $\Bigg\} \Rightarrow$ the initial value problem has a unique solution.

<u>Remark</u> : The general n^{th}-order differential equation is equivalent to a system of n 1st-order equations. It follows that the uniqueness theorem extends to the n^{th}-order initial value problem.

Homogeneous linear equations

Def: A homogeneous linear ODE is an equation of the form
$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \cdots + p_0(x)\, y = 0.$$

Def: Let $\{y_1(x), y_2(x), \ldots, y_k(x)\}$ be a set of functions.
We say that they are linearly independent $\Leftrightarrow$
$$\cancel{\forall (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{C}^k \setminus \{0\}}$$
$$(\lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_k y_k(x) = 0 \Rightarrow (\lambda_1, \lambda_2, \ldots, \lambda_k) = 0.$$

Thm: There exist n linearly independent functions $y_1, y_2, \ldots, y_n$ such that all solutions to the equation
$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \cdots + p_0(x)\, y = 0$$
can be written as
$$y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x)$$
with $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$.

Wronskian and linear indepedence

Def: Let $\{y_1, y_2, \ldots, y_n\}$ be n functions that are n-differentiable
The Wronskian $W(x) \stackrel{\circ}{=} W[y_1, y_2, \ldots, y_n]$ is defined by:
$$W[y_1, y_2, y_{n-1}, y_n] = \det \begin{bmatrix} y_1(x) & y_2(x) & y_{n-1}(x) & y_n(x) \\ y_1'(x) & y_2'(x) & y_{n-1}'(x) & y_n'(x) \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & y_{n-1}^{(n-1)}(x) & y_n^{(n-1)}(x) \\ y_1^{(n)}(x) & y_2^{(n)}(x) & y_{n-1}^{(n)}(x) & y_n^{(n)}(x) \end{bmatrix}$$

Thm: Let $\{y_1, y_2, \ldots, y_n\}$ be n functions that are n-differentiable.
$\{y_1, y_2, \ldots, y_n\}$ are linearly independent $\Leftrightarrow W[y_1, y_2, \ldots, y_n] \neq 0$
$\cancel{\forall x \in \mathbb{C} - \{\lambda_1, \lambda_2, \ldots,}$
except for isolated points.

<u>Thm</u>: Let $\{y_1, y_2, \dots, y_n\}$ be linearly indepedent solutions to

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_0(x)y = 0$$

and $W(x) = W[y_1, y_2, \dots, y_n](x)$.

Then,

$$\exists c \in \mathbb{R} : W(x) = \exp\left[-\int_c^x p_{n-1}(t)\,dt\right]$$

- <u>Initial value problems</u>

Consider the ODE $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_0(x)y = 0$ with general solution:

$$y(x) = \sum_{j=1}^{n} \lambda_j y_j(x)$$

Impose initial conditions: $y^{(k)}(x_0) = a_k$, $k \in \{0, \dots, n-1\}$. $\Leftrightarrow$

$$\Leftrightarrow \sum_{j=1}^{n} \lambda_j y_j^{(k)}(x) = a_k,\ \forall k \in \{0, \dots, n-1\} \Leftrightarrow A\lambda = a \quad (1).$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, $a = (a_1, a_2, \dots, a_n)$ and $A_{ij} = y_j^{(i)}(x_0)$

The ODE has a unique solution $\Leftrightarrow$ (1) has a unique solution $\Leftrightarrow$ $\Leftrightarrow \det A \neq 0$.

Note that: $\det A = \det[y_j^{(i)}(x_0)] = W[y_1, y_2, \dots, y_n](x_0) = W(x_0)$

so we obtain that

~~The OD~~

The initial value problem has a unique solution $\Leftrightarrow W(x_0) \neq 0$.

Solution techniques for 1st order ODEs.

There is no general solution to the 1st order ODE problem:

$$y' = f(x,y)$$
$$y(a) = b.$$

There are solutions methods for the following special cases however:

Seperable ODEs

Def: A 1st order seperable ODE is an equation of the form.

$$\frac{dy}{dx} = g(x)h(y). \quad , \quad y(a) = b.$$

Solution method:

$$y' = g(x)h(y) \Leftrightarrow \frac{1}{h(y)}\frac{dy}{dx} = g(x) \Leftrightarrow \int_b^y \frac{dy}{h(y)} = \int_a^b g(x)dx.$$

Then compute the y integral $\rightsquigarrow$ transcedental equation which defines the solution.

example: Solve $y' = \frac{y\cos x}{1+2y^2}$ with $y(0) = 1$.

$$y' = \frac{y\cos x}{1+2y^2} \Leftrightarrow \frac{(1+2y^2)y'}{y} = \cos x \Leftrightarrow \int_1^y \frac{1+2\tilde{y}^2}{\tilde{y}}d\tilde{y} = \int_0^x \cos\tilde{x}\,d\tilde{x}.$$

$$\int_1^y \frac{1+2\tilde{y}^2}{\tilde{y}}dy = \int_1^y \frac{d\tilde{y}}{\tilde{y}} + 2\int_1^y \tilde{y}\,d\tilde{y} = \ln|y| + (y^2-1)$$

so $\boxed{\ln|y| + (y^2-1) = \sin x.}$

• Homogeneous ODEs

Def : A homogeneous ODE is an equation of the form

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right).$$

Solution method

Let $y(x) = xu(x)$. $\Rightarrow$ $\frac{dy}{dx} = x\frac{du}{dx} + u = g(u) \Leftrightarrow x\frac{du}{dx} = g(u) - u \Leftrightarrow$

$$\Leftrightarrow \frac{1}{g(u)-u}\frac{du}{dx} = \frac{1}{x} \Leftrightarrow \int_{u_0}^{u} \frac{d\tilde{u}}{g(\tilde{u}) - \tilde{u}} = \int_{x_0}^{x} \frac{d\tilde{x}}{\tilde{x}} = \ln\left(\frac{x}{x_0}\right)$$

example

$$\frac{dy}{dx} = \frac{y^2+2xy}{x^2} = \left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right).$$

Let $u = y/x$.

$$x\frac{du}{dx} + u = u^2+2u \Leftrightarrow x\frac{du}{dx} = u(u+1) \Leftrightarrow \int \frac{du}{u(u+1)} = \int \frac{dx}{x} = \ln|x| + \ln|c|.$$

Since $\int \frac{du}{u(u+1)} = \int \left(\frac{1}{u} - \frac{1}{u+1}\right) du = \ln|u| - \ln|u+1|$ so

$$\ln|u| - \ln|u+1| = \ln|x| + \ln|c| \Leftrightarrow cx = \frac{u}{u+1} \Leftrightarrow cx(u+1) = u \Leftrightarrow$$

$$\Leftrightarrow (cx-1)u = -cx \Leftrightarrow u = \frac{cx}{1-cx} \Leftrightarrow \frac{y}{x} = \frac{cx}{1-cx} \Leftrightarrow$$

$$\Leftrightarrow y(x) = \frac{cx^2}{1-cx}$$

Use initial condition to find c.

- <u>Integrating Factors Method</u>

This method can be applied to ODEs of the form:

$$\boxed{y' + f(x)y = g(x)}$$

<u>Solution</u>: Multiply by $h(x)$: $h(x)y' + h(x)f(x)y = h(x)g(x)$.
If $h'(x) = h(x)f(x)$ then,

$$h(x)y' + h'(x)y = h(x)g(x) \Leftrightarrow \frac{d}{dx}\left[h(x)y(x)\right] = h(x)g(x) \Leftrightarrow$$

$$\Leftrightarrow h(x)y = \int h(x)g(x)dx + C \Leftrightarrow$$

$$\Leftrightarrow y(x) = \frac{1}{h(x)}\int h(x)g(x)dx + \frac{C}{h(x)}$$

To find $h(x)$ solve $h'(x) = h(x)f(x) \Leftrightarrow \frac{h'}{h} = f \Leftrightarrow \ln h = \int f\,dx \Leftrightarrow$

$$\Leftrightarrow h(x) = \exp\left(\int f(x)dx + c\right).$$

We call $h(x)$ the <u>integrating factor.</u>
Note that $y(x) = 1/h(x)$ is the solution to $y' + f(x)y = 0$.
We call it the <u>homogeneous term.</u>
The integral term is called the particular solution.

<u>example</u>: Solve $y' + xy = x^2$.

Use integrating factor $h(x) = \exp\left(\int x\,dx\right) = e^{x^2/2}$.
Thus,

$$e^{x^2/2}y' + xe^{x^2/2}y = x^2e^{x^2/2} \Leftrightarrow \frac{d}{dx}\left(e^{x^2/2}y\right) = x^2e^{x^2/2} \Leftrightarrow$$

$$\Leftrightarrow e^{x^2/2}y - c = \int_0^x \xi^2 e^{\xi^2/2}\,d\xi \Leftrightarrow y = e^{-x^2/2}\int_0^x \xi^2 e^{\xi^2/2}\,d\xi + ce^{-x^2/2}.$$

Exact ODEs.

<u>Def</u>: An <u>exact ODE</u> is an equation of the form.

$$M(x,y) + y' N(x,y) = 0.$$

<u>Solution method</u>

Consider a function $\Psi(x,y)$. Since y is a function of x, the total derivative of Ψ wrt to x is:

$$\frac{d\Psi}{dx} = \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx}$$

Suppose that a Ψ exists such that

$$\frac{\partial \Psi(x,y)}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \Psi(x,y)}{\partial y} = N(x,y)$$

Then $M(x,y) + y'N(x,y) = 0 \Leftrightarrow \frac{d\Psi(x,y)}{dx} = 0 \Leftrightarrow \underline{\Psi(x,y(x)) = c}$ ~ implicit solution.

Q: When does Ψ exist?

<u>Thm</u>: Let $M(x,y)$ and $N(x,y)$ be two functions such that

$$\boxed{\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}}$$

Then a function $\Psi(x,y)$ exists such that

$$\frac{\partial \Psi(x,y)}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \Psi(x,y)}{\partial y} = N(x,y)$$

and is given by

$$\Psi(x,y) = \int M(x,y)\,dx + g(y)$$

$$g(y) = \int \left\{ N(x,y) - \int \frac{\partial M}{\partial y}\,dx \right\} dy.$$ ~ must be a function of y only.

Proof:

Suppose that $\frac{\partial \Psi(x,y)}{\partial x} = M(x,y) \Leftrightarrow \Psi(x,y) = \int M(x,y)\,dx + g(y)$

Then $\frac{\partial \Psi(x,y)}{\partial y} = \int \frac{\partial M}{\partial y}\,dx + g'(y) = N(x,y) \Leftrightarrow$

$\Leftrightarrow \int \frac{\partial M}{\partial y}\,dx - N(x,y) = -g'(y)$

This can only be true if the left-hand-side is not a function of x

$\frac{\partial}{\partial x}\left[\int \frac{\partial M}{\partial y}\,dx - N(x,y)\right] = 0 \Leftrightarrow$

$\Leftrightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\int \frac{\partial M}{\partial y}\,dx = \frac{\partial M}{\partial y}$

therefore Ψ exists $\Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

To obtain Ψ:

$\frac{\partial \Psi}{\partial y} = \int \frac{\partial M}{\partial y}\,dx + g'(y) = N(x,y) \Leftrightarrow g'(y) = N(x,y) - \int \frac{\partial M}{\partial y}\,dx \Leftrightarrow$

$\Leftrightarrow g(y) = \int \left\{ N(x,y) - \int \frac{\partial M}{\partial y}\,dx \right\} dy.$

therefore $\Psi(x,y) = \int M(x,y)\,dx + \int \left\{ N(x,y) - \int \frac{\partial M}{\partial y}\,dx \right\} dy.$ $\square$

example: Solve $(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0$.

$M(x,y) = y\cos x + 2xe^y$

$N(x,y) = \sin x + x^2e^y - 1$

Check:

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\cos x + 2xe^y\right) = \cos x + 2xe^y \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\sin x + x^2e^y - 1\right) = \cos x + 2xe^y \end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$$

$\Rightarrow \Psi(x,y)$ exists such that $\frac{\partial \Psi}{\partial x} = M$ and $\frac{\partial \Psi}{\partial y} = N$.

$$\frac{\partial \Psi}{\partial x} = y\cos x + 2xe^y \Leftrightarrow \Psi(x,y) = y\int \cos x\,dx + e^y\int 2x\,dx + g(y) =$$
$$= y\sin x + x^2e^y + g(y).$$

also $\dfrac{\partial \Psi}{\partial x} = \sin x + x^2e^y - 1$

and $\dfrac{\partial \Psi}{\partial y} = \dfrac{\partial}{\partial y}\left(y\sin x + x^2e^y + g(y)\right) = \sin x + x^2e^y + g'(y).$

so $g'(y) = -1 \Leftrightarrow g(y) = -y$. therefore $\Psi(x,y) = y\sin x + x^2e^y - y = \text{const}.$

- <u>Integrating factors for exact equations</u>

If Ψ does not exist, it may be possible to make it exist by using an integrating factor $\mu(x,y)$:

$M(x,y) + N(x,y)y' = 0 \Leftrightarrow \mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$

Want $\mu(x,y)$ such that:

$$\frac{\partial}{\partial y}\left[\mu(x,y)M(x,y)\right] = \frac{\partial}{\partial x}\left[\mu(x,y)N(x,y)\right] \Leftrightarrow$$
$$\Leftrightarrow \mu_y M + \mu M_y = \mu_x N + \mu N_y \Leftrightarrow M\mu_y - N\mu_x + (M_y - N_x)\mu = 0.$$

This is a partial differential equation

a) Hard to solve.

b) Solution is not unique.

There are some results which will cover special cases.

A) <u>The Homogeneous case</u>

<u>Def</u>: A function $F(x,y)$ is called <u>homogeneous</u> and of degree n
$\Leftrightarrow F(\lambda x, \lambda y) = \lambda^n F(x,y)$, $\forall x,y \in \mathbb{R}$.

<u>Prop</u> : If $F(x,y)$ homogeneous of order n $\Big\} \Longrightarrow x\dfrac{\partial F}{\partial x} + y\dfrac{\partial F}{\partial y} = nF.$

<u>Thm</u> : Let $M(x,y) + N(x,y)\,y' = 0$ be an ODE with $M(x,y)$, $N(x,y)$ homogeneous. Then

$$\mu(x,y) = \frac{1}{xM(x,y) + yN(x,y)}$$

is an integrating factor.

<u>Proof</u>

$$\frac{\partial(\mu M)}{\partial y} = \mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial M}{\partial y} - \mu^2 M\left(x\frac{\partial M}{\partial y} + y\frac{\partial N}{\partial y} + N\right).$$

$$\frac{\partial(\mu N)}{\partial x} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x} = \mu\frac{\partial N}{\partial x} - \mu^2 N\left(x\frac{\partial M}{\partial x} + y\frac{\partial N}{\partial x} + M\right)$$

So $\dfrac{\partial(\mu M)}{\partial y} = \dfrac{\partial(\mu N)}{\partial x} \Longleftrightarrow$

$$\Longleftrightarrow \frac{\partial M}{\partial y} - \mu M\left(x\frac{\partial M}{\partial y} + y\frac{\partial N}{\partial y}\right) = \frac{\partial N}{\partial x} - \mu N\left(x\frac{\partial M}{\partial x} + y\frac{\partial N}{\partial x}\right).$$

Multiply by $xM + yN$ and simplify:

$$N\left(x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y}\right) = M\left(x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y}\right).$$

$M(x,y)$ homogeneous $\Longrightarrow x\dfrac{\partial M}{\partial x} + y\dfrac{\partial M}{\partial y} = nM$

$N(x,y)$ homogeneous $\Longrightarrow x\dfrac{\partial N}{\partial x} + y\dfrac{\partial N}{\partial y} = nN$

Thus $N(nM) = M(nN)$ which is true. $\square$.

B) One variable integrating factors.

If $M(x,y), N(x,y)$ are not homogeneous, then it's worth trying the following:

Thm: If $\frac{1}{N(x,y)}\left(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right) = g(x)$, then

$$\mu(x,y) = C\exp\left[\int g(x)dx\right]$$

is an integrating factor to $M(x,y) + N(x,y)y' = 0$.

Proof

The integrating factor satisfies: $M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} + \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu = 0$

Suppose that $\frac{\partial \mu}{\partial y} = 0$. Then

$$-N\frac{\partial \mu}{\partial x} + \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu = 0 \Leftrightarrow \frac{\partial \mu}{\partial x} = \frac{M_y - N_x}{N}\mu$$

By hypothesis $\frac{M_y - N_x}{N} = g(x) \rightarrow$ independent of y as it should be

so $\frac{\partial \mu}{\partial x} = g(x)\mu \Leftrightarrow \mu(x,y) = C\exp\left[\int g(x)dx\right]$. $\square$

Thm: If $\frac{1}{M(x,y)}\left(\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right) = g(y)$, then

$$\mu(x,y) = C\exp\left[\int g(y)dy\right]$$

is an integrating factor to $M(x,y) + N(x,y)y' = 0$.

c) <u>Other special integrating factors.</u>

Thm: If $M(x,y) = y f(xy)$ and $N(x,y) = x g(xy)$ then
$$\mu(x,y) = \frac{1}{xM(x,y) - yN(x,y)}$$
is an integrating factor.

Thm: If $\begin{cases} \dfrac{\partial M(x,y)}{\partial x} = \dfrac{\partial N(x,y)}{\partial y} \\ \dfrac{\partial M(x,y)}{\partial y} = -\dfrac{\partial N(x,y)}{\partial x} \end{cases} \Rightarrow \mu(x,y) = \dfrac{1}{M^2(x,y) + N^2(x,y)}$

Thm: If $\begin{matrix} M(x,y) = y + x f(x^2+y^2) \\ N(x,y) = -x + y f(x^2+y^2) \end{matrix} \Rightarrow \mu(x,y) = \dfrac{1}{x^2+y^2}$.

Thm: (Goursat)

If $\begin{matrix} M(x,y) = (a + \alpha x^m y^n) y \\ N(x,y) = (b + \beta x^m y^n) x \end{matrix} \Rightarrow \mu(x,y) = x^p y^q$

where p,q are given by: $\begin{cases} bp - aq = a - b \\ \beta p - \alpha q = \alpha(n+1) - \beta(m+1) \end{cases}$

▼ Special non-linear 1st order equations

● Bernoulli's equation

Def : An ODE of the form $\boxed{y' = f(x)y + g(x)y^a}$ is called a Bernoulli equation.

Solution method : Substitute $u(x) = y^{1-a}(x)$. $\Rightarrow$

$$\Rightarrow u'(x) = (1-a)y^{1-a-1}(x)\,y'(x) = (1-a)\,y^{-a}(x)\,y'(x) =$$

$$= (1-a)\,y^{-a}(x)\left[f(x)y + g(x)y^a\right] =$$

$$= (1-a)\left[f(x)y^{1-a}(x) + g(x)\right] = (1-a)\left(f(x)u(x) + g(x)\right).$$

This is a linear ODE 1st-order and it can be solved by integrating factor.

● Clairaut's equation

Def : An ODE of the form $\boxed{y = xy' + f(y')}$ is called a Clairaut equation

Solution Method

This equation has a general and a singular solution

Differentiate once:

$$y' = (xy' + f(y'))' = y' + xy'' + f'(y')y'' \Leftrightarrow y''[x + f'(y')] = 0 \Leftrightarrow$$

$$\Leftrightarrow y'' = 0 \lor x + f'(y') = 0.$$

The solution of $y'' = 0 \Leftrightarrow y = ax + b$. and

$y = xy' + f(y') \Leftrightarrow ax + b = ax + f(a) \Leftrightarrow b = f(a)$.

Therefore, the <u>general solution</u> is $y = cx + f(c)$.
To obtain the singular solution:

$$\begin{cases} y' = c \\ x + f'(y') = 0 \end{cases} \iff x + f'(c) = 0 \iff c = (f')^{-1}(-x)$$

so $y(x) = x(f')^{-1}(-x) + f((f')^{-1}(-x))$.

<u>Remark</u>: The general version of what we did is to start from a family of curves

$$F(x,y,c)$$

and eliminate c between

$$F(x,y,c) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial c} F(x,y,c) = 0$$

The curve obtained is called the <u>envelope</u> and it's equation is called c-discriminant equation.
The singular solution is the envelope of the family of the general solutions <u>and</u> it is itself a solution.

<u>example</u>: Solve $y = xy' + \cos y'$

The general solution is $y = cx + \cos c$.
For the singular solution

$$\begin{cases} y' = c \\ x - \sin y' = 0 \end{cases} \iff x - \sin c = 0 \iff c = \arcsin x.$$

so $y = x \arcsin x + \cos(\arcsin x) =$
$= x \cdot \arcsin x + \sqrt{1 - x^2}$

Chrystal's equation

Def: An ODE of the form $(y')^2 + axy' + by + cx^2 = 0$ is called a Chrystal's equation.

Thm: Define z in terms of the transformation

$$4by = (a^2 - 4c - z^2)x^2$$

and let ρ_1, ρ_2 be the roots of either of

$$z^2 + bz + (4c - ab - a^2) = 0$$

$$z^2 - bz + (4c - ab - a^2) = 0 .$$

A family of solutions is given by $x(z-\rho_1)^m (z-\rho_2)^n = c$ where

$$m = \frac{\rho_1}{\rho_1 - \rho_2} \quad \text{and} \quad n = -\frac{\rho_2}{\rho_1 - \rho_2} .$$

Proof

$$(y')^2 + axy' + by + cx^2 = 0 \Leftrightarrow y' = -\frac{1}{2}ax \pm \frac{1}{2}\left(a^2x^2 - by - 4cx^2\right)^{1/2} \Leftrightarrow$$

$$\Leftrightarrow xzz' = a^2 + ab - 4c \pm bz - z^2 \Leftrightarrow \frac{z}{z^2 \mp bz + (4c - ab - a^2)} \frac{dz}{dx} = -\frac{1}{x}$$

$$\Leftrightarrow \int \frac{z\,dz}{(z-\rho_1)(z-\rho_2)} = -\int \frac{dx}{x} \Leftrightarrow \dots \Leftrightarrow x(z-\rho_1)^m (z-\rho_2)^n = c.$$

Thm : Let $Ly = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y$ and let $y_1, y_2, \ldots, y_n$ be two solutions to the homogeneous problem $Ly = 0$.

a) The Green's function is given by

$$G(x,a) = \begin{cases} \sum_{k=1}^{n} B_k(a)\, y_k(x) & , x \geq a \\ 0 & , x < a \end{cases}$$

and a particular solution by:

$$y_p(x) = \sum_{k=1}^{n} y_k(x) \int^{x} f(t)\, B_k(t)\, dt$$

where $B_1(a), B_2(a), \ldots, B_n(a)$ are given by:

$$\begin{bmatrix} B_1(a) \\ B_2(a) \\ \vdots \\ B_n(a) \end{bmatrix} = W[y_1, y_2, \ldots, y_n]^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Proof

The Green's function has the form

$$G(x,a) = \sum_{k=1}^{n} A_k(a)\, y_k(x) \quad , \forall x < a$$

$$G(x,a) = \sum_{k=1}^{n} B_k(a)\, y_k(x) \quad , \forall x > a$$

$G, \partial G/\partial x, \ldots, \partial^{n-2} G/\partial x^{n-2}$ are continuous at $x = a \Leftrightarrow$

$$\Leftrightarrow \sum_{k=1}^{n} A_k(a)\, y_k^{(m)}(x) = \sum_{k=1}^{n} B_k(a)\, y_k^{(m)}(x) \quad , \forall m \in \{0, \ldots, n-2\} \Leftrightarrow$$

$$\Leftrightarrow \sum_{k=1}^{n} (B_k(a) - A_k(a))\, y_k^{(m)}(x) = 0 \quad , \forall m \in \{0, \ldots, n-2\}. \quad (1).$$

$\partial^{n-1} G/\partial x^{n-1}$ has a jump discontinuity of magnitude 1 at $x = a \Leftrightarrow$

$$\Leftrightarrow \sum_{k=1}^{n} B_k(a)\, y_k^{(n-1)}(x) - \sum_{k=1}^{n} A_k(a)\, y_k^{(n-1)}(x) = 1.$$

Linear systems of 1st order equations.

Let $A \in \mathbb{R}^{n\times n}$ be a non-diagonal matrix and $x \in \mathbb{R}^n$ a vector. The equation

$$\frac{dx}{dt} = Ax$$

is called a <u>linear system of first-order equations</u> and it is equivalent to a set of n ODEs:

$$\frac{dx_j}{dt} = \sum_{k=1}^{n} a_{jk} x_k \, , \quad j \in [n].$$

<u>Case 1:</u> A has <u>real-distinct eigenvalues.</u>

Let $\lambda(A) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ with eigenvectors $v_1, v_2, \dots, v_n$. $\Rightarrow$
$\Rightarrow$ $P = [v_1 \; v_2 \cdots v_n]$ has an inverse and

$$P^{-1}AP = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n).$$

We use this fact to reduce to an uncoupled system.
Let $y = P^{-1}x \Leftrightarrow x = Py$. Then

$$\frac{dy}{dt} = P^{-1}\frac{dx}{dt} = P^{-1}Ax = P^{-1}APy = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)\, y. \Leftrightarrow$$

$$\Leftrightarrow \forall j \in [n] : \frac{dy_j}{dt} = \lambda_j y_j \Leftrightarrow \forall j \in [n] : y_j(t) = y_j(0)\exp(\lambda_j t).$$

The initial condition can be obtained from $y(0) = P^{-1}x(0)$ and $x(t) = Py(t)$.

<u>example</u>
$$\begin{cases} \dot{x}_1 = -x_1 - 3x_2 \\ \dot{x}_2 = 2x_2. \end{cases} \Leftrightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

λ_1, λ_2 eigenvalues of $A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix} \Leftrightarrow \det(\lambda I - A) = 0 \Leftrightarrow$

$$\Leftrightarrow \begin{vmatrix} \lambda+1 & 3 \\ 0 & \lambda-2 \end{vmatrix} = 0 \Leftrightarrow (\lambda+1)(\lambda-2) = 0 \Leftrightarrow \lambda^2 - \lambda - 2 = 0 \Leftrightarrow$$

$$\cancel{\Leftrightarrow \lambda^2 - \lambda - 5 = 0} \qquad \Leftrightarrow \lambda_1 = -1 \lor \lambda_2 = 2.$$

$\Delta = 1 + 20$

The eigenvectors are $v_1 = (1,0)$ and $v_2 = (-1,1)$ and
$P = [v_1 v_2] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Let $y = P^{-1}x$.

$$\begin{cases} \dot{y}_1 = \lambda_1 y_1 \\ \dot{y}_2 = \lambda_2 y_2 \end{cases} \Leftrightarrow \begin{cases} \dot{y}_1 = -y_1 \\ \dot{y}_2 = 2y_2 \end{cases} \Leftrightarrow \begin{cases} y_1 = y_1(0)\, e^{-t} = c_1 e^{-t} \\ y_2 = y_2(0)\, e^{2t} = c_2 e^{2t} \end{cases}.$$

so $x(t) = Py(t) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{2t} \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} - c_2 e^{2t} \\ c_2 e^{2t} \end{bmatrix}$

$$\Leftrightarrow \begin{cases} x_1(t) = c_1 e^{-t} - c_2 e^{2t} \\ x_2(t) = c_2 e^{2t} \end{cases}.$$

<u>Def</u>: Let $A \in \mathbb{R}^{n \times n}$ with distinct eigenvalues and in particular k negative eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ and $n-k$ positive eigenvalues $\lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_n$.
Let $v_1, v_2, \dots, v_k, \dots, v_n$ be the corresponding eigenvectors.

a) The <u>stable subspace</u> of A is
$$E^s = \text{span}\{v_1, v_2, \dots, v_k\}$$

b) The <u>unstable subspace</u> of A is:
$$E^n = \text{span}\{v_{k+1}, \dots, v_n\}.$$

<u>Remark</u>: If the initial condition lies in E^s, the solution will remain bounded. Otherwise it will not.

• <u>Matrix Exponentials</u>

<u>Def</u>: Let $A \in \mathbb{R}^{n \times n}$. We define the <u>matrix exponential</u> by:
$$e^A = \sum_{k=0}^{+\infty} \frac{A^k}{k!}$$

Thm: Let $P, T \in \mathbb{R}^{n\times n}$ with $\det P \neq 0$ and $S = PTP^{-1}$.
Then $\exp(S) = P\exp(T)P^{-1}$.

Proof

$$\exp(S) = \sum_{k=0}^{+\infty} \frac{S^k}{k!} = \sum_{k=0}^{+\infty} \frac{(PTP^{-1})^k}{k!} = \sum_{k=0}^{+\infty} P\,\frac{T^k}{k!}\,P^{-1} = P\left\{\sum_{k=0}^{+\infty}\frac{T^k}{k!}\right\}P^{-1} =$$
$$= P\exp(T)P^{-1}. \qquad \square.$$

Remark: Special case $\rightsquigarrow$ $PAP^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$
Then
$$\exp(tA) = P\exp(\mathrm{diag}(\lambda_1 t, \lambda_2 t, \dots, \lambda_n t))P^{-1} =$$
$$= P\,\mathrm{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t})P^{-1}.$$

Thm (special 2x2 case).

If $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in \mathbb{R}^{2\times 2} \Rightarrow \exp(A) = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$

Proof

Let $\lambda \in \mathbb{C}$ with $\lambda = a + bi$. By induction we find that

$$A^k = \begin{bmatrix} \mathrm{Re}(\lambda^k) & -\mathrm{Im}(\lambda^k) \\ \mathrm{Im}(\lambda^k) & \mathrm{Re}(\lambda^k) \end{bmatrix} \Rightarrow$$
$$\Rightarrow \exp(A) = \sum_{k=0}^{+\infty} \frac{1}{k!} A^k = \sum_{k=0}^{+\infty} \begin{bmatrix} \mathrm{Re}(\lambda^k/k!) & -\mathrm{Im}(\lambda^k/k!) \\ \mathrm{Im}(\lambda^k/k!) & \mathrm{Re}(\lambda^k/k!) \end{bmatrix} =$$
$$= \begin{bmatrix} \mathrm{Re}(e^\lambda) & -\mathrm{Im}(e^\lambda) \\ \mathrm{Im}(e^\lambda) & \mathrm{Re}(e^\lambda) \end{bmatrix} = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}. \qquad \square$$

<u>Thm</u> : Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. Then: $\frac{d}{dt} e^{At} = Ae^{At} = e^{At}A$

<u>Proof</u>

$$\frac{d}{dt} e^{At} = \frac{d}{dt} \sum_{k=0}^{+\infty} \frac{t^k A^k}{k!} = \sum_{k=0}^{+\infty} \frac{d}{dt}\left(\frac{t^k A^k}{k!}\right) = \sum_{k=1}^{+\infty} k \frac{t^{k-1}A^k}{k!} =$$

$$= \begin{cases} A \sum_{k=1}^{+\infty} \frac{t^{k-1}A^{k-1}}{(k-1)!} = A e^{At} \\ \cancel{A}\left(\sum_{k=1}^{+\infty} \frac{t^{k-1}A^{k-1}}{(k-1)!}\right) A = e^{At} A \end{cases} \Rightarrow \frac{d}{dt} e^{At} = Ae^{At} = e^{At} A . \quad \square$$

<u>Thm</u> : Let $A \in \mathbb{R}^{n\times n}$. Then $e^{tA} e^{-tA} = I$.
so $(e^{tA})^{-1} = e^{-tA}$.

<u>Proof</u>
Define $F(t) = e^{tA} e^{-tA} . \Rightarrow$

$$\Rightarrow F'(t) = e^{tA}(e^{-tA})' + (e^{tA})' e^{-tA} = e^{tA}(-Ae^{-tA}) + Ae^{tA}e^{-tA} =$$
$$= -Ae^{tA}e^{-tA} + Ae^{tA}e^{-tA} = 0 \Rightarrow$$
$$\Rightarrow F(t) = F(0) = e^{0A}e^{0A} = I. \quad , \forall t \in \mathbb{R}. \quad \square.$$

<u>Thm</u> : Let $A, B \in \mathbb{R}^{n\times n}$ and $F(t) \in \mathbb{R}^{n\times n}$, $\forall t \in \mathbb{R}$.

$$\begin{cases} F'(t) = AF(t) \\ F(0) = B \end{cases} \Leftrightarrow F(t) = e^{At} B .$$

<u>Proof</u>
<u>($\Leftarrow$)</u> $F(t) = e^{At} B \Rightarrow F'(t) = Ae^{At} B = AF(t)$.

<u>($\Rightarrow$)</u> Let $F(t)$ be a solution to $F'(t) = AF(t)$ and $F(0) = B$.
Define $G(t) = e^{-At} F(t)$.

$$G'(t) = e^{-tA} F'(t) - Ae^{-At} F(t) = e^{-tA} AF(t) - Ae^{-At} F(t) =$$
$$= (e^{-tA} A - Ae^{-tA}) F(t) = 0 \Rightarrow G(t) = G(0) = F(0) = B , \forall t > 0 \Rightarrow$$
$$\Rightarrow B = e^{-At} F(t) \Rightarrow F(t) = e^{At} B .$$

<u>Thm</u> : Let $A, B \in \mathbb{R}^{n\times n}$. $AB = BA \Rightarrow e^{A+B} = e^A e^B$.
<u>Proof</u>

By induction $AB = BA \Rightarrow A^k B = BA^k$. From this, it follows that:

$$e^{tA}B = \left\{\sum_{k=0}^{+\infty} \frac{t^k A^k}{k!}\right\} B = \sum_{k=0}^{+\infty} \frac{t^k}{k!} A^k B = \sum_{k=0}^{+\infty} \frac{t^k}{k!} BA^k =$$
$$= B\left\{\sum_{k=0}^{+\infty} \frac{t^k A^k}{k!}\right\} = Be^{tA}.$$

So: $e^{tA}B = Be^{tA}$. Define $F(t) = e^{t(A+B)} - e^{tA}e^{tB}$.

$$F'(t) = (A+B)e^{t(A+B)} - Ae^{tA}e^{tB} - e^{tA}Be^{tB} = \quad * \; e^{tA}B = Be^{tA}.$$
$$= (A+B)e^{t(A+B)} - (A+B)e^{tA}e^{tB} =$$
$$= (A+B)\left(e^{t(A+B)} - e^{tA}e^{tB}\right) = (A+B)F(t) \Rightarrow$$
$$\Rightarrow F(t) = e^{t(A+B)}F(0) = e^{t(A+B)}\,\mathbf{0} = \mathbf{0}, \; \forall t > 0 \Rightarrow e^{t(A+B)} = e^{tA}e^{tB}.$$

It follows that for $t=1$, $e^{A+B} = e^A e^B$.

- <u>Computing the matrix exponential</u>

We showed that the solution to
$$\frac{dx}{dt} = Ax \quad \text{with } x(0) = b \quad \text{is} \quad x(t) = e^{At}b.$$

It remains to show how to compute e^{At} in practice. The most general method (Putzer's method) follows from the following result:

<u>Thm</u> : Let $A \in \mathbb{R}^{n\times n}$ and
$$f(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$
Then $A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = \mathbf{0}$.

It follows that A^{n+1}, A^{n+2}, etc are linear combinations of $I, A, A^2, \ldots, A^{n-1}$, so e^{tA} should be of the form:

$$e^{tA} = \sum_{k=0}^{n-1} q_k(t) A^k.$$

Thm (Putzer)
Let $A \in \mathbb{R}^{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.
Define $B_0 = I$, $B_k = \prod_{m=1}^{k} (A - \lambda_m I)$, $k \in [n]$

and $r_1(t), r_2(t), \dots, r_n(t)$ by the initial value problems:

$$r_1'(t) = \lambda_1 r_1(t) \quad , \quad r_1(0) = 1$$

$$r_{k+1}'(t) = \lambda_{k+1} r_{k+1}(t) + r_k(t) \quad , \quad r_{k+1}(0) = 0 \quad , \quad k \in [n-1]$$

Then e^{tA} is given by:
$$e^{tA} = \sum_{k=0}^{n-1} r_{k+1}(t) B_k$$

The following results can be used for special case:

1) All the eigenvalues are equal to λ.

Thm : If $A \in \mathbb{R}^{n\times n}$ with $\lambda(A) = \{\lambda\}$, then

$$e^{tA} = e^{\lambda t} \sum_{k=0}^{n-1} \frac{t^k}{k!} (A - \lambda I)^k.$$

2) The eigenvalues are distinct.

Thm : If $A \in \mathbb{R}^{n\times n}$ with n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ then

$$e^{tA} = \sum_{k=1}^{n} e^{\lambda_k t} L_k$$

where $L_k = \prod_{j \in [n] - \{k\}} \frac{1}{\lambda_k - \lambda_j} (A - \lambda_j I)$ for $k \in [n]$.

3) Two distinct eigenvalues – one with multiplicity 1.

Thm : Let $A \in \mathbb{R}^{n\times n}$ be a matrix with $n \geq 3$, with two distinct eigenvalues λ, μ, where $\lambda \rightsquigarrow$ multiplicity $n-1$
$\mu \rightsquigarrow$ multiplicity 1

Then,
$$e^{tA} = e^{\lambda t} \sum_{k=0}^{n-2} \frac{t^k}{k!} (A-\lambda I)^k + \left\{ \frac{e^{\mu t}}{(\mu-\lambda)^{n-1}} - \frac{e^{\lambda t}}{(\mu-\lambda)^{n-1}} \sum_{k=0}^{n-2} \frac{t^k}{k!} (\mu-\lambda)^k \right\} (A-\lambda I)^{n-1}$$

These results cover the 2×2 and 3×3 cases.

- The 2×2 case

Let $A \in \mathbb{R}^{2\times 2}$ with eigenvalues λ_1, λ_2.

a) If $\lambda_1 = \lambda_2 = \lambda$, then
$$e^{tA} = e^{\lambda t}(I + t(A-\lambda I)).$$

b) If $\lambda_1 \neq \lambda_2$, then
$$e^{tA} = \frac{A-\lambda_2 I}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + \frac{A-\lambda_1 I}{\lambda_2-\lambda_1} e^{\lambda_2 t}.$$

- The 3×3 case

Let $A \in \mathbb{R}^{3\times 3}$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

a) If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, then
$$e^{tA} = e^{\lambda t}\left[I + t(A-\lambda I) + \frac{1}{2} t^2 (A-\lambda I)^2 \right]$$

b) If $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$ then if $\lambda(A) = \{\lambda, \mu, \nu\}$
$$e^{tA} = \frac{(A-\mu I)(A-\nu I)}{(\lambda-\mu)(\lambda-\nu)} e^{\lambda t} + \frac{(A-\lambda I)(A-\nu I)}{(\mu-\lambda)(\mu-\nu)} e^{\mu t} + \frac{(A-\lambda I)(A-\mu I)}{(\nu-\lambda)(\nu-\mu)} e^{\nu t}.$$

c) If A has eigenvalues λ, λ, μ with $\lambda \neq \mu$, then
$$e^{tA} = e^{\lambda t}[I + t(A-\lambda I)] + \frac{e^{\mu t} - e^{\lambda t}}{(\mu-\lambda)^2}(A-\lambda I)^2 - \frac{t e^{\lambda t}}{\mu - \lambda}(A-\lambda I)^2.$$

Nonhomogeneous linear systems.

<u>Thm</u> : Let $A \in \mathbb{R}^{n \times n}$ and $b(t) \in \mathbb{R}^n$. Then

$$\begin{cases} \dfrac{dy}{dt} = Ay + b(t) \\ y(a) = c \end{cases} \iff y(t) = e^{(t-a)A}c + e^{tA}\int_a^t e^{-\tau A} b(\tau)\,d\tau$$

<u>Proof</u>

$$\frac{dy(t)}{dt} = Ay(t) + b(t) \iff \frac{dy(t)}{dt} - Ay(t) = b(t) \iff$$

$$\iff e^{-tA}\left[\frac{dy(t)}{dt} - Ay(t)\right] = e^{-tA}b(t) \iff \frac{d}{dt}\left[e^{-tA}y(t)\right] = e^{-tA}b(t) \iff$$

$$\iff e^{-tA}y(t) - e^{-aA}y(a) = \int_a^t e^{-\tau A}b(\tau)\,d\tau \iff$$

$$\iff e^{-tA}y(t) = e^{-aA}c + \int_a^t e^{-\tau A}b(\tau)\,d\tau \iff$$

$$\iff y(t) = e^{(t-a)A}c + e^{tA}\int_a^t e^{-\tau A}b(\tau)\,d\tau. \qquad \square$$

▼ Higher-order ODEs

● Homogeneous linear equations

<u>Def</u>: A homogeneous linear ODE is an equation of the form

$$y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \dots + p_1(x)\,y' + p_0(x)\,y = 0.$$

A general solution is not known. However, a general solution can be characterized by the following theorem:

<u>Def</u>: Let $S = \{y_1, y_2, \dots, y_n\}$ be a set of functions. We say that S is <u>linearly independent</u> $\Leftrightarrow$

$$(\lambda_1 y_1(x) + \lambda_2 y_2(x) + \dots + \lambda_n y_n(x) = 0,\ \forall x \Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0.$$

<u>Thm</u>: There exist n linearly independent functions $y_1, y_2, \dots, y_n$ such that all solutions to

$$y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \dots + p_1(x)\,y' + p_0(x)\,y = 0$$

can be written as

$$y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \dots + \lambda_n y_n(x).$$

► <u>Case 1</u>: Constant coefficient case: $p_k(x) = c_k$

$$y^{(n)} + c_{n-1}\,y^{(n-1)} + \dots + c_1 y' + c_0 y = 0$$

Let $p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$ be the characteristic polynomial. with roots ~~$\lambda_1, \lambda_2, \dots, \lambda_n$.~~ $\rho_1, \rho_2, \dots, \rho_n$.

a) If the roots are distinct then

$$y_k(x) = \exp(\rho_k x)$$

and

$$y(x) = \sum_{k=1}^{n} \lambda_k e^{\rho_k x}$$

b) If a root has multiplicity m then it corresponds to the following solutions:

$$e^{\rho x},\ x e^{\rho x},\ \dots,\ x^{m-1} e^{\rho x}.$$

▶ Case 2 : Equidimensional case : $p_K(x) = c_K x^K$

$$x^n y^{(n)} + c_{n-1} x^{n-1} y^{(n-1)} + \dots + c_1 x y' + c_0 y = 0.$$

Let L be the operator corresponding to this equation, such that $Ly = 0$.

Compute the characteristic polynomial by

$$L(x^r) = P(r)\, x^{r-n}.$$

Let $\rho_1, \rho_2, \dots, \rho_n$ be the roots of $P(r)$.

a) If the roots are distinct then

$$y(x) = \sum_{k=1}^{n} \lambda_k x^{\rho_k}$$

b) A root ρ of multiplicity m corresponds to the following solutions:

$$x^\rho,\ x^\rho \ln x,\ \dots,\ x^\rho (\ln x)^{m-1}.$$

▶ The general case can only be approached by asymptotic methods.

● Inhomogeneous ODEs (linear)

Def : An inhomogeneous linear ODE is an equation of the form:

$$L(y(x)) = f(x)$$

where

$$L(y(x)) = y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_1(x)\, y' + p_0(x)\, y.$$

Thm : There exist n linearly indepedent solutions $y_1, y_2, \dots, y_n$ and a particular solution y_p such that

a) All solutions to $Ly = 0$ are :

$$y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \dots + \lambda_n y_n(x)$$

b) All solutions to $Ly = f(x)$ are :

$$y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \dots + \lambda_n y_n(x) + y_p(x).$$

- The objective here is to find the particular solution. assuming the homogeneous solutions $y_1, y_2, \dots, y_n$ are known. A general method for doing this is with Green's functions.

<u>Def</u>: The <u>Green's function</u> $G(x,a)$ associated with an inhomogeneous linear ODE. of the form $Ly = f(x)$ is a solution to the ODE: $LG(x,a) = \delta(x-a)$.

<u>Thm</u>: Let $G(x,a)$ be a Green function to L. Then a particular solution is given by

$$y(x) = \int_{-\infty}^{+\infty} f(a)\, G(x,a)\, da.$$

<u>Proof</u>

$$Ly(x) = L\int_{-\infty}^{+\infty} f(a)\,G(x,a)\,da = \int_{-\infty}^{+\infty} f(a)\, L G(x,a)\,da =$$

$$= \int_{-\infty}^{+\infty} f(a)\,\delta(x-a)\,da = f(x) \quad \square.$$

To compute $G(x,a)$ we use the following theorem:

<u>Thm</u>: Suppose that $y_1, y_2, \dots, y_n$ are n linearly indepedent solutions to the homogeneous propblem $Ly=0$. of Then

$$G(x,a) = \begin{cases} A_1(a)\,y_1(x) + A_2(a)\,y_2(x) + \dots + A_n(a)\,y_n(x), & x<a \\ B_1(a)\,y_1(x) + B_2(a)\,y_2(x) + \dots + B_n(a)\,y_n(x), & x>a \end{cases}$$

such that

a) $G(x,a), \partial G/\partial x, \dots, \partial^{n-2} G/\partial x^{n-2}$ are continuous at $x=a$

b) $\partial^{n-1} G/\partial x^{n-1}$ has a finite jump discontinuity of magnitude 1.

Proof

Note that for $x<a$

$LG(x,a)=\delta(x-a)=0 \Leftrightarrow G(x,a)=\sum_{k=1}^{n} A_k(a)\,y_k(x)$, $\forall x<a$

Similarly for $x>a$

$LG(x,a)=\delta(x-a)=0 \Leftrightarrow G(x,a)=\sum_{k=1}^{n} B_k(a)\,y_k(x)$, $\forall x>a$.

Note that:

$$LG(x,a)=\delta(x-a) \Leftrightarrow \frac{\partial^n G}{\partial x^n}+p_{n-1}(x)\frac{\partial^{n-1}G}{\partial x^{n-1}}+\cdots+p_1(x)\frac{\partial G}{\partial x}+p_0(x)G=\delta(x-a)$$

a) If any of $G, \partial G/\partial x, \ldots, \partial^{n-2}G/\partial x^{n-2}$ is not continuous then $\partial^n G/\partial x^n$ is more singular than a delta function $\rightsquigarrow$ contradiction. So they must be continuous.

b) Also $\dfrac{\partial^n G}{\partial x^n}-\delta(x-a)=F(x)$ must be less singular than a delta function.

Then $\displaystyle\int F(x)\,dx=\frac{\partial^{n-1}G}{\partial x^{n-1}}-h(x-a)$ is continuous $\Rightarrow$

$\Rightarrow \partial^{n-1}G/\partial x^{n-1}$ has a finite jump discontinuity of magnitude 1. $\square$

Prop (2nd order inhomogeneous linear ODEs)

Consider the problem $y''+p_1(x)y'+p_0(x)y=f(x)$ and let $y_1(x), y_2(x)$ be two solutions to the homogeneous problem.

Define $W(x)=y_1(x)y_2'(x)-y_1'(x)y_2(x)$.

a) A Green's function is given by:

$$G(x,a)=\begin{cases}\dfrac{-y_2(a)y_1(x)+y_1(a)y_2(x)}{y_1(a)y_2'(a)-y_1'(a)y_2(a)} & , x\geq a\\ 0 & , x<a\end{cases}$$

b) A particular solution is given by:

$$y_p(x)=-y_1(x)\int^x\frac{f(t)y_2(t)}{W(t)}\,dt+y_2(x)\int^x\frac{f(t)y_1(t)}{W(t)}\,dt.$$

and the general solution is

$$y(x)=c_1y_1(x)+c_2y_2(x)+y_p(x).$$

Proof

a) The Green's function has the form:

$$G(x,a) = \begin{cases} A_1(a)\, y_1(x) + A_2(a)\, y_2(x) & , x<a \\ B_1(a)\, y_1(x) + B_2(a)\, y_2(x). & , x>a \end{cases}$$

G is continuous at $x=a \Leftrightarrow$

$\Leftrightarrow A_1 y_1(a) + A_2 y_2(a) = B_1 y_1(a) + B_2 y_2(a) \Leftrightarrow$

$\Leftrightarrow (B_1 - A_1) y_1(a) + (B_2 - A_2) y_2(a) = 0$, (1).

$\partial G/\partial x$ has a jump discontinuity of magnitude 1 at $x=a \Leftrightarrow$

$\Leftrightarrow (B_1 y_1'(a) + B_2 y_2'(a)) - (A_1 y_1'(a) + A_2 y_2'(a)) = 1 \Leftrightarrow$

$\Leftrightarrow (B_1 - A_1) y_1'(a) + (B_2 - A_2) y_2'(a) = 1.$ (2)

To force uniqueness, choose $A_1 = A_2 = 0$. Then

$$\begin{cases} B_1 y_1(a) + B_2 y_2(a) = 0 \\ B_1 y_1'(a) + B_2 y_2'(a) = 1 \end{cases} \Leftrightarrow \begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{y_1(a)y_2'(a) - y_1'(a)y_2(a)} \begin{bmatrix} y_2'(a) & -y_2(a) \\ -y_1'(a) & y_1(a) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Leftrightarrow B_1(a) = -\frac{y_2(a)}{y_1(a)y_2'(a) - y_1'(a)y_2(a)} = -\frac{y_2(a)}{W(a)}$$

and

$$B_2(a) = \frac{y_1(a)}{y_1(a)y_2'(a) - y_1'(a)y_2(a)} = \frac{y_1(a)}{W(a)}$$

b) The particular solution is:

$$y_p(x) = \int_{-\infty}^{+\infty} f(t)\, G(x,t)\, dt = \int_{-\infty}^{x} f(t) \left[B_1(t)\, y_1(x) + B_2(t)\, y_2(x) \right] dt =$$

$$= y_1(x) \int_{-\infty}^{x} B_1(t) f(t)\, dt + y_2(x) \int_{-\infty}^{x} B_2(t) f(t)\, dt =$$

$$= -y_1(x)\int_{-\infty}^{x}\frac{f(t)\,y_2(t)}{W(t)}\,dt + y_2(x)\int_{-\infty}^{x}\frac{f(t)\,y_1(t)}{W(t)}\,dt \quad \square .$$

▶ Green's function for the general inhomogeneous linear ODE.

Def: Let $y_1, y_2, \ldots, y_n$ be n functions. The Wronskian matrix $W[y_1, \ldots, y_n]$ is defined by:

$$W[y_1, y_2, \ldots, y_n] = \left[\, y_j^{(i-1)}(x) \right].$$

example: $W[y_1, y_2](x) = \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}$

Thm: (determinant of the Wronskian matrix).
Let $y_1, y_2, \ldots, y_n$ be n linearly indepedent solutions to the homogeneous problem

$$y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \cdots + p_1(x)\,y' + p_0(x)\,y = 0$$

and let

$$w(x) = \det W[y_1, y_2, \ldots, y_n](x)$$

Then

a) $w'(x) = -p_{n-1}(x)\,w(x)$

b)
$$w(x) = \exp\left[-\int^{x} p_{n-1}(t)\,dt\right]$$

Thm: (Criterion for linear indepedence).
Let $y_1, y_2, \ldots, y_n$ be any n functions.
$y_1, y_2, \ldots, y_n$ are linearly indepedent $\Leftrightarrow$ $\det W[y_1, y_2, \ldots, y_n]$ vanishes only at isolated points.

These facts about the Wronskian matrix can help us determine a Green's function and a particular solution for the general case.

To force uniqueness choose $A_k(a) = 0$, $\forall k \in \{1, \ldots, n\}$. Then:

$$\begin{cases} \sum_{k=1}^{n} B_k(a)\, y_k^{(m)}(x) = 0 & , \forall m \in \{0, \ldots, n-2\} \\ \sum_{k=1}^{n} B_k(a)\, y_k^{(n-1)}(x) = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow W[y_1, y_2, \ldots, y_n] \begin{bmatrix} B_1(a) \\ B_2(a) \\ \vdots \\ B_n(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} B_1(a) \\ B_2(a) \\ \vdots \\ B_n(a) \end{bmatrix} = W[y_1, y_2, \ldots, y_n]^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

It follows that

$$y_p(x) = \int_{-\infty}^{+\infty} f(t)\, G(x,t)\, dt =$$

$$= \int_{-\infty}^{x} f(t) \left[\sum_{k=1}^{n} B_k(t)\, y_k(x) \right] dt =$$

$$= \sum_{k=1}^{n} y_k(x) \int_{-\infty}^{x} f(t)\, B_k(t)\, dt$$

2nd order nonlinear ODEs.

In this part we consider ODEs of the form

$$y'' = A + By + Cy^2 + Dy^3$$

where A, B, C, D are constants.

The solution is expressed in terms of elliptic functions which we define:

<u>Def</u>: The elliptic integral of the first kind is defined as

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}, \quad \forall \varphi > 0, \ \forall k^2 < 1$$

<u>Def</u>: (elliptic functions of Jacobi)

a) $sn(x,k) = \sin\varphi \Leftrightarrow x = F(\varphi,k)$.

b) $cn(x,k) = \cos\varphi \Leftrightarrow x = F(\varphi,k)$

c) $tn(x,k) = \dfrac{sn(x,k)}{cn(x,k)}$

d) $dn(x,k) = \sqrt{1-k^2\sin\varphi} \Leftrightarrow x = F(\varphi,k)$

e) $am(x,k) = \varphi \Leftrightarrow x = F(\varphi,k)$.

▶ Solution technique

$$y'' = A + By + Cy^2 + Dy^3 \Leftrightarrow 2y'y'' = 2Ay' + 2By'y + 2Cy^2y' + 2Dy^3y' \Leftrightarrow$$

$$\Leftrightarrow (y')^2 = a + 2Ay + By^2 + \frac{2}{3}Cy^3 + \frac{1}{2}Dy^4$$

Rewrite more conveniently as

$$(y')^2 = a + by + cy^2 + dy^3 + ey^4$$

where a is a constant to be determined by the initial conditions.

~~Define $\Delta(y)^2 =$~~

Factor the polynomial in y:

$$a+by+cy^2+dy^3+ey^4 = h^2(y-a)(y-\beta)(y-\gamma)(y-\delta)$$

and define $\Delta^2(y) = (y-a)(y-\beta)(y-\gamma)(y-\delta)$.

Then:

$$(y')^2 = h^2(y-a)(y-\beta)(y-\gamma)(y-\delta) = h^2\Delta^2(y) \Leftrightarrow y' = h\Delta(y) \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{h\Delta(y)}\frac{dy}{dx} = 1$$

To solve this, define the constants:

$$k^2 = \frac{(\beta-\gamma)(a-\delta)}{(a-\gamma)(\beta-\delta)}, \quad M^2 = \frac{(\beta-\delta)(a-\gamma)}{4}$$

and

$$z^2 = \frac{(\beta-\delta)(y-a)}{(a-\delta)(y-b)}$$

Then: $$\frac{M}{\Delta(y)}\frac{dy}{dx} = \frac{1}{\sqrt{(1-z^2)(1-k^2z^2)}}\frac{dz}{dx}$$

so

$$\frac{1}{h\Delta(y)}\frac{dy}{dx} = 1 \Leftrightarrow \frac{1}{h\sqrt{(1-z^2)(1-k^2z^2)}}\frac{dz}{dx} = M \Leftrightarrow$$

$$\Leftrightarrow \int\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = hMx \Leftrightarrow$$

$$\Leftrightarrow \int_0^z \frac{dz'}{\sqrt{(1-z'^2)(1-k^2z'^2)}} = hM(x-x_0) \Leftrightarrow$$

$$\Leftrightarrow \int_0^z \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = \sin(hM(x-x_0)) \Leftrightarrow$$

$$\Leftrightarrow z = \mathrm{sn}(hM(x-x_0),k) \Leftrightarrow \frac{(\beta-\delta)(y-a)}{(a-\delta)(y-b)} = \mathrm{sn}^2(hM(x-x_0),k) \Leftrightarrow$$

$$\Leftrightarrow \frac{y-a}{y-b} = \frac{a-\delta}{\beta-\delta}\,\mathrm{sn}^2(hM(x-x_0),k) \equiv \Gamma \Leftrightarrow y-a = \Gamma(y-b) \Leftrightarrow$$

$$\Leftrightarrow (1-\Gamma)y = a - b\Gamma \Leftrightarrow$$

$$\Leftrightarrow y = \frac{a-b\Gamma}{1-\Gamma} = \frac{a - b\,\frac{a-\delta}{b-\delta}\,sn^2(hM(x-x_0),k)}{1-\frac{a-\delta}{\beta-\delta}\,sn^2(hM(x-x_0),k)} =$$

$$= a(\beta-\delta) - \cancel{\beta}a$$

$$= \frac{a(\beta-\delta)-\beta(a-\delta)\,sn^2(hM(x-x_0),k)}{\cancel{a}(\beta-\delta)-(a-\delta)\,sn^2(hM(x-x_0),k)}.$$

Conclusion:

The solution to $(y')^2 = h^2(y-a)(y-b)(y-\gamma)(y-\delta)$ is given by.

$$y = \frac{a(b-\delta)-b(a-\delta)\,sn^2(hM(x-x_0),k)}{(b-\delta)-(a-\delta)\,sn^2(hM(x-x_0),k)}$$

where $k^2 = \frac{(b-\gamma)(a-\delta)}{(a-\gamma)(b-\delta)}$ and $M^2 = \frac{(b-\delta)(a-\gamma)}{4}$

► Solution technique (specialized)

Consider the problem:

$$y'' = A + By + Cy^2 \Leftrightarrow 2y'y'' = 2Ay' + 2Byy' + 2Cy^2y' \Leftrightarrow$$

$$\Leftrightarrow (y')^2 = a + 2Ay + By^2 + \frac{2}{3}Cy^3.$$

Rewrite as:

$$(y')^2 = a + \beta y + cy^2 + dy^3 = h^2(y-a)(y-b)(y-\gamma).$$

and define

$$(\Delta(y))^2 = (y-a)(y-b)(y-\gamma).$$

Then

$$(y')^2 = h^2\Delta^2(y) \Leftrightarrow y' = h\Delta(y) \Leftrightarrow \frac{1}{h\Delta(y)}\frac{dy}{dx} = 1.$$

Define the constants

$$k^2 = \frac{b-\gamma}{a-\gamma} \quad \text{and} \quad M^2 = \frac{a-\gamma}{4}$$

and also let $z^2 = \frac{a-\gamma}{y-\gamma}$.

Then $\frac{1}{\Delta(y)}\frac{dy}{dx} = -\frac{1}{M\sqrt{(1-z^2)(1-k^2z^2)}}\frac{dz}{dx}$ so

$$\frac{1}{h\Delta(y)}\frac{dy}{dx} = 1 \Leftrightarrow -\frac{1}{hM\sqrt{(1-z^2)(1-k^2z^2)}}\frac{dz}{dx} = 1 \Leftrightarrow$$

$$\Leftrightarrow \int \frac{dz}{\cancel{M}\sqrt{(1-z^2)(1-k^2z^2)}} = -hMx \Leftrightarrow \cancel{z = sn(-hMx,k)}$$

$$\Leftrightarrow z = sn(-hM(x-x_0),k).$$

By $z^2 = \frac{a-\gamma}{y-\gamma} \Leftrightarrow y-\gamma = \frac{a-\gamma}{z^2} \Leftrightarrow y = \gamma + \frac{a-\gamma}{z^2} \Leftrightarrow$

$$\Leftrightarrow y = \gamma + \frac{a-\gamma}{sn^2(-hM(x-x_0),k)}.$$

<u>Conclusion</u>

The solution to ~~$(y')^2 = a+by+cy$~~ $(y')^2 = h(y-a)(y-b)(y-\gamma)$ is given by

$$y = \gamma + \frac{a-\gamma}{sn^2(-hM(x-x_0),k)}$$

where $M^2 = \frac{a-\gamma}{4}$ and $k^2 = \frac{b-\gamma}{a-\gamma}$.

▶ <u>Other special results.</u>

1) The equation $y'' = 6y^2$ has solution

$$y(x) = C^2\left[\frac{-k^2}{1+k^2} + \frac{1}{sn^2(C(x-x_1),k)}\right]$$

where C, x_1 determined by initial conditions and k is a root of $1-k^2+k^4=0$.

2) The equation $y'' = Ay + By^3$ has the general solution:

$$y(x) = C\,\mathrm{sn}(\lambda(x - x_0), k)$$

where λ, x_0 are determined by the initial conditions and

$$k^2 = -\frac{\lambda^2 + A}{\lambda^2}, \quad C^2 = -\frac{2(\lambda^2 + A)}{B}$$

Since k, C are both functions of λ, only one of them can be chosen arbitrarily.

3) The pendulum equation $\cancel{\frac{d^2y}{dx^2}}$ $y'' = -n^2 \sin y$ has the general solution

$$y = 2\,\mathrm{Arcsin}\left[k\,\mathrm{sn}(nx, k) \right]$$

where $k = \sin(w/2)$ and

w = maximum displacement of pendulum.

<u>Remark</u>: The pendulum equation can be reduced to the form $y'' = Ay + By^3$ with the transformation $y = \sin(z/2)$.

Non-linear systems of ODEs

Def: A non-linear system of ODEs is a system of equations of the form:

$$\begin{aligned}
\dot{y}_1 &= f_1(y_1, y_2, \ldots, y_n) \\
\dot{y}_2 &= f_2(y_1, y_2, \ldots, y_n) \\
&\;\; - - - - - - \\
\dot{y}_n &= f_n(y_1, y_2, \ldots, y_n)
\end{aligned}$$

where $y_1, y_2, \ldots, y_n$ are functions of t.

Remark: The solution is a trajectory on an n-dimensional space (called phase space) which is parametrized by:

$$y_1 = y_1(t) \quad , \quad y_2 = y_2(t) \quad , \quad \ldots \quad , \quad y_n = y_n(t).$$

No two such trajectories can cross otherwise the tangent vector at a crossing point wouldn't be unique.

Def: The solutions to the system

$$\begin{aligned}
f_1(y_1, y_2, \ldots, y_n) &= 0 \\
f_2(y_1, y_2, \ldots, y_n) &= 0 \\
&\;\; - - - - - - \\
f_n(y_1, y_2, \ldots, y_n) &= 0
\end{aligned}$$

as points in phase space are called critical points.

Remark: Even though an exact solution is not known, the behaviour near critical points can be described in broad terms.

• Local analysis at critical points

Let $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))$ and $f(y) = (f_1(y), f_2(y), \ldots, f_n(y))$ and rewrite the non-linear system of ODEs as.

$$\frac{dy_k}{dt} = f_k(y_1, y_2, \ldots, y_n), \forall k \in [n] \Leftrightarrow \frac{dy}{dt} = f(y).$$

Let y_0 be a critical point solution (i.e. a solution with initial condition on top of a critical point).

Then $\frac{dy_0}{dt} = 0$, $\forall t > 0$ and $f(y_0) = 0 \rightsquigarrow$ the solution will remain constant. Now suppose that we perturb the solution:

$$y(t) = y_0(t) + \varepsilon y_1(t)$$

Then

$$\frac{d}{dt}\left(y_0(t) + \varepsilon y_1(t)\right) = f(y_0 + \varepsilon y_1) \Leftrightarrow$$

$$\Leftrightarrow \cancel{\frac{dy_0}{dt}}^{0} + \varepsilon \frac{dy_1}{dt} = \cancel{f(y_0)}^{0} + [Df(y_0)]\varepsilon y_1 \Leftrightarrow$$

$$\Leftrightarrow \varepsilon \frac{dy_1}{dt} = [Df(y_0)]\varepsilon y_1 \Leftrightarrow \boxed{\frac{dy_1}{dt} = [Df(y_0)]\, y_1}$$

where $Df(y_0)$ = the Jacobian of f at $y = y_0$.

$$[Df]_{ij} = \frac{\partial f_i}{\partial y_j}$$

The linearized system of ODEs: $\frac{dy_1}{dt} = [Df(y_0)]y_1$ can be used to tell whether y_1 will vanish, explode or oscillate.

▶ Compute the eigenvalues of $Df(y_0)$.

a) If all eigenvalues $\forall \lambda \in \lambda(Df(y_0)) : \lambda > 0 \Rightarrow$ $\Rightarrow$ the point y_0 is a <u>source</u>.
i.e. all trajectories move away from y_0.

b) If $\forall \lambda \in \lambda(Df(y_0)) : \lambda < 0 \Rightarrow$ the point y_0 is a <u>sink</u>.
i.e. all trajectories move towards y_0.

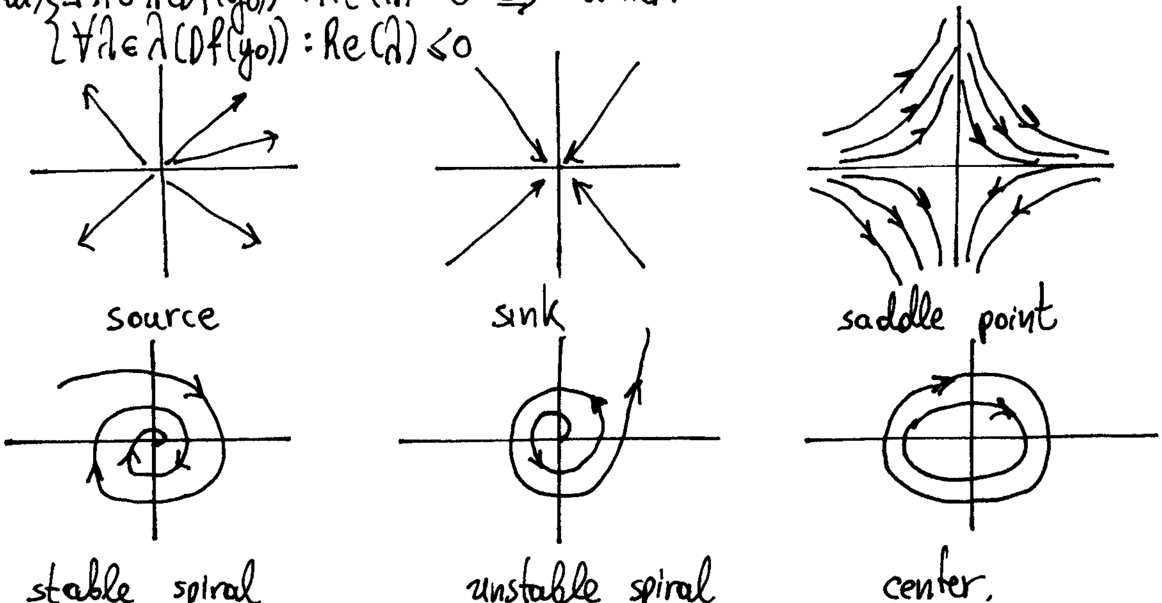
c) If $\exists \lambda_1, \lambda_2 \in \lambda(Df(y_0)) : \lambda_1 < 0 \wedge \lambda_2 > 0 \Rightarrow y_0$ is a saddle point.

d) If some eigenvalues are complex then trajectories form spirals or a center.

i) $\forall \lambda \in \lambda(Df(y_0)) : Re(\lambda) < 0 \Rightarrow$ stable spiral

ii) $\exists \lambda \in \lambda(Df(y_0)) : Re(\lambda) > 0 \Rightarrow$ unstable spiral.

iii) $\begin{cases} \exists \lambda \in \lambda(Df(y_0)) : Re(\lambda) = 0 \\ \forall \lambda \in \lambda(Df(y_0)) : Re(\lambda) \leq 0 \end{cases} \Rightarrow$ center.



example: Analyze $\begin{cases} y_1' = y_1(3-y_1-y_2) \\ y_2' = y_2(y_1-1) \end{cases}$

Critical points are given by:

$$\begin{cases} y_1(3-y_1-y_2) = 0 \\ y_2(y_1-1) = 0 \end{cases} \Leftrightarrow \begin{cases} y_1 = 0 \\ y_2 = 0 \end{cases} \vee \begin{cases} 3-y_1-y_2 = 0 \\ y_2 = 0 \end{cases} \vee \begin{cases} 3-y_1-y_2 = 0 \\ y_1 - 1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y_1 = 0 \\ y_2 = 0 \end{cases} \vee \begin{cases} y_1 = 3 \\ y_2 = 0 \end{cases} \vee \begin{cases} y_1 = 1 \\ y_2 = 2 \end{cases}$$

so the critical points are $\{(0,0), (1,2), (3,0)\}$.

The Jacobian

$$Df(y_1,y_2) = \begin{bmatrix} \partial f_1/\partial y_1 & \partial f_1/\partial y_2 \\ \partial f_2/\partial y_1 & \partial f_2/\partial y_2 \end{bmatrix} = \begin{bmatrix} 3-2y_1-y_2 & -y_1 \\ y_2 & y_1-1 \end{bmatrix}$$

At (0,0): $Df(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ has eigenvalues $\lambda = 3$ and $\lambda = -1$

$\Rightarrow (0,0)$ is a saddle point.

At (1,2): $Df(1,2) = \begin{bmatrix} 3-2-2 & -1 \\ 2 & 1-1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}$

eigenvalues: $\det(\cancel{Df}\,\lambda I - Df(1,2)) = 0 \Leftrightarrow \det \begin{bmatrix} \lambda+1 & 1 \\ -2 & \lambda \end{bmatrix} = 0 \Leftrightarrow$

$\Leftrightarrow \lambda(\lambda+1) + 2 = 0 \Leftrightarrow \lambda^2 + \lambda + 2 = 0 \Rightarrow \lambda_{1,2} = \dfrac{-1 \pm i\sqrt{7}}{2}$

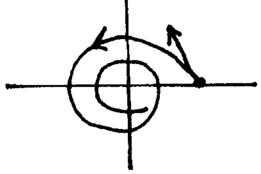
$\Delta = 1 - 8 = -7$

Since $Re(\lambda_1) < 0 \wedge Re(\lambda_2) < 0 \Rightarrow (1,2)$ is a stable spiral.

To determine whether the spiral is clockwise or counterclockwise consider the linearized system of ODEs in the neighborhood of (1,2):

$$\begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow$$

$\Rightarrow$ the spiral is a counterclockwise stable spiral.



At (3,0): $Df(3,0) = \begin{bmatrix} 3-6-0 & -3 \\ 0 & 3-1 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 0 & 2 \end{bmatrix}$

eigenvalues: $\det(\lambda I - Df(3,0)) = 0 \Leftrightarrow \begin{vmatrix} \lambda+3 & 3 \\ 0 & \lambda-2 \end{vmatrix} = 0 \Leftrightarrow$

$\Leftrightarrow (\lambda+3)(\lambda-2) = 0 \Leftrightarrow \lambda_1 = -3,\ \lambda_2 = 2$

so (3,0) is a saddle point.

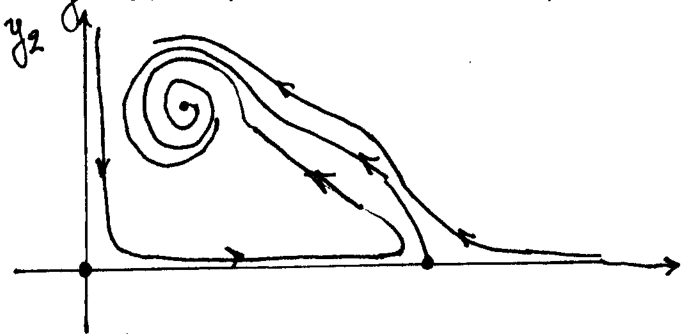
eigenvectors $v_1 = (1,0)$ for $\lambda_1 = -3$

$v_2 = (-3,5)$ for $\lambda_2 = 2$

$$\begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} -3 \\ 5 \end{bmatrix} e^{2t}$$

so the ingoing trajectory ~~has~~ is $\parallel$ to the y_1-axis and the outgoing trajectory has asymptotic slope of $-5/3$ as $t \to \infty$.

Using this information we make the following trajectory profile



- Limit cycles ⇝ in some cases, an entire closed trajectory acts as a "critical point".

Such trajectories are called limit cycles.

example

$$\begin{cases} y_1' = y_1 + y_2 - y_1(y_1^2+y_2^2) \\ y_2' = -y_1 + y_2 - y_2(y_1^2+y_2^2) \end{cases}$$

Critical point at $(0,0)$ is an unstable clockwise spiral.
However at infinity:

$$\begin{cases} y_1' = -y_1(y_1^2+y_2^2) < 0 \\ y_2' = -y_2(y_1^2+y_2^2) < 0 \end{cases} \Rightarrow \text{distant trajectories move towards the origin.}$$

There must be at least one trajectory that neither moves inward nor outward.

Note that if $z = y_1^2+y_2^2$ then

$$z' = (y_1^2+y_2^2)' = 2y_1y_1' + 2y_2y_2' =$$

$$= 2y_1\left[y_1+y_2-y_1(y_1^2+y_2^2)\right] + 2y_2\left[-y_1+y_2-y_2(y_1^2+y_2^2)\right] =$$

$$= 2y_1^2 + \cancel{2y_1y_2} - 2y_1^2(y_1^2+y_2^2) - \cancel{2y_1y_2} + 2y_2^2 - 2y_2^2(y_1^2+y_2^2) =$$

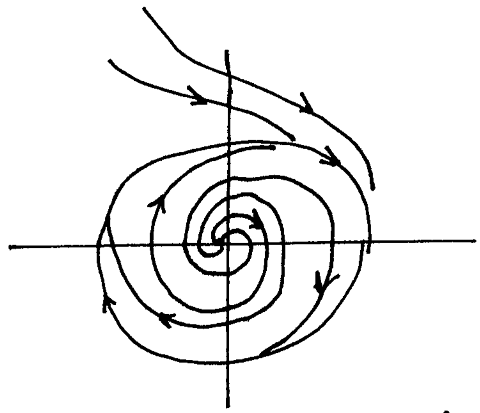
$$= 2(y_1^2+y_2^2) - 2(y_1^2+y_2^2)(y_1^2+y_2^2) = 2z - 2z^2.$$

Steady state is at

$$2z - 2z^2 = 0 \Leftrightarrow z = 1$$

so the orbit $y_1^2+y_2^2 = 1$ is a fixed point (limit cycle)

Let $f(z) = 2z - 2z^2 \Rightarrow \frac{\partial f}{\partial z} = 2 - 4z \Big|_{z=1} = -2 \Rightarrow$ attracting cycle !!



Problems with critical point analysis.

1) Nonlinear critical points

a) Some trajectories are ingoing whereas others are outgoing.
b) Trajectories close to the critical point are curved as $t \to \infty$.
i.e. they do not approach being a line

example

$$\begin{cases} \dot{y}_1 = y_1^2 + y_2^2 + y_1^4 \\ \dot{y}_2 = \sin(y_1^4 + y_2^4). \end{cases}$$

A critical point at $(0,0)$. In the neighbourhood of the origin

$$\dot{\varepsilon}_1 \sim \varepsilon_1^2 + \varepsilon_2^2 \quad \rightsquigarrow \text{problem: these are still nonlinear.}$$
$$\dot{\varepsilon}_2 \sim \varepsilon_1^4 + \varepsilon_2^4$$

The Jacobian is $J = 0$.

2) Centers : we may have closed orbits around a critical point.

example or... we may not!

$$\begin{cases} \dot{y}_1 = -y_2 + y_1(y_1^2 + y_2^2) \\ \dot{y}_2 = -y_1 + y_2(y_1^2 + y_2^2) \end{cases}$$

Critical point at $(y_1, y_2) = (0,0)$.
However:

$$\begin{cases} y_1 \dot{y}_1 = y_1^2(y_1^2 + y_2^2) - y_1 y_2 \\ y_2 \dot{y}_2 = y_2^2(y_1^2 + y_2^2) + y_1 y_2 \end{cases} \Rightarrow$$

$$\Rightarrow y_1\dot{y}_1 + y_2\dot{y}_2 = y_1^2(y_1^2+y_2^2) + y_2^2(y_1^2+y_2^2) = (y_1^2+y_2^2)^2 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} r^2 = y_1^2+y_2^2 \\ \frac{1}{2}\frac{d}{dt}(r^2) = r^4 \end{cases} \Leftrightarrow r(t) = \frac{r(0)}{\sqrt{1-2r^2(0)t}}$$

Radius increases as $t \to \infty$ and becomes infinite in finite time: $t = 1/2r^2(0)$. $\rightsquigarrow$ not a center

To show that we have closed orbits we construct an energy integral which is called Liapunov function.

<u>example 1</u>

$$\begin{cases} \dot{y}_1 = -y_2 - y_2^3 \\ \dot{y}_2 = y_1 \end{cases} \qquad \text{Critical points } \{(0,0)\}. \rightsquigarrow \begin{cases} \dot{\varepsilon}_1 = -\varepsilon_2 \\ \dot{\varepsilon}_2 = \varepsilon_1 \end{cases}$$

It could be a center.

$$y_1\dot{y}_1 + y_2\dot{y}_2 = y_1(-y_2-y_2^3) + y_2y_1 = -y_1y_2 - y_1y_2^3 + y_1y_2 = -y_1y_2^3$$

$$= -y_2^3\dot{y}_2 \Leftrightarrow \frac{d}{dt}\left(y_1^2+y_2^2+\frac{1}{4}y_2^4\right) = 0 \Leftrightarrow$$

$$\Leftrightarrow y_1^2 + \frac{5}{4}y_2^2 = c. \rightsquigarrow \text{concetric closed curves around } (0,0)$$

therefore $(0,0)$ is a center.

<u>example 2</u>

$$\begin{cases} \dot{y}_1 = y_1 - y_1y_2 \\ \dot{y}_2 = -y_2 + y_1y_2 \end{cases}$$

Critical points $\begin{cases} y_1 - y_1y_2 = 0 \\ y_2 - y_1y_2 = 0 \end{cases} \Leftrightarrow \begin{cases} y_1(1-y_2) = 0 \\ y_2(1-y_1) = 0 \end{cases}$

Consider the point $(y_1,y_2) = (1,1)$.

Multiply by $\frac{1-y_1}{y_1}$, $\frac{1-y_2}{y_2}$:

$$\frac{1-y_1}{y_1}\dot{y}_1 + \frac{1-y_2}{y_2}\dot{y}_2 = \frac{1-y_1}{y_1}(y_1-y_1y_2) + \frac{1-y_2}{y_2}(-y_2+y_1y_2) =$$

$$= (1-y_1)(1-y_2) + (1-y_2)(y_1-1) = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{d}{dt}\left[\ln(y_1y_2)-(y_1+y_2)\right]=0 \Leftrightarrow y_1+y_2-\ln(y_1y_2)=c$$

so there are eccentric closed curves containing $(1,1) \Rightarrow$
$\Rightarrow (1,1)$ is a center.

● When does linearization work

<u>Def</u> : Consider a system of ODEs : $\dot{y}=f(y)$. A critical point is called a <u>hyperbolic point</u> if and only if $f(y_0)=0$ and none of the eigenvalues of $Df(y_0)$ have $Re\lambda=0$.
<u>example</u>: a stable/unstable spiral is a hyperbolic point.
a center is a non-hyperbolic point.

<u>Remark</u> : If y_0 is a hyperbolic point then the local behaviour of the nonlinear system is topologically equivalent to the local behaviour of the linearized equations. In that case local analysis can be applied safely.
Non-hyperbolic points are current research and sometimes can be treated with Liapunov functions.
Another result known for ~~hyperbo~~ non-hyperbolic points is "center-manifold" reduction.

Center-manifold reduction

Let $\dot{y} = f(y)$ be a system of n-ODEs and y_0 a critical point with $f(y_0) = 0$.
Let k = # of eigenvalues of $Df(y_0)$ with $Re\lambda < 0$
j = # of eigenvalues of $Df(y_0)$ with $Re\lambda > 0$
Define $m = n - k - j$ = # of eigenvalues of $Df(y_0)$ with $Re\lambda = 0$.
If $m = 0 \Rightarrow$ the point is hyperbolic and we know how to analyze it.
Suppose that $m > 0$. Then the following result characterizes the behaviour of the ODEs only when $j = 0$.

Thm : (Center-manifold theorem).
Suppose we have a system of n ODEs $\dot{z} = F(z)$ that can be split and ~~Tay.~~ linearized as:

$$\dot{x} = Ax + f(x,y) \qquad \text{where } (x,y) \in \mathbb{R}^n \times \mathbb{R}^m$$
$$\dot{y} = By + g(x,y) \qquad A = n\times n \text{ matrix with } Re\lambda = 0$$
$$B = m \times m \text{ matrix with } Re\lambda < 0$$

Then there exists a manifold $W^c = \{(x,y) \mid y = h(x)\}$ such that $h(0) = Dh(0) = 0$ called the center manifold.
The solution of the non-linear system converges to the center-manifold as $t \to +\infty$.

Remarks

a) Once h is known, the solution can be approximated by the center-manifold approximation:

$$\dot{x} = Ax + f(x, h(x))$$
$$y = h(x).$$

b) The relation $y = h(x)$ is called the "slaving" principle

c) The condition $h(y_0) = Dh(y_0) = 0$ means that W^c is tangent to the stable subspace E^c which is given by:
$E^c = \text{span}\{\lambda \in \lambda(A)\}$.

d) $f(x,y)$ and $g(x,y)$ are the nonlinear parts of the RHS.

Methodology.

Let $\dot{y} = F(y)$ be a system of n ODEs and y_0 be a critical point $(F(y_0) = 0)$.
Suppose that $DF(y_0)$ has eigenvalues such that $\mathrm{Re}\,\lambda \leq 0$.
To split the system to a stable set of equations and a center set of equations, we do the following:

•1 Linearlize the equations around y_0:

$$\dot{y} = DF(y_0)\, y + G(y).$$

•2 Diagonalize $DF(y_0) = P\, \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) P^{-1}$.

•3 Let $x = P^{-1} y \Leftrightarrow y = Px$. Then

$$\dot{x} = P^{-1}\dot{y} = P^{-1}\left(DF(y_0)\, y + G(y)\right) = P^{-1}\left(DF(y_0) P x + G(Px)\right) =$$

$$= \left[P^{-1} DF(y_0) P\right] x + P^{-1} G(Px) =$$
$$= \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)\, x + P^{-1} G(Px).$$

which reduces to:

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 + g_1(x_1, x_2, \ldots, x_n) \\ \dot{x}_2 = \lambda_2 x_2 + g_2(x_1, x_2, \ldots, x_n) \\ - - - - - - - - \\ \dot{x}_n = \lambda_n x_n + g_n(x_1, x_2, \ldots, x_n) \end{cases}$$

If $\mathrm{Re}\,\lambda_j = 0$ for $j \in [k]$ and $\mathrm{Re}\,\lambda_j < 0$ for $j \in [n] - [k]$ then the first k equations drive the system and the other $n-k$ equations are driven by "slaving principles":

$$x_{k+1} = h_1(x_1, x_2, \ldots, x_k)$$
$$x_{k+2} = h_2(x_1, x_2, \ldots, x_k)$$
$$- - - - - - - - - -$$
$$x_n = h_{n-k}(x_1, x_2, \ldots, x_k).$$

and the center manifold is given from the slaving principles by

$$W^c = \{(x_1, x_2, \ldots, x_n) : \cancel{x_j = h}\ x_{k+j} = h_j(x_1, x_2, \ldots, x_k),\ \forall j \in [n-k]\}.$$

We obtain the following result:

<u>Thm</u>: Suppose we have n ODEs of the form:

$$\dot{x}_1 = \lambda_1 x_1 + g_1(x_1, x_2, \dots, x_n).$$
$$\dot{x}_2 = \lambda_2 x_2 + g_2(x_1, x_2, \dots, x_n).$$
$$- - - - - - - - - - -$$
$$\dot{x}_n = \lambda_n x_n + g_n(x_1, x_2, \dots, x_n).$$

withe $\mathrm{Re}\lambda_1 = \mathrm{Re}\lambda_2 = \dots = \mathrm{Re}\lambda_k = 0$

$\mathrm{Re}\lambda_j < 0 \quad , \forall j > k$

and x_0 is a critical point.

Then, all solutions with initial condition close enough to x_0 will approach a center-manifold W^c as $t \to +\infty$ given by

$W^c = \{(x_1, x_2, \dots, x_n) : x_{k+j} = h_j(x_1, x_2, \dots, x_k), \forall j \in [n-k]\}$

where $h_1, h_2, \dots, h_{n-k}$ are given by $n-k$ PDEs in terms of $x_1, x_2, \dots, x_k$: ~~(define $G_j(x, h(x))$)~~

Define $x = (x_1, x_2, \dots, x_k)$ and

$$G_j(x, h(x)) = g_j(x_1, x_2, \dots, x_k, h_1(x), h_2(x), \dots, h_{n-k}(x)).$$

Then the PDEs are:

$$\sum_{j=1}^{k} \left[\lambda_j x_j + G_j(x, h(x))\right] \frac{\partial h_i}{\partial x_j} = \lambda_{k+i} h_i - G_{k+i}(x, h(x)). \quad , \forall i \in [n-k].$$

$$\left.\begin{aligned} & h_1(x_0) = h_2(x_0) = h_{n-k}(x_0) = 0 \\ & \left.\frac{\partial h_i}{\partial x_j}\right|_{x_0} = 0 \quad , \forall i, j. \end{aligned}\right\} \text{boundary conditions}$$

$\square$.

The only way to solve these PDEs is by Taylor expansion.

This result is called the "spectral gap theorem"

We describe now a method to obtain h_j and W^c.
Let $u = (x_1, x_2, \dots, x_k)$ and $v = (x_{k+1}, \dots, x_n)$.

~~Then~~ $\cancel{\dot{u} = Au + f(u,v)}$

$\dot{v} =$

Then
$$\dot{u} = Au + G_1(u,v)$$
$$\dot{v} = Bv + G_2(u,v)$$
where $A = \mathrm{diag}(\lambda_1, \dots, \lambda_k)$, $B = \mathrm{diag}(\lambda_{k+1}, \dots, \lambda_n)$
and G_1, G_2 are
$$G_1 = (g_1, g_2, \dots, g_k)$$
$$G_2 = (g_{k+1}, g_{k+2}, \dots, g_n).$$
The slaving principle can be written as
$$v = h(u).$$
Differentiate: $\dot{v} = Dh(u)\,\dot{u} = Dh(u)\left[Au + G_1(u,v)\right] = Dh(u)\left[Au + G_1(u,h(u))\right]$

also $\dot{v} = Bv + G_2(u,v) = Bh(u) + G_2(u,h(u))$.

It follows that:
$$Dh(u)\left[Au + G_1(u,h(u))\right] = Bh(u) + G_2(u,h(u)).$$
Define the N operator by:
$$N(h(u)) = Dh(u)\left[Au + G_1(u,h(u))\right] - Bh(u) - G_2(u,h(u)).$$
Also let $x_0 = P^{-1}y_0 = (u_0, v_0)$ be the critical point.
The the initial value problem
$$\begin{cases} N(h(u)) = 0 \\ h(u_0) = Dh(u_0) = 0 \end{cases}$$
will determine h (and as a consequence W^c).
Now let us expand this problem in terms of components.
$N(h(u))$ is an $n-k$ vector. A, B are diagonal matrices and
G_1, G_2 are just $g_1, g_2, \dots, g_n$. Also
$$(Dh(u))_{ij} = \frac{\partial h_i}{\partial u_j}$$
is an $(n-k)\times k$ matrix. It follows that
$$N_i = \sum_{j=1}^{k}\left[(\lambda_j x_j + g_j(x))\frac{\partial h_i}{\partial x_j}\right] - (\lambda_{k+i} h_i - g_{k+i}(x)).$$

<u>example</u>: Let us consider a simple case with a master equation and a slave equation.

$$\begin{cases} \dot{u} = \lambda u + f(u,v) & \text{with } \operatorname{Re}\lambda = 0 \\ \dot{v} = \mu v + g(u,v) & \operatorname{Re}\mu < 0 . \end{cases}$$

Then $k=1$, $n=2$, $n-k=1$ ⇝ only one PDE ⇝ an ODE:

$$N(u) = (\lambda u + f(u,h(u)))\frac{\partial h}{\partial u} - (\mu h(u) - g(u,h(u))) = 0$$

Boundary condition: If (u_0,v_0) is the critical ~~poit~~ point then $h(u_0) = h'(u_0) = 0$.

So $h(u) = \gamma_2(u-u_0)^2 + \gamma_3(u-u_0)^3 + \dots + \gamma_k(u-u_0)^k + \dots$ around u_0. We get:

$$N(u) = (\lambda u + f(u,h(u)))[2\gamma_2(u-u_0) + 3\gamma_3(u-u_0)^2 + \dots] -$$
$$- \mu[\gamma_2(u-u_0)^2 + \gamma_3(u-u_0)^3 + \dots] + g(u,h(u)) = 0.$$

The coefficients $\gamma_2, \gamma_3, \dots$ can be determined by doing a full expansion on $N(u)$ and matching terms.

<u>example</u>: The Lorenz equations

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = x - y - xz \\ \dot{z} = -bz + xy. \end{cases} \quad \text{where } \sigma > 0, \ b > 0.$$

Critical points:

$$\begin{cases} \sigma(y-x) = 0 \\ x - y - xz = 0 \\ -bz + xy = 0 \end{cases} \Leftrightarrow \begin{cases} y = x \\ x - x - xz = 0 \\ -bz + x^2 = 0 \end{cases} \Leftrightarrow \begin{cases} y = x \\ xz = 0 \\ x^2 - bz = 0 \end{cases} \Leftrightarrow \begin{cases} y = x \\ x(x^2/b) = 0 \\ z = x^2/b \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0. \end{cases}$$

Let $f_1(x,y,z) = \sigma(y-x)$, $f_2(x,y,z) = x - y - xz$, $f_3(x,y,z) = -bz + xy$

$$Df(x,y,z) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1-z & -1 & -x \\ y & x & -b \end{bmatrix} \Rightarrow Df(0,0,0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

It follows that

$$\dot{u} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial t} = f'(v)\dot{v} = \frac{-\sigma}{\sigma+1}u(v-\sigma w)f'(v) = \frac{-\sigma}{\sigma+1}f(v)(v-\sigma g(v))f'(v)$$

$$\dot{w} = \frac{\partial w}{\partial v}\frac{\partial v}{\partial t} = g'(v)\dot{v} = \frac{-\sigma}{\sigma+1}u(v-\sigma w)g'(v) = \frac{-\sigma}{\sigma+1}f(v)(v-\sigma g(v))g'(v).$$

Also
$$\dot{u} = -bu + (v-\sigma w)(v+w) = -bf(v) + (v-\sigma g(v))(v+g(v)).$$
$$\dot{w} = -(\sigma+1)w - \frac{(v-\sigma w)u}{\sigma+1} = -(\sigma+1)g(v) - \frac{(v-\sigma g(v))f(v)}{\sigma+1}$$

therefore

$$\frac{-\sigma}{\sigma+1}f(v)(v-\sigma g(v))f'(v) = -bf(v) + (v-\sigma g(v))(v+g(v)). \qquad (1)$$

$$\frac{-\sigma}{\sigma+1}f(v)(v-\sigma g(v))g'(v) = -(\sigma+1)g(v) - \frac{(v-\sigma g(v))f(v)}{\sigma+1} \qquad (2)$$

Assume the expansions

$$f(v) = A_1 v^2 + A_2 v^3 + O(v^4)$$
$$g(v) = B_1 v^2 + B_2 v^3 + O(v^4).$$

and substitute to (1) and (2).

$$(1) \Leftrightarrow (bA_1 - 1)v^2 + (bA_2 + \sigma B_1 - B_1)v^3 + O(v^4) = 0$$
$$(2) \Leftrightarrow B_1(\sigma+1)v^2 + \left((\sigma+1)B_2 + \frac{A_1}{\sigma+1}\right)v^3 + O(v^4) = 0$$

It follows that

$$\begin{cases} bA_1 - 1 = 0 \\ bA_2 + (\sigma-1)B_1 = 0 \\ B_1(\sigma+1) = 0 \\ (\sigma+1)B_2 + \dfrac{A_1}{\sigma+1} = 0 \end{cases} \Leftrightarrow \begin{cases} A_1 = 1/b \\ bA_2 = 0 \\ B_1 = 0 \\ (\sigma+1)B_2 + \dfrac{1/b}{\sigma+1} = 0 \end{cases} \Leftrightarrow \begin{cases} A_1 = 1/b \\ B_2 = \dfrac{-1}{b(\sigma+1)^2} \end{cases} \quad B_1 = A_2 = 0$$

so
$$f(v) = v^2/b + O(v^4)$$
$$g(v) = \frac{-v^3}{b(\sigma+1)^2} + O(v^4)$$

The slaving principle is then:

Eigenvectors / Eigenvalues $\quad \lambda_1 = -b \qquad v_1 = (0,0,1)$

$\lambda_2 = 0 \qquad v_2 = (1,1,0)$

$\lambda_3 = -\sigma-1 \qquad v_3 = (-\sigma,1,0).$

Define

$$P = [v_1\, v_2\, v_3] = \begin{bmatrix} 0 & 1 & -\sigma \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{\sigma+1}\begin{bmatrix} 0 & 0 & \sigma+1 \\ 1 & \sigma & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

and let $(x,y,z) = P(u,v,w) \Leftrightarrow (u,v,w) = P^{-1}(x,y,z)$.

$$\begin{cases} x = v - \sigma w \\ y = v + w \\ z = u \end{cases} \quad \text{and} \quad \begin{cases} u = z \\ v = (x+\sigma y)/(\sigma+1) \\ w = -(x-y)/(\sigma+1) \end{cases} \quad \text{and} \quad \begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = x - y - xz \\ \dot{z} = -bz + xy. \end{cases}$$

Now we rewrite the ODEs in terms of u,v,w:

$$\dot{u} = \dot{z} = -bz + xy = -bu + (v-\sigma w)(v+w).$$

$$\dot{v} = \frac{\dot{x}+\sigma\dot{y}}{\sigma+1} = \cancel{\frac{(v-\sigma w)+\sigma(v+w)}{\sigma+1}}\ \frac{\sigma(y-x)+\sigma(x-y-xz)}{\sigma+1} = \frac{-\sigma xz}{\sigma+1} =$$

$$= \frac{-\sigma(v-\sigma w)u}{\sigma+1} = -\frac{\sigma}{\sigma+1}u(v-\sigma w).$$

$$\dot{w} = -\frac{\dot{x}-\dot{y}}{\sigma+1} = -\frac{\sigma(y-x)-(x-y-xz)}{\sigma+1} =$$

$$= \frac{-1}{\sigma+1}\left[\sigma(w+\sigma w) + \sigma w + w + (v-\sigma w)u\right] =$$

$$= -(\sigma+1)w - \frac{(v-\sigma w)u}{\sigma+1}$$

To summarize:

$$\dot{u} = -bu + (v-\sigma w)(v+w)$$
$$\dot{v} = -\frac{\sigma}{\sigma+1}u(v-\sigma w)$$
$$\dot{w} = -(\sigma+1)w - \frac{(v-\sigma w)u}{\sigma+1}$$

v is the leading variable and u, w are slaves.

Let the slaving principle be $u = f(v)$

$w = g(v).$

$$u = \frac{v^2}{b} + O(v^{\cancel{3}\,4}). \quad \text{and} \quad w = \frac{-v^3}{b(\sigma+1)^2} + O(v^4).$$

$$\text{and} \quad \dot{v} = -\frac{\sigma}{\sigma+1}\, u(v - \sigma w) = -\frac{\sigma}{\sigma+1}\left(\frac{v^2}{b} + O(v^{\cancel{3}\,4})\right)\left(v + \sigma \frac{v^3}{b(\sigma+1)^2} + O(v^4)\right) =$$

$$= -\frac{\sigma}{\sigma+1}\,\frac{v^2}{b}\left(v + \sigma \frac{v^3}{b(\sigma+1)^2}\right) + O(v^4) =$$

$$= -\frac{\sigma}{\sigma+1}\,\frac{v^3}{b} + O(v^4). \rightsquigarrow \text{3rd order approximation. } \square$$

Floquet theory.

The objective of Floquet theory is to analyze the stability of a system of ODEs of the form

$$\dot{x} = A(t)x \quad \text{with} \quad x \in \mathbb{C}^n \text{ and } A \in \mathbb{C}^{n\times n}.$$

where $A(t)$ is continous and T-periodic $\rightsquigarrow A(t+T) = A(t)$.

<u>Def</u> : Let $\dot{x} = A(t)x$ be a linear system of n ODEs. We call a <u>fundamental solution</u> $\Phi(t)$ an <u>nxn matrix</u> such that:

$$\dot{\Phi}(t) = A(t)\Phi(t).$$

<u>Thm</u> : Let $\Phi(t)$ be a fundamental solution to $\dot{x} = A(t)x$. Then the exact solution to the initial value problem

$$\dot{x} = A(t)x + b \quad \text{with } x(0) = x_0$$

is given by

$$x(t) = \Phi(t)\Phi^{-1}(0)x_0 + \int_0^t \Phi(t)\Phi^{-1}(\tau)\, b(\tau)\, d\tau$$

<u>Remark.</u> : In the simple case where $A(t)$ is not time dependent (i.e. $A(t) = A$), the fundamental solution is known and given by $\Phi(t) = \exp(tA)$.
When $A(t)$ is time dependent, $\Phi(t)$ is not generally know. Floquet theory provides with a ~~real~~ result in the case where $A(t)$ is periodic.

<u>Thm</u> : (Floquet).
Let $\dot{x} = A(t)x$ be a system of n ODEs. If $A(t)$ is continuous and T-periodic then the fundamental solution can be written as

$$\Phi(t) = P(t)\exp(tB)$$

where $B \in \mathbb{C}^{n\times n}$ constant matrix and $P(t)$ is T-periodic.

<u>Remark</u>: From this result, it follows that:

$$x(t) = \Phi(t)\Phi^{-1}(0)x_0 + \int_0^t \Phi(t)\Phi^{-1}(\tau)b(\tau)d\tau =$$

$$= \left(P(t)e^{Bt}\right)\left(P^{-1}(0)e^{0}\right)^{-1}x_0 + \int_0^t \left(P(t)e^{Bt}\right)\left(P(\tau)e^{B\tau}\right)^{-1}b(\tau)d\tau =$$

$$= P(t)e^{Bt}P^{-1}(0)x_0 + \int_0^t P(t)e^{B(t-\tau)}P^{-1}(\tau)b(\tau)d\tau.$$

<u>Proof of Floquet thm</u>

Let $\Phi(t)$ be a fundamental solution to $\dot{x} = A(t)x$.
Since the ODE is invariant under the transformation $\tau = t+T \Rightarrow$
$\Rightarrow \Phi(t+T)$ is also a fundamental solution.
It follows that there is a non-singular matrix C such that

$$\Phi(t+T) = \Phi(t)C.$$

C = the <u>monodromy matrix.</u>
Let B be a matrix such that $\exp(BT) = C$. and
define $P(t) = \Phi(t)e^{-Bt}$.
Claim: $P(t)$ is periodic.

$$P(t+T) = \Phi(t+T)e^{-B(t+T)} = \Phi(t)Ce^{-B(t+T)} = \Phi(t)e^{BT}e^{-B(t+T)} =$$
$$= \Phi(t)e^{B(T-t-T)} = \Phi(t)e^{-Bt} = P(t).$$

It follows then that $P(t) = \Phi(t)e^{-Bt} \Rightarrow \Phi(t) = P(t)e^{Bt}$
with $P(t)$ periodic.

<u>Def</u>: The eigenvalues of the matrix B are <u>Floquet exponents.</u>
and they determine stability.

<u>Thm</u>: The system of ODEs $\dot{x} = A(t)x$ is
a) asymptotically stable $\Leftrightarrow$ All Floquet exponents are $\mathrm{Re}\lambda < 0$
b) stable $\Leftrightarrow$ All Floquet exponents have $\mathrm{Re}\lambda \leq 0$ and
$\mathrm{Re}\lambda = 0 \Rightarrow \lambda$ has multiplicity 1.

problem : In general there is no way to compute B and $P(t)$. However, they can be obtained numerically as follows:

1) Use numerical integration to timestep $\phi(t)$ to $\phi(t+T)$
   Then $\phi(t+T) = \phi(t) C \Rightarrow C = \phi^{-1}(t)\phi(t+T)$.
2) With C known, stability can be determined. If stable continue.
3) Compute a B such that $C = e^{BT}$.
4) Now compute $P(t) = \phi(t) e^{-Bt}$ using the sampled values of $\phi(t)$ over one period.
5) Now we can follow the evolution of the system to very large times without depending on numerical integration which is very limited.

Stability and monodromy matrix

Def : The eigenvalues ρ of the monodromy matrix C are called "characteristic multipliers"

Remark : The relation between a characteristic multiplier ρ and the corresponding Floquet exponent λ is $\rho = \exp(\lambda T)$.

Thm : The system of ODEs $\dot{x} = A(t)x$ is
a) asymptotically stable $\Leftrightarrow$ All characteristic multipliers are $|\rho| < 1$.
b) stable $\Leftrightarrow$ All characteristic multipliers are $|\rho| \leq 1$ and
$|\rho| = 1 \Rightarrow \rho$ has multiplicity 1.

In some cases, it is easy to prove instability using the following result.

Thm : Let $\dot{x} = A(t)x$ be a system of ODEs with $A(t)$ T-periodic. The characteristic multipliers $\{\rho_1, \rho_2, \dots, \rho_n\}$ satisfy:

$$\rho_1 \rho_2 \cdots \rho_n = \exp\left[\int_0^T \operatorname{Tr}(A(t))\,dt\right]$$

Remark : Of course if $|\rho_1 \rho_2 \cdots \rho_n| > 1 \Rightarrow$ the equations are unstable. Otherwise we don't have sufficient information.

<u>example</u> : $\dot{x} = A(t)x$. with

$$A(t) = \begin{bmatrix} -2\cos^2 t & -1-2\sin 2t & 0 \\ 1-\sin 2t & -2\sin^2 t & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

Fundamental matrix solution

$$\phi(t) = \begin{bmatrix} e^{-2t}\cos t & -\sin t & 0 \\ e^{-2t}\sin t & \cos t & 0 \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

Decomposes to $\phi(t) = P(t)e^{Bt}$ where

$$P(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

The Floquet exponents are $-2, 0, \lambda$ and the system is stable $\Leftrightarrow \lambda < 0$.

- <u>Results for remarkable special cases.</u>

<u>Prop.</u> (1d case).
Consider the ODE $x'(t) = f(t)x(t)$ with $f(t)$ a T-periodic scalar function. Then

a) $x(t) = P(t)\exp(tB)$ with

$$P(t) = \int_0^t f(\tau)d\tau - \frac{t}{T}\int_0^T f(\tau)d\tau \quad \text{and}$$

$$B = \frac{1}{T}\int_0^T f(t)d\tau.$$

b) The ODE is stable $\Leftrightarrow \int_0^T f(\tau)d\tau \leq 0$.

<u>Proof</u>

The equation can be solved exactly as follows.

$$x'(t) = f(t)x(t) \Leftrightarrow \frac{x'(t)}{x(t)} = f(t) \Leftrightarrow \int_{x_0}^{x} \frac{dx}{x} = \int_0^t f(\tau)d\tau \Leftrightarrow$$

$$\Leftrightarrow \ln x(t) - \ln x_0 = \int_0^t f(\tau)\,d\tau \Leftrightarrow \frac{x(t)}{x_0} = \exp\left[\int_0^t f(\tau)d\tau\right] \Leftrightarrow$$

$$\Leftrightarrow x(t) = x_0 \exp\left[\int_0^t f(\tau)d\tau\right]$$

▸ Rewrite the integral as:

$$\int_0^t f(\tau)d\tau = \int_0^t f(\tau)d\tau - \frac{t}{T}\int_0^T f(\tau)d\tau + \frac{t}{T}\int_0^T f(\tau)d\tau$$

and define

$$P(t) = \int_0^t f(\tau)\,d\tau - \frac{t}{T}\int_0^T f(\tau)d\tau.$$

Claim: $P(t)$ is periodic.

Let $t \in \mathbb{R}$ be given. Define $n \in \mathbb{Z}$ such that $t = nT + t_0$ with $t_0 \in [0,T)$.

$$P(t) = \int_0^t f(\tau)d\tau - \frac{t}{T}\int_0^T f(\tau)d\tau =$$

$$= \cancel{n\int_0^T f(\tau)d\tau} + \int_{nT}^t f(\tau)d\tau - \cancel{n\int_0^T f(\tau)d\tau} - \frac{t-nT}{T}\int_0^T f(\tau)d\tau =$$

$$= \int_0^{t_0} f(\tau)d\tau - \frac{t_0}{T}\int_0^T f(\tau)d\tau = P(t_0).$$

Rewrite the solution:

$$x(t) = x_0 \exp\left[P(t) + \frac{t}{T}\int_0^T f(\tau)d\tau\right] =$$

$$= x_0 \exp\left(P(t)\right) \exp\left[\frac{t}{T}\int_0^T f(\tau)d\tau\right]$$

It follows that $\mathcal{P}(t) = x_0 e^{P(t)}$ and $B = \frac{1}{T}\int_0^T f(\tau)d\tau$.

The ODE is stable $\Leftrightarrow B \leq 0 \Leftrightarrow \int_0^T f(\tau)d\tau \leq 0$.

Prop : (special n-dim case)

Let $M \in \mathbb{C}^{n\times n}$ be a constant matrix and $\dot{x} = f(t)Mx$ a system of n ODEs with $f(t)$ periodic with period T.
Let $\lambda(M) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$.
The ODEs are stable $\Leftrightarrow \lambda_j \int_0^T f(\tau)d\tau \leq 0$, $\forall j \in [n]$.

Proof

Let Y be the diagonalizing matrix to M. such that

$$Y^{-1}MY = \operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$$

Define $y(t) = Y^{-1}x(t) \Leftrightarrow x(t) = Yy(t)$.

$$\dot{y}(t) = Y^{-1}\dot{x}(t) = Y^{-1}(f(t)Mx) = f(t)\,Y^{-1}Mx = f(t)\,Y^{-1}MYy = $$
$$= f(t)\operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}\,y.$$

therefore

$$\dot{y}_1(t) = \lambda_1 f(t) y_1(t)$$
$$\dot{y}_2(t) = \lambda_2 f(t) y_2(t)$$
$$\cdots$$
$$\dot{y}_n(t) = \lambda_n f(t) y_n(t)$$

Using the previous result, these equations are stable $\Leftrightarrow$

$$\lambda_j \int_0^T f(\tau)d\tau \leq 0$$

Boundary value problems.

Sturm-Liouville problems.

<u>Def</u>: A boundary value problem of the form

$$Ly = -\lambda p(t) y$$
$$M_1 y = 0 \quad , \quad M_2 y.$$

where $Ly = (k(t)y')' + g(t)y \quad , \forall t \in [a,b]$
with g, k, p real valued
and k, p positive
and $M_1 y = d_{11} y(a) + d_{12} y'(a) - c_{11} y(b) - c_{12} y'(b) = 0$
$M_2 y = d_{21} y(a) + d_{22} y'(a) - c_{21} y(b) - c_{22} y'(b) = 0$
are called <u>Sturm-Liouville problems.</u>

<u>Remark</u>: To solve these problems one must find all λ for which a solution exists and that solution $y(t)$
$\lambda \rightsquigarrow$ eigenvalue
$y(t) \rightsquigarrow$ corresponding eigenfunction.

<u>notation</u>: Define $D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$ and $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$.
We may then write the boundary conditions as:

$$D \begin{bmatrix} y(a) \\ y'(a) \end{bmatrix} = C \begin{bmatrix} y(b) \\ y'(b) \end{bmatrix}.$$

Self-adjointness.

<u>Def</u>: (inner product).
Let $y(t)$ and $z(t)$ be two ~~fcnts~~ functions in $[a,b]$.
Then the <u>inner product</u> is defined by.

$$(y,z) = \int_a^b y(t) z^*(t) dt.$$

The <u>weighted inner product</u> is defined by

$$(y,z)_p = \int_a^b y(t) z^*(t) p(t) dt.$$

Define $\phi(t) = \begin{bmatrix} z^*(t) & y(t) \\ z^{*\prime}(t) & y'(t) \end{bmatrix}$.

Then $(Ly,z) - (y,Lz) = K(b)\det\phi(b) - k(a)\det\phi(a)$. (1).

Note that $C\begin{bmatrix} y(a) \\ y'(a) \end{bmatrix} = D\begin{bmatrix} y(b) \\ y'(b) \end{bmatrix}$ and $C\begin{bmatrix} z(a) \\ z'(a) \end{bmatrix} = D\begin{bmatrix} z(b) \\ z'(b) \end{bmatrix} \Rightarrow$

$\cancel{\Rightarrow C\phi(b) = C\begin{bmatrix} z^*(b) & y(b) \\ z^{*\prime}(b) & y'(b) \end{bmatrix}}$ $\quad C\phi(b) = D\phi(a)$

therefore

$$(Ly,z) - (y,Lz) = k(b)\det\phi(b) - k(a)\det\phi(a) =$$

$$= K(b)\frac{\det(C\phi(b))}{\det C} - K(a)\det\phi(a) =$$

$$= k(b)\frac{\det(D\phi(a))}{\det C} - k(a)\det\phi(a) =$$

$$= k(b)\frac{\det D\cdot\det\phi(a)}{\det C} - k(a)\det\phi(a) =$$

$$= \left[K(b)\frac{\det D}{\det C} - k(a)\right]\det\phi(a) = \det C\left[k(b)\det D - K(a)\det C\right]\det\phi(a) = 0$$

$\Leftrightarrow$ $\underline{k(b)\det D - k(a)\det C = 0.}$ $\square$.

<u>Corollary.</u>: The Sturm-Liouville problem is self-adjoint for the following types of boundary conditions:

a) $M_1 y = y(a) = 0.$
   $M_2 y = y(b) = 0.$

b) $M_1 y = y'(a) = 0$
   $M_2 y = y'(b) = 0.$

<u>Thm</u>: <u>(Condition for existence of solutions to Sturm-Liouville problem)</u>

Let $\varphi(t) = c_1\varphi_1(t) + c_2\varphi_2(t)$ be the general solution to $Ly = 0$.
where $Ly = (ky')' + gy$ is a Sturm-Liouville operator.
Let $M_0 y = D\begin{bmatrix} y(a) \\ y'(a) \end{bmatrix} - C\begin{bmatrix} y(b) \\ y'(b) \end{bmatrix}$ be the boundary condition

and define

$$\Phi(t) = \begin{bmatrix} \varphi_1(t) & \varphi_2(t) \\ \varphi_1'(t) & \varphi_2'(t) \end{bmatrix}, \quad \Delta(\varphi_1,\varphi_2) = \det(D\Phi(a) - C\Phi(b))$$

<u>Def</u> : An operator L and a set of boundary conditions $My = 0$ are called <u>self-adjoint</u> $\Leftrightarrow$ $(Ly, z) = (y, Lz)$, $\forall y, z$.

<u>Thm</u> : The Sturm-Liouville problem $Ly = -\lambda \rho y$ with boundary conditions

$$D \begin{bmatrix} y(a) \\ y'(a) \end{bmatrix} = C \begin{bmatrix} y(b) \\ y'(b) \end{bmatrix}$$

and $Ly = (ky')' + gy$ is <u>self-adjoint</u> $\Leftrightarrow$ $\boxed{K(b) \det D = K(a) \det C}$

<u>Proof</u>

$$(Ly, z) - (y, Lz) = \int_a^b \left[(Ly) z^* - y (Lz)^* \right] dt =$$

$$= \int_a^b \left[\left((ky')' + gy \right) z^* - y \left((kz')' + gz \right)^* \right] dt =$$

g real

$$= \int_a^b \left[(ky')' z^* - y (kz')'^* + gyz^* - ygz^* \right] dt =$$

$$= \int_a^b \left[(ky')' z^* - y (kz')'^* \right] dt =$$

$$= \left[(ky') z^* - y (kz')^* \right]_a^b - \int_a^b \left[(ky')(z^*)' - y' (kz')^* \right] dt =$$

$$= \left[(ky') z^* - y (kz')^* \right]_a^b =$$

$$= K(b) y'(b) z^*(b) - K(b) y(b) z'^*(b) - k(a) y'(a) z^*(a) + k(a) y(a) z^{*\prime}(a) =$$

$$= K(b) \det \begin{bmatrix} z^*(b) & y(b) \\ z^{*\prime}(b) & y'(b) \end{bmatrix} - K(a) \det \begin{bmatrix} z^*(a) & y(a) \\ z^{*\prime}(a) & y'(a) \end{bmatrix}$$

then,

a) $\begin{cases} Ly=0 \\ My=0 \end{cases}$ has a solution $\Leftrightarrow \Delta(\varphi_1,\varphi_2)=0$

b) If $\Delta(\varphi_1,\varphi_2)\neq 0 \to \forall f, p$: $\begin{cases} Ly=f \\ My=p \end{cases}$ has a unique solution.

<u>Proof</u>

a) $\begin{cases} Ly=0 \\ My=0 \end{cases}$ has a solution $\Leftrightarrow \exists c_1\neq 0, c_2\neq 0$: $\varphi(t)=c_1\varphi_1(t)+c_2\varphi_2(t)$ satisfies the boundary condition $M\varphi=0$.

$$M\varphi = D\begin{bmatrix}\varphi(a)\\ \varphi'(a)\end{bmatrix} - C\begin{bmatrix}\varphi(b)\\ \varphi'(b)\end{bmatrix} =$$

$$= D\begin{bmatrix} c_1\varphi_1(a)+c_2\varphi_2(a) \\ c_1\varphi_1'(a)+c_2\varphi_2'(a)\end{bmatrix} - C\begin{bmatrix} c_1\varphi_1(b)+c_2\varphi_2(a) \\ c_1\varphi_1'(b)+c_2\varphi_2'(b)\end{bmatrix} =$$

$$= D\begin{bmatrix}\varphi_1(a) & \varphi_2(a)\\ \varphi_1'(a) & \varphi_2'(a)\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} - C\begin{bmatrix}\varphi_1(b) & \varphi_2(b)\\ \varphi_1'(b) & \varphi_2'(b)\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} =$$

$$= (D\Phi(a)-C\Phi(b))\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$ has a nontrivial solution $\Leftrightarrow$

$\Leftrightarrow \det(D\Phi(a)-C\Phi(b))=0 \Leftrightarrow \Delta(\varphi_1,\varphi_2)=0$. $\square$.

b) Define $$\varphi_p(t)=\int_a^t \frac{[\varphi_1(t)\varphi_2(s)-\varphi_2(t)\varphi_1(s)]f(s)}{W(\varphi_1,\varphi_2)}\,ds$$

$\varphi_p(t)$ is a particular solution and satisfies $L\varphi_p=f$.
Want c_1, c_2 such that $\varphi(t)=c_1\varphi_1(t)+c_2\varphi_2(t)+\varphi_p(t)$ also satisfies the boundary conditions $M\varphi=p$.

$$M\varphi = D\begin{bmatrix}\varphi(a)\\ \varphi'(a)\end{bmatrix} - C\begin{bmatrix}\varphi(b)\\ \varphi'(b)\end{bmatrix} =$$

$$= D\begin{bmatrix} c_1\varphi_1(a)+c_2\varphi_2(a)+\varphi_p(a) \\ c_1\varphi_2'(a)+c_2\varphi_2'(a)+\varphi_p'(a)\end{bmatrix} - C\begin{bmatrix} c_1\varphi_1(b)+c_2\varphi_2(b)+\varphi_p(b) \\ c_1\varphi_1'(b)+c_2\varphi_2'(b)+\varphi_p'(b)\end{bmatrix} =$$

$$= (D\Phi(a)-C\Phi(b))\begin{bmatrix}c_1\\ c_2\end{bmatrix} + D\begin{bmatrix}\varphi_p(a)\\ \varphi_p'(a)\end{bmatrix} - C\begin{bmatrix}\varphi_p(b)\\ \varphi_p'(b)\end{bmatrix} = \begin{bmatrix}p_1\\ p_2\end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow (D\Phi(a)-C\Phi(b))\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}p_1\\ p_2\end{bmatrix} - D\begin{bmatrix}\varphi_p(a)\\ \varphi_p'(a)\end{bmatrix} + C\begin{bmatrix}\varphi_p(b)\\ \varphi_p'(b)\end{bmatrix}$$

has a nontrivial solution $\Leftrightarrow \det(D\Phi(a)-C\Phi(b))\neq 0 \Leftrightarrow \Delta(\varphi_1,\varphi_2)\neq 0$ $\square$.

Properties of self-adjoint problems.

First, note that:

Prop: Let y, z be functions.

a) $(y+z, w) = (y, w) + (z, w)$.

b) $a(y, z) = (ay, z) = (y, a^* z)$.

c) $(y, z) = (z, y)^*$

d) $(y, y) > 0 \quad , \forall y \neq 0$.

Def: Let $\begin{cases} Ly = \lambda \rho y \\ My = 0 \end{cases}$ be a self-adjoint problem.

a) A pair (λ, y) that is a solution is called
λ = eigenvalue
y = eigenfunction corresponding to λ.

Thm: If $Ly = \lambda \rho y$, $My = 0$ is a self-adjoint problem ~~is and only if~~ then all eigenvalues are real.

Proof

Let λ be an eigenvalue with eigenfunction φ.

$$\lambda(\varphi, \varphi)_\rho = (\lambda \rho \varphi, \varphi) = (-L\varphi, \varphi) = -(L\varphi, \varphi) = -(\varphi, L\varphi) =$$

$$= (\varphi, -L\varphi) = (\varphi, \lambda\rho\varphi) = \lambda^*(\varphi, \rho\varphi) = \lambda^*(\varphi, \varphi)_\rho \Rightarrow \lambda = \lambda^* \Rightarrow \operatorname{Im}\lambda = 0 \quad \square.$$

Thm: If φ_m is eigenfunction to eigenvalue λ_m and φ_n is eigenfunction to eigenvalue λ_n then $\lambda_m \neq \lambda_n \Longrightarrow (\varphi_m, \varphi_n)_\rho = 0$.

Proof

$$\lambda_m(\varphi_m, \varphi_n)_\rho = (\lambda_m \rho \varphi_m, \varphi_n) = (-L\varphi_m, \varphi_n) = (\varphi_m, -L\varphi_n) =$$

$$= (\varphi_m, \lambda_n \rho \varphi_n) = \lambda_n^*(\varphi_m, \varphi_n)_\rho = \lambda_n(\varphi_m, \varphi_n)_\rho \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l}(\lambda_m - \lambda_n)(\varphi_m, \varphi_n)_\rho = 0 \\ \lambda_m \neq \lambda_n\end{array}\right\} \Rightarrow (\varphi_m, \varphi_n)_\rho \neq 0.$$

• Problems that are not self-adjoint

<u>Def</u> : Let $Ly = \lambda \rho y$, $My = 0$ be a boundary value problem. An operator L^+ is called the <u>adjoint</u> of L iff
$$(Ly, z) = (y, L^+ z) + c, \quad \forall y, z$$
where c is boundary terms.

<u>Remark</u> : If $L = L^+ \Rightarrow L$ is self adjoint.
Otherwise it is a non self-adjoint problem.
With non-self-adjoint problems eigenvalues may be complex but and eigenfunctions are not orthogonal. The following results apply,

<u>Thm</u> : (<u>biorthogonality</u>).

Let (λ, y) be an eigenvalue, eigenfunction of $Ly = \lambda \rho y$ and (μ, z) be an eigenvalue, eigenfunction $L^+ z = \mu \rho z$.
Then $\lambda \neq \mu^* \Rightarrow (y, z)_\rho = 0$.

<u>Proof</u>

$$\lambda (y,z)_\rho = (\lambda \rho y, z) = -(Ly, z) = -(y, L^+ z) = (y, \mu \rho z) = \mu^* (y,z)_\rho$$

$$\Rightarrow (\lambda - \mu^*)(y,z)_\rho = 0 \Rightarrow (y,z)_\rho = 0 \quad \square.$$

Asymptotic methods for ODEs.

1) Local analysis: Provide a solution which is accurate only in a local region.
2) Global methods: Provide an approximate solution valid on the entire domain.
   a) Boundary layer theory.
   b) Multiple scale analysis.

Asymptotic methods are centered around the following concepts:

Def: Let f, g be functions.

a) $f(x) \ll g(x)$ as $x \to x_0 \Leftrightarrow \lim_{x \to x_0} (f(x)/g(x)) = 0$

b) $f(x) \sim g(x)$ as $x \to x_0 \Leftrightarrow \lim_{x \to x_0} (f(x)/g(x)) = 1$.

examples.

a) $e^x + x \sim e^x$ as $x \to +\infty$.
b) $x^3 \ll x^2$ as $x \to 0$.
c) $x^3 \gg x^2$ as $x \to +\infty$.

Def: A power series defined a sequence a_n is asymptotic to a function $y(x)$ iff

$$\forall N > 0 : \left| y(x) - \sum_{n=0}^{N} a_n (x - x_0)^n \right| \ll (x - x_0)^N \text{ as } x \to x_0.$$

notation: $y(x) \sim \sum_{n=0} a_n (x - x_0)^n$.

This definition generalizes to:

Def: $$f(x) \sim \sum_n \phi_n(x) \Leftrightarrow \begin{cases} \phi_{n+1}(x) \ll \phi_n(x), \ \forall n > 0 \\ \left| f(x) - \sum_{n=1}^{N} \phi_n(x) \right| \ll \phi_N(x). \end{cases}$$

Def: An asymptotic series $\sum_n \phi_n(x)$ is convergent iff

$$\lim_{N \to +\infty} \sum_{n=N}^{+\infty} \phi_n(x) = 0.$$

<u>Prop.</u> : Let $f(x)$ be a function such that
$$f(x) \sim \sum_n a_n (x-x_0)^{\gamma n}$$
Then $f(x) \sim \sum_n b_n (x-x_0)^{\delta n} \Rightarrow a_n = b_n$ and $\gamma_n = \delta_n$. (uniqueness).
The unique coefficients are given by:
$$a_0 = \lim_{x \to x_0} f(x)$$
$$a_1 = \lim_{x \to x_0} \frac{f(x) - a_0}{(x-x_0)^{\gamma}}$$
$$a_n = \lim_{x \to x_0} \frac{f(x) - \sum_{i=1}^{n-1} a_i (x-x_0)^{i\gamma}}{(x-x_0)^{n\gamma}}.$$

<u>Remark</u> : Although each $f(x)$ has a unique asymptotic expansion each expansion is asymptotic to many functions.

● <u>Properties of asymptotic expansions.</u>

<u>Thm</u> : Let $f(x) \sim \sum_n a_n (x-x_0)^n$ and $g(x) = \sum_n b_n (x-x_0)^n$. as $x \to x_0$
Then:

a) $\lambda f(x) + \mu g(x) \sim \sum_n (\lambda a_n + \mu b_n)(x-x_0)^n$.

b) $f(x) g(x) \sim \sum_n c_n (x-x_0)^n$ with $c_n = \sum_{n=0}^{n} a_m b_{n-m}$

c) $\dfrac{f(x)}{g(x)} \sim \sum_n d_n (x-x_0)^n$ with $d_n = \dfrac{a_n - \sum_{m=0}^{n-1} d_m b_{n-m}}{b_0}$

as $x \to x_0$ $\qquad d_0 = a_0/b_0$.

<u>Thm</u> : (<u>Integration Abelian theorem</u>)

If $f(x) \sim \sum_n a_n (x-x_0)^n \Rightarrow \int_{x_0}^{x} f(t)\,dt \sim \sum_n \frac{a_n}{n+1} (x-x_0)^{n+1}$ as $x \to x_0$.

Remark. : Differentiation of asymptotic expansions doesn't always work. There is a complicated set of theorems called "Tauberian theorems" that can be applied to justify differentiation. For ~~to~~ local analysis of ODEs, the following result is the most useful:

Thm : If $y(x)$ is solution to

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_1(x) y' + p_0(x) y = 0$$

and $p_K(x)$ ~~have~~ are asymptotic to power series:

$$p_K(x) \sim \sum_n b_{nK} (x - x_0)^K \quad \text{as} \quad x \to x_0$$

then

$$y(x) \sim \sum_n \phi_n(x) \Rightarrow y'(x) = \sum_n \phi_n'(x).$$

Regular singular points.

Thm (Fuchs). The ODE, with $p_k(x)$ analytic at $x=x_0$

$$y^{(n)}+\frac{p_{n-1}(x)}{x-x_0}y^{(n-1)}+\frac{p_{n-2}(x)}{(x-x_0)^2}y^{(n-2)}+\cdots+\frac{p_0(x)}{(x-x_0)^n}y=0.$$

has n linearly indepedent solutions of the form:

$$y_k(x)=(x-x_0)^{\gamma_k}\sum_{j=1}^{k-1}[\ln|x-x_0|]^j F_j(x). \ , \ \forall k\in\{1,2,\dots,n\}.$$

where $\gamma_1,\gamma_2,\dots,\gamma_n$ are constants and $F_i(x)$ analytic at $x=x_0$. The γ_k are the roots of the indicial equation.

$$\varphi(\gamma)=\prod_{k=0}^{n-1}(\gamma-k)+\sum_{m=0}^{n-1}p_{n-1}(x_0)\left\{\prod_{k=0}^{m-1}(\gamma-k)\right\}=0$$

In the $n=2$ case the specialized theory of Frobenius applies:

Thm: The ODE $y''+\frac{p(x)}{x-x_0}y'+\frac{q(x)}{(x-x_0)^2}y=0$ has two solutions y_1,y_2 that are characterized according to the roots a_1,a_2 of the indicial equation: $\varphi(t)=t(t-1)+p(x_0)t+q(x_0)$.

Case 1: If $a_1-a_2\notin\mathbb{Z}$ then

$$y_1(x)=|x-x_0|^{a_1}F_1(x)$$
$$y_2(x)=|x-x_0|^{a_2}F_2(x)$$

where F_1,F_2 are analytic in a neighborhood of $x=x_0$.

Case 2: If $a_1-a_2\in\mathbb{Z}$ then

$$y_1(x)=|x-x_0|^{a_1}F_1(x)$$
$$y_2(x)=|x-x_0|^{a_2}F_2(x)+Cy_1(x)\log|x-x_0|.$$

with F_1,F_2 analytic in $x=x_0$.

• Irregular singular points.

There is no theory. There is a method called method of dominant balance

Methodology.: Suppose that $y''+p(x)y'+q(x)y=0$ has an irregular singular point x_0.

- •1 Define $S(x)$: $y(x)=\exp(S(x))$

  Then $y'(x)=S'(x)\exp(S(x))=S'(x)y(x)$

  $$y''(x)=[S''(x)+(S'(x))^2]\exp(S(x))=$$
  $$=[S''(x)+(S'(x))^2]y(x).$$

  and the ODE becomes:

  $$y''+p(x)y'+q(x)y=0 \Leftrightarrow [S''(x)+(S'(x))^2+p(x)S'(x)+q(x)S(x)]y(x)=0$$

  $$\Leftrightarrow S''(x)+(S'(x))^2+p(x)S'(x)+q(x)S(x)=0. \qquad (*)$$

- •2 Guess: Since x_0 is irregular, perhaps $S''(x)\ll(S'(x))^2$ as $x\to x_0$. Assume so.
- •3 Then $(*)\Rightarrow (S')^2\sim -p(x)S'-q(x)$ as $x\to x_0$.
- •4 Solve for $S(x)$ and verify that $S''\ll(S')^2$. $\rightsquigarrow S(x)\sim S_1(x)$ as $x\to x_0$
- •5 Assume that $S(x)\sim S_1(x)+S_2(x)$ with $S_2(x)\ll S_1(x)$ as $x\to x_0$. Substitute to $(*)$ and obtain an equation for $S_2(x)$.
- •6 Simplify by dropping terms that are $\ll$ by other terms.
- •7 Apply the same method to find $S_2(x)$.
- •8 Repeat until $S(x)\sim a\ln x$ (weakest singularity possible)

Remark.: Obtaining $S_2(x),\dots$ is a black art. There is no theory to state any guarantees.

Remark: Suppose that $y(x)=\exp[a(x-x_0)^{-b}]$ with $b>0$.
y has an essential singularity at $x=x_0$.

$$S(x)=a(x-x_0)^{-b} \text{ and } \left.\begin{aligned} S''(x)&=ab(b+1)(x-x_0)^{-(b+2)}\\ [S'(x)]^2&=a^2b^2(x-x_0)^{-2b-2}\end{aligned}\right\}\Rightarrow S''\ll(S')^2.$$

This is the justification for trying $S''\ll(S')^2$.

<u>example</u> : $y'' + \frac{2}{x} y' - \frac{1}{x^4} y = 0$.

Let $y(x) = \exp(S(x))$: $S'' + (S')^2 + \frac{2}{x} S' - \frac{1}{x^4} = 0$.

- Try to balance two terms ~~and~~ assuming the others are subdominant, then check for consistency.

Try 3rd and 4th term:

$$\frac{2}{x} S' \sim \frac{1}{x^4} \Leftrightarrow S' \sim \frac{1}{2x^3}$$

But then $(S')^2 \sim \frac{1}{4x^6} \gg S'$ as $x \to 0$, therefore this is an <u>inconsistent</u> balance

<u>Remark:</u> As a first guess always try $S'' \ll (S')^2$ and $S' \ll (S')^2$.

$$(S')^2 \sim \frac{1}{x^4} \Leftrightarrow S' \sim \frac{\pm 1}{x^2} \Leftrightarrow S \sim \pm \frac{1}{x}$$

Check for consistency: $S'' \sim \pm \frac{2}{x^3} \ll \frac{1}{x^4} \sim (S')^2$ as $x \to 0$

and $S' \sim \pm \frac{1}{x^2} \ll \frac{1}{x^4} \sim (S')^2$ as $x \to 0$

Therefore $y(x) \sim e^{\pm 1/x}$ as $x \to 0$. $\square$.

<u>Remark.</u>: For regular singular points we've seen solutions of the form:

$$y(x) = |x - x_0|^{\gamma} F(x)$$

where $F(x)$ is analytic and can be determined by a power series:

$$F(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n$$

With irregular singular points, this form must be generalized to

$$y(x) \simeq l(x) F(x)$$

where $l(x)$ has an essential singularity at $x = x_0$ and $F(x)$ analytic at x_0. We show how $l(x)$ can be found with the method of dominant balance. Once $l(x)$ is known, substitute:

$$F(x) \simeq \sum_{n=0}^{+\infty} a_n x^{an}$$

and solve for a, a_n.

<u>example</u>: Airy's equation $y''=xy$. as $x\to\infty$.

Let $y(x)=e^{S(x)} \Rightarrow S''+(S')^2 = x$.
Dominant balance: $S'' \ll (S')^2$
$(S')^2 \sim x \Leftrightarrow S' \sim \pm x^{1/2} \Leftrightarrow S \sim \pm \frac{2}{3} x^{3/2}$.
Note that $S\sim -\frac{2}{3} x^{3/2}$ as $x\to+\infty$ decays to 0, therefore not singular.
Take then $S(x) \sim \frac{2}{3} x^{3/2}$.

Consistency: $S'' \sim (\pm x^{1/2})' \sim \pm\frac{1}{2} x^{-1/2} \ll (S')^2 \sim x$ as $x\to+\infty$.
To obtain the full $l(x)$ write:
$S(x) = -\frac{2}{3} x^{3/2} + C(x)$. with $C(x) \ll x^{3/2}$.

$$S''+(S')^2 = x \Leftrightarrow -\frac{1}{2} x^{-1/2} + C'' + \left(- x^{1/2} + C'\right)^2 = x \Leftrightarrow$$

$$\Leftrightarrow -\frac{1}{2} x^{-1/2} + C'' + x - 2C'x^{1/2} + (C')^2 = x$$

$$\Leftrightarrow -\frac{1}{2} x^{-1/2} + C'' - 2C'x^{1/2} + (C')^2 = 0.$$

~~Since $\lim_{x\to+\infty}\left(-\frac{1}{2} x^{-1/2}\right) = 0$ and $C(x) \ll x^{3/2} \Rightarrow C'(x) \ll x^{1/2} \Rightarrow$~~

~~$\Rightarrow 2x^{1/2}C' \ll x \sim (C')^2 \Rightarrow \underline{2x^{1/2}C' \ll (C')^2}$~~
~~it follows that~~
~~$2C'x^{1/2} \sim$~~

Since
$C(x) \ll x^{3/2} \Rightarrow C'(x) \ll x^{1/2} \Rightarrow 2x^{1/2}C'(x) \ll x \sim (C')^2 x^{\varepsilon} \ll (C')^2$
(where ε very small perhaps) $\Rightarrow \underline{2x^{1/2}C'(x) \ll (C')^2}$.
Also $\underline{C'' \ll x^{-1/2}}$
It follows that $2C'x^{1/2} \sim -\frac{1}{2} x^{-1/2} \Leftrightarrow C' \sim \frac{-1}{4x} \Leftrightarrow C(x) = -\frac{1}{4} \ln x$.

So $S(x) \sim \frac{2}{3} x^{2/3} - \frac{1}{4} \ln x \Rightarrow y(x) \sim x^{-1/4} \exp\left(-\frac{2}{3} x^{3/2}\right)$. as $x\to\infty$

Let $y(x) = D x^{-1/4} \exp\left(-\frac{2}{3} x^{3/2}\right) w(x)$ such that $\lim_{x\to+\infty} w(x) = 1$

Then:

$$y'' = xy \Leftrightarrow x^{-1/4} w'' - \left(\frac{1}{2} x^{-5/4} + 2x^{1/4}\right) w' + \frac{5}{16} x^{-9/4} w = 0 \Leftrightarrow$$

$$\Leftrightarrow w'' - \left(2x^{1/2} + \frac{1}{2x}\right) w' + \frac{5}{16} \frac{1}{x^2} w = 0.$$

Assume $w(x) = 1 + \varepsilon(x)$ with $\varepsilon(x) \ll 1$ as $x \to +\infty$.

$$\cancel{\varepsilon''} - \left(2x^{1/2} + \cancel{\frac{1}{2x}}\right) \varepsilon' + \frac{5}{16x^2} (1 + \cancel{\varepsilon}) = 0 \quad \text{as } x \to +\infty.$$

Apply dominant balance: $2x^{1/2} \varepsilon' \sim \frac{5}{16x^2} \Leftrightarrow$

$$\Leftrightarrow \varepsilon' \sim \frac{5}{32} x^{-5/2} \Leftrightarrow \varepsilon \sim -\frac{5}{48} x^{-3/2}. \quad \text{So:}$$

$$y(x) \sim D x^{-1/4} \exp\left(-\frac{2}{3} x^{3/2}\right)\left(1 - \frac{5}{48} x^{-3/2} + \cdots\right) \quad D.$$

A remarkable result that may be useful is that:

<u>Thm</u>: Consider the ODE $y^{(n)} = q(x) y$.

a) If $\lim_{x \to x_0} |(x-x_0)^n q(x)| = \infty \Rightarrow$

$$y(x) \sim c\,[q(x)]^{(1-n)/2n} \exp\left\{ \omega \int^x [q(t)]^{1/n} dt \right\} \quad \text{as } x \to x_0$$

b) If $\lim_{x \to \infty} |x^n q(x)| = \infty \Rightarrow$

$$y(x) \sim c\,[q(x)]^{(1-n)/2n} \exp\left\{ \omega \int^x [q(t)]^{1/n} dt \right\} \quad \text{as } x \to \infty.$$

where ω an n^{th} root of unity.

Try to balance: $x^2 y \sim \sin x \Leftrightarrow y(x) \sim \frac{\sin x}{x^2}$

$y'(x) \sim \frac{x^2\cos x - 2x\sin x}{x^4} \sim \frac{\cos x}{x^2} - \frac{2\sin x}{x^3} \sim \frac{\cos x}{x^2} \Rightarrow$

$\Rightarrow y''(x) \sim \frac{-x^2\sin x - 2x\cos x}{x^4} \sim -\frac{\sin x}{x^2} - \frac{2\cos x}{x^3} \sim -\frac{\sin x}{x^2} \ll \sin x.$

Let $y(x) = \frac{\sin x}{x^2} + y_1(x)$ with $y_1(x) \ll \frac{\sin x}{x^2}$.

Then: $y'(x) = \frac{\cos x}{x^2} - \frac{2\sin x}{x^3} + y_1'(x).$

$y''(x) = -\frac{\sin x}{x^2} - \frac{4\cos x}{x^3} + \frac{6\sin x}{x^4} + y_1''(x).$

So $y'' + x^2 y = \sin x \Leftrightarrow -\frac{\sin x}{x^2} - \frac{4\cos x}{x^3} + \frac{6\sin x}{x^4} + y_1''(x) + \sin x + x^2 y_1 = \sin x \Leftrightarrow$

$-\frac{\sin x}{x^2} - \frac{4\cos x}{x^3} + \frac{6\sin x}{x^4} + y_1'' + x^2 y = 0$

Since: $-\frac{\sin x}{x^2} \gg \frac{4\cos x}{x^3} \gg \frac{6\sin x}{x^4}$ we have $y_1'' + x^2 y_1 \sim \frac{\sin x}{x^2}$.

Assume $y_1'' \ll x^2 y_1$
Then $x^2 y_1 \sim \frac{\sin x}{x^2} \Leftrightarrow y_1 \sim \frac{\sin x}{x^4}$

Consistency:
$y_1 \sim \frac{\sin x}{x^4} \Rightarrow y_1' \sim \frac{-4x^3\sin x + x^4\cos x}{x^8} \sim \frac{-4\sin x}{x^5} + \frac{\cos x}{x^4} \sim \frac{\cos x}{x^4} \Rightarrow$

$\Rightarrow y_1'' \sim \dots \sim \frac{-\sin x}{x^4} \ll \frac{\sin x}{x^2} \sim x^2 y_1 \Rightarrow y_1'' \ll x^2 y_1.$

Therefore $y_p(x) \sim \frac{\sin x}{x^2} + \frac{\sin x}{x^4}$.

Boundary Layer Theory.

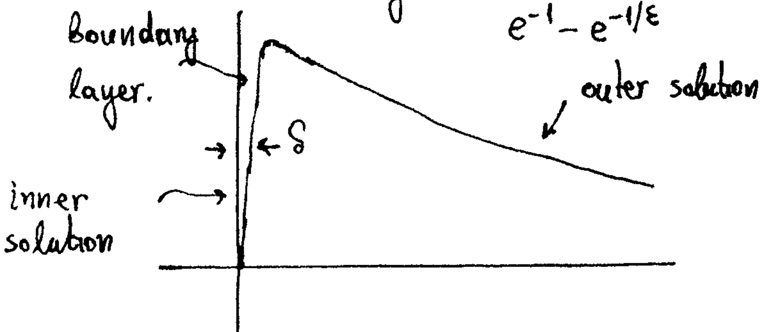
A <u>boundary layer</u> is a narrow region where the solution of an ODE changes rapidly.

<u>example:</u> (exactly soluble boundary-layer problem).

$$\varepsilon y'' + (1+\varepsilon) y' + y = 0$$

The exact solution is:

$$y(x) = \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}$$



In the limit $\varepsilon \to 0$, $y(x)$ becomes discontinuous.

<u>Boundary layer theory</u> is a collection of methods for solving ODEs whose solutions have boundary layers.

Typical problems with boundary layers are problems of the form:

$$\varepsilon y'' + a(x) y' + b(x) y = 0 \quad , \; y(x_0) = A, \;\; y(x_1) = B.$$

with $\varepsilon \to 0$. Also various non-linear problems.

<u>Methodology.</u>

We want to find an <u>outer solution</u> $y_{out}(x)$ valid away from the boundary layer and an <u>inner solution</u> $y_{in}(x)$ valid in the boundary layer region. Then we need to patch them together and obtain a global approximation of the form:

$$y(x) = y_{out}(x) + y_{in}(x) - y_{match}(x).$$

- 1 Suppose that $y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + O(\varepsilon^3)$
  Substitute into ODE and solve for $y_0(x)$. $\rightsquigarrow$ this is the outer solution.
- 2 Try to make $y_0(x)$ satisfy the boundary conditions.
  The boundary conditions that fail to be satisfied are where the boundary layers are located.
- 3 Suppose there is a boundary layer at $x = x_0$. Let δ be the thickness of the boundary layer.

Let $x = x_0 \pm \delta X$ and $y(x) = y(x_0 \pm \delta X) = Y(X)$.

- 4 Write down an ODE for $Y(X)$.
- 5 Use dominant balance to obtain an asymptotic relation between δ and ε. Balance a term with ε and a term without ε.
- 6 Substitute the relation for δ and prove consistency.
- 7 Write $Y(X) = Y_0(X) + \varepsilon Y_1(X) + \varepsilon^2 Y_2(X) + O(\varepsilon^3)$, substitute and solve for $Y_0(X)$.
- 8 We now have $y_{out}(x) = y_0(x)$ and $y_{in}(x) = Y_0(X)$.
  Match: $\lim_{x \to x_0} y_{out}(x) = \lim_{X \to +\infty} y_{in}(X) = y_{match}$

A global approximation to the solution is then:
$y(x) \approx y_{out}(x) + y_{in}(x) - y_{match}$. $\rightsquigarrow$ leading order approximation.

<u>example.</u>: Solve $\varepsilon y'' + x^{1/3} y' + y^3 = 0$, $y(0) = 0$, $y(1) = 1/2$.

Outer solution: let $y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + O(\varepsilon^3)$.
Then: $x^{1/3} y_0' + y_0^3 = 0$ and $y_0(0) = 0$, $y_0(1) = 1/2$.
$\Updownarrow$

$$\frac{y_0'}{y_0^3} = -x^{-1/3} \Leftrightarrow \int \frac{dy_0}{y_0^3} = -\int^x t^{-1/3}\,dt \Leftrightarrow -\frac{1}{2y_0^2} = -\frac{x^{2/3}}{2/3} + c \Leftrightarrow$$

$$\Leftrightarrow 2y_0^2 = \frac{1}{\frac{3x^{2/3}}{2} - c} \Leftrightarrow y_0^2 = \frac{1}{3x^{2/3} - 2c} \Leftrightarrow y_0 = \frac{\pm 1}{\sqrt{3x^{2/3} - c}}$$

Want $y_0(0) = 0$ and $y_0(1) = 1/2$.
The boundary condition $y_0(1) = 1/2$ forces the choice of

$$y_0(x) = \frac{1}{\sqrt{3x^{2/3} - c}}$$

$$y_0(1) = 1/2 \Leftrightarrow \frac{1}{\sqrt{3 - c}} = \frac{1}{2} \Leftrightarrow 3 - c = 4 \Leftrightarrow c = -1.$$

therefore $$y_0(x) = \frac{1}{\sqrt{3x^{2/3} + 1}}$$

At $x=0$: $y_0 = 1 \neq 0$, so the boundary condition at $x=0$ can not be satisfied $\Rightarrow$ we have a boundary layer at $x=0$.
Let δ = boundary layer thickness.
Define $x = \delta X \Rightarrow y(x) = Y(X) = Y(x/\delta)$.
Then

$$\varepsilon y'' + x^{1/3} y' + y^3 = 0 \Leftrightarrow \frac{\varepsilon}{\delta^2} Y'' + \frac{x^{1/3}}{\delta} + Y^3 = 0.$$

Dominant balance: Suppose that 1st + 2nd terms dominate. Then

$$\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta} \Leftrightarrow \varepsilon \sim \frac{\delta^2}{\delta} \Leftrightarrow \varepsilon \sim \delta.$$

The ODE becomes: $\frac{1}{\varepsilon} Y'' + \frac{x^{1/3}}{\varepsilon} Y' + Y^3 = 0 \Leftrightarrow Y'' + x^{1/3} Y' + \varepsilon Y^3 = 0$.
The balance is consistent because the 1st + 2nd terms indeed dominate the 3rd term.
Let $Y(X) = Y_0(X) + \varepsilon Y_1(X) + \varepsilon^2 Y_2(X) + O(\varepsilon^3)$.
Then $Y_0'' + x^{1/3} Y_0' = 0 \Leftrightarrow \frac{Y_0''}{Y_0} = -x^{1/3} \Leftrightarrow \int \frac{dY_0'}{Y_0'} = -\int^x t^{1/3} dt \Leftrightarrow$

$$\Leftrightarrow \ln Y_0' = -\frac{3x^{4/3}}{4} + c_1 \Leftrightarrow Y_0' = c_1 \exp\left(-\frac{3x^{4/3}}{4}\right) \Leftrightarrow$$

$$\Leftrightarrow Y_0(X) = c_1 \int_0^X \exp\left(-\frac{3t^{4/3}}{4}\right) dt + c_2$$

At $x=0$: $Y_0(0) = 0 \Leftrightarrow c_2 = 0$. so $Y_0(X) = c_1 \int_0^X \exp\left(-\frac{3t^{4/3}}{4}\right) dt$.

<u>Remark.</u> : c_1 is determined by <u>asymptotic matching.</u>:
As $\varepsilon \to 0$, $X = x/\delta = x/\varepsilon \to +\infty$. Therefore:

$$\lim_{X \to +\infty} Y_0(X) = \lim_{x \to 0} y_{outer} = \lim_{x \to 0} \frac{1}{\sqrt{3x^{3/2}+1}} = 1.$$

$$\lim_{X \to +\infty} Y_0(X) = c_1 \int_0^{+\infty} \exp\left(-\frac{3t^{4/3}}{4}\right) dt = \frac{3^{1/4}\Gamma(3/4)}{\sqrt{2}} c_1$$

Therefore $\frac{3^{1/4}\Gamma(3/4)}{\sqrt{2}} c_1 = 1 \Leftrightarrow c_1 = \frac{\sqrt{2}}{3^{1/4}\Gamma(3/4)}$

and the inner solution is:

$$y_{inner}(x) = Y_0(X/\varepsilon) = \frac{\sqrt{2}}{3^{1/4}\,\Gamma(3/4)} \int_0^{X/\varepsilon} \exp\left(-\frac{3t^{4/3}}{4}\right) dt.$$

and the uniform solution is:

$$y_{unif}(x) = y_{out}(x) + y_{inner}(x) - y_{match} =$$

$$= \frac{1}{\sqrt{3x^{2/3}+1}} + \frac{\sqrt{2}}{3^{1/4}\,\Gamma(3/4)} \int_0^{X/\varepsilon} \exp\left(-\frac{3t^{4/3}}{4}\right) dt. - 1. \quad \square.$$

● Remarks for linear case

Suppose that $\varepsilon y'' + a(x)y' + b(x)y = 0$; $y(0) = A$, $y(1) = B$, $0 < \varepsilon \ll 1$.
The presense of a boundary layer ultimately depends on the boundary conditions and the ability of $y_{out}(x)$ to match both of them.

If there is a boundary layer then
  for $a(x) > 0$, $\forall x \in [0,1] \Rightarrow$ Boundary layer on $x = 0$.
  for $a(x) < 0$, $\forall x \in [0,1] \Rightarrow$ Boundary layer on $x = 1$

Also: If $a(x)$ vanishes then expect a boundary layer at $x = x_0$ where $a(x_0) = 0$. (internal layer or corner layer).

If there is no boundary layer, then a global approximation is obtained by the perturbation expansion:

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + O(\varepsilon^3).$$

● Boundary layers of thickness other than ε.

In general we will find that $\delta \sim \varepsilon^n$ where $n \in (0, +\infty)$.
Then, the inner solution $Y(x)$ must be expanded in powers of δ:

$$Y(x) = Y_0(x) + \delta Y_1(x) + \delta^2 Y_2(x) + O(\delta^3)$$

Otherwise we proceed normally as before.

<u>example</u> : $\varepsilon y'' + x^2 y' - y = 0$ on $x \in [0,1]$.
$y(0) = 1$, $y(1) = 1$.

Outer solution: Let $y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + O(\varepsilon^3)$.

$$x^2 y_0' - y_0 = 0 \Leftrightarrow \frac{y_0'}{y_0} = \frac{1}{x^2} \Leftrightarrow \int \frac{dy_0}{y_0} = \int \frac{dx}{x^2} = -\frac{1}{x} + c \Leftrightarrow$$

$$\Leftrightarrow \ln y_0 = -\frac{1}{x} + c \Leftrightarrow y_0 = a e^{-1/x}.$$

Apply boundary condition at $x=1$: $y_0(1) = 1 \Leftrightarrow a e^{-1} = 1 \Leftrightarrow a = e$

therefore $y_0 = e^{1-1/x}$.

At $x=0$, $\lim_{x \to 0^+} y_0(x) = \lim_{x \to 0^+} e^{1-1/x} = 0 \neq 1$, therefore there is a boundary layer at $x=0$.

Let $x = \delta X$, $\delta \ll 1$ ($X = O(1)$ in boundary layer) Let $y(x) = Y(X) = Y(x/\delta)$

Then

$$\varepsilon y'' + x^2 y' - y = 0 \Leftrightarrow \frac{\varepsilon}{\delta^2} Y'' - (\delta X)^2 \frac{1}{\delta} Y' - Y = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{\varepsilon}{\delta^2} Y'' - \delta X^2 Y' - Y = 0.$$

Dominant balance: try 1st + 2nd terms.

$$\frac{\varepsilon}{\delta^2} Y'' \sim \delta X^2 Y' \Leftrightarrow \frac{\varepsilon}{\delta^2} \sim \delta \Leftrightarrow \varepsilon \sim \delta^3.$$

substitute: $\delta Y'' - \delta X^2 Y - Y = 0 \rightsquigarrow$ not consistent balance because
$\delta Y'' \ll -Y$ and $-\delta X^2 Y \ll -Y$.

Try 1st + 3rd terms:

$$\frac{\varepsilon}{\delta^2} Y'' \sim -Y \Leftrightarrow \frac{\varepsilon}{\delta^2} \sim 1 \Leftrightarrow \delta \sim \sqrt{\varepsilon}.$$

substitute: $Y'' - \sqrt{\varepsilon}\, X^2 Y' - Y = 0 \rightsquigarrow$ consistent since $Y'' \gg \sqrt{\varepsilon}\, X^2 Y'$
and $Y \gg \sqrt{\varepsilon}\, X^2 Y'$

Let $Y(X) = Y_0(X) + \delta Y_1(X) + \delta^2 Y_2(X) + O(\delta^3)$.

$Y_0'' - Y_0 = 0 \Leftrightarrow Y_0(X) = A e^X + B e^{-X}$

$Y_0(0) = 1 \Leftrightarrow A + B = 1 \Leftrightarrow A = 1 - B$, therefore

$Y_0(X) = (1-B)e^{X} + Be^{-X}$

Asymptotic matching:

$$\lim_{X \to +\infty} Y_0(X) = \lim_{x \to 0} y_{out}(x) = 0 \iff 1 - B = 0 \iff B = 1$$

so $\quad y_{unif} = e^{1-1/x} + e^{-x/\sqrt{\varepsilon}} + o(\varepsilon).$

▼ Multiple scale analysis.

In certain ODEs a regular perturbation method does not work because we encounter secular terms.

example : $y'' + y + \varepsilon y^3 = 0.$, $y(0) = 1$, $y'(0) = 0.$

Try a perturbation expansion:

$$y(t) = \sum_{n=0}^{+\infty} \varepsilon^n y_n(t).$$

Substituting to the ODE we obtain:

$$y_0'' + y_0 = 0$$

$$y_1'' + y_1 = -y_0^3$$

and the boundary conditions:

$$y_0(0) = 1 \quad , \quad y_0'(0) = 0$$

$$y_1(0) = 0 \quad , \quad y_1'(0) = 0.$$

The solution is: $\quad y_0(t) = \cos t$

$$y_1(t) = \cancel{A\cos t}\ \frac{\cos t - \cos 3t}{32} - \frac{3t}{8}\sin t$$

Note that $y_1(t)$ blows up as $t \to +\infty$. This approach does not yield an approximation which is valid as $t \to +\infty$.

When we encounter such situation we use the method of multiple scale analysis.