Numerical Methods for Partial Differential Equations

1. Finite Difference Schemes.

▼ Relevent definitions

Def : An s-step linear finite difference operator is an operator of the form:

$$P v_m^n = \sum_{\nu=0}^{s} \sum_{\mu=-\ell}^{r} a_{\mu\nu} v_{m+\mu}^{n+1-\nu}$$

where $a_{\mu\nu}$ are constants with $a_{00} \neq 0$, $\exists \nu_1 : a_{-\ell,\nu_1} \neq 0$
$\exists \nu_2 : a_{r,\nu_2} \neq 0$.

a) P is explicit $\Leftrightarrow a_{\mu 0} = 0, \forall \mu \neq 0$

b) P is implicit $\Leftrightarrow \exists \mu \neq 0 : a_{\mu 0} \neq 0$.

Remark.

A partial differential equation usually has the form

$$Pu = f$$

where u = the unknown function
P = a partial differential operator.
f = an inhomogeneity.

Suppose that $u = u(x,y)$ and $f = f(x,y)$. Define

$$u_m^n = u(mh, nk).$$
$$f_m^n = f(mh, nk)$$

The objective is to approximate the PDE with a "convergent" different scheme of the form:

$$P_{k,h} v_m^n = R_{k,h} f_m^n$$

where $P_{k,h}, R_{k,h}$ are linear finite difference operators.
Next we define "convergence" for difference schemes.

Def : (Well-posedness)
We say that a PDE is __well-posed__ iff
$\forall t > 0, \exists C_t > 0 : \forall$ initial conditions : $\|u(t,\cdot)\| \leq C_t \|u(0,\cdot)\|$.

Def : (interpolation operator)
Let v be a sequence v_m. Define $\hat{v}(\xi)$ by:

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi}\, \hat{v}(\xi)\, d\xi .$$

The __interpolation operator__ S is defined by:

$$u = Sv \Leftrightarrow u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{ix\xi}\, \hat{v}(\xi)\, d\xi \ , \ \forall x \in \mathbb{R}. \quad \text{D.}$$

__Remark__ : This definition enables us to compare continuous functions with discretized functions. To generalize to many variables, apply an interpolation operator on every variable

~~**Def : (convergence of difference schemes)**~~
~~A finite difference scheme~~ $P_{k,h} v_m^n = R_{k,h} f_m^n$ ~~is called convergent iff~~ $\forall$ ~~well-posed PDEs~~

~~**Def : (convergence of difference schemes)**~~
~~We say that a finite difference scheme~~ $P_{k,h} v_m^n = R_{k,h} f_m^n$ ~~with solutions~~ $v_m^n(k,h)$ ~~is convergent to a well-posed PDE with solution~~ $u(x,t)$ ~~iff~~

$$\lim_{(k,h)\to 0} \left[S v_m^n(k,h) - u(x,t) \right] = 0$$

$$\lim_{(k,h)\to 0} \left[v(\cdot,t) - u(\cdot,t) \right] = 0 \quad \text{with } v(x,t) = S v_m^n(k,h).$$

Def : (Convergent finite difference schemes)
Suppose that v_m^n is the solution to the finite difference scheme $P_{k,h} v_m^n = R_{k,h} f_m^n$ and $u(x,t)$ the solution to a well-posed PDE.
Let $v_{k,h}(x,t) = S v_m^n$ for given steps k, h.
We say that the scheme converges to the PDE iff:

$$\lim_{(k,h)\to 0} \left[v_{k,h}(\cdot,0) - u(\cdot,0) \right] = 0 \Longrightarrow \lim_{(k,h)\to 0} \left[v_{k,h}(\cdot,t) - u_{k,h}(\cdot,t) \right] = 0.$$

Remark : Note that the limits must converge uniformly for all values of x.

▼ Conditions for convergence

Def : (Consistency).
A scheme $P_{k,h} v = R_{k,h} f$ is consistent with the PDE $Pu = f$ iff:
$$\lim_{(k,h)\to 0} \left[P_{k,h} \varphi - R_{k,h} P\varphi \right] = 0, \quad \forall \text{ smooth functions } \varphi.$$

Def : (Stability)
A finite difference scheme $P_{k,h} v_m^n = R_{k,h} f_m^n$ is stable in a stability region Λ iff: $\forall (k,h) \in \Lambda$:

$$\exists J \in \mathbb{N}^* : \forall T > 0, \exists C_T > 0 : h \sum_{m=-\infty}^{+\infty} |v_m^n|^2 \leqslant C_T h \sum_{j=0}^{J} \sum_{m=-\infty}^{+\infty} |v_m^j|^2, \forall n : 0 \leqslant nk \leqslant T$$

Thm : (Lax-Richtmyer Equivalence Theorem)
If $P_{k,h} v = R_{k,h} f$ is a scheme consistent to a well-posed PDE $Pu = f$ then,
the scheme is convergent $\Longleftrightarrow$ the scheme is stable

Order of accuracy.

<u>Def</u> : A scheme $P_{k,h} v = R_{k,h} f$ consistent with a PDE $Pu = f$ is <u>(p,q) order accurate</u> iff

$\forall$ smooth functions φ : $$\lim_{(k,h)\to 0} \frac{P_{k,h}\varphi - R_{k,h}P\varphi}{k^p h^q} = 0.$$

<u>Def</u> : A scheme $P_{k,h} v = R_{k,h} f$ with $k = \Lambda(h)$ consistent with the PDE $Pu = f$ is <u>accurate of order r</u> iff:

$\forall$ smooth functions φ : $$\lim_{(k,h)\to 0} \frac{P_{k,h}\varphi - R_{k,h}P\varphi}{h^r} = 0.$$

Conditions for consistency.

<u>Def</u> : Let $P_{k,h}$ be a finite difference operator. The <u>symbol of $P_{k,h}$</u> is a function $S(s,\xi;P_{k,h})$ defined by:

$$S(s,\xi;P_{k,h}) = \frac{P_{k,h}(e^{skn}e^{imh\xi})}{e^{skn}e^{imh\xi}}$$

<u>Def</u> : Let P be a partial differential operator. The <u>symbol of P</u> is a function $S(s,\xi;P)$ defined by:

$$\cancel{S(s,\xi;P) = \frac{P(e^{skn}e^{imh\xi})}{e^{skn}e^{imh\xi}}}$$

$$S(s,\xi;P) = \frac{P(e^{st}e^{i\xi x})}{e^{st}e^{i\xi x}}.$$

<u>Thm</u> : A scheme $P_{k,h} v = R_{k,h} f$ is consistent with $Pu = f$ and has order of accuracy (p,q) iff

$$S(s,\xi;P_{k,h}) - S(s,\xi;P)\,S(s,\xi;R_{k,h}) = O(k^p) + O(h^q)$$

Thm : A scheme $P_{k,h} v = R_{k,h} f$ with $k = \Lambda(h)$ is consistent with $Pu = f$ and is accurate of order r, iff:

$$\$(s,\xi; P_{k,h}) - \$(s,\xi; P)\,\$(s,\xi; R_{k,h}) = O(h^r).$$

▼ Spatial / Temporal difference / average operators.

Def : The following operators are called fundamental operators and are defined as follows:

a) Time-shift operator: $Z v_m^n = v_m^{n+1}$

b) Space-shift operator: $K v_m^n = v_{m+1}^n$

c) Identity operator: $I v_m^n = v_m^n$.

Def : The spatial difference operators are:

a) Forward difference $\quad \delta_+ = \frac{1}{h}(K - I)$

b) Backwards difference $\quad \delta_- = \frac{1}{h}(I - K^{-1})$

c) Center difference $\quad \delta_0 = \frac{1}{2h}(K - K^{-1})$

d) 2nd order difference: $\quad \delta_x = \frac{1}{h^2}(K - 2I + K^{-1})$

Def : The spatial averaging operators are

a) Forward average $\quad \mu_+ = (I + K)/2$

b) Backwards average $\quad \mu_- = (K^{-1} + I)/2$

c) Center average $\quad \mu_0 = (K^{-1} + K)/2$.

Def : The temporal difference operators are:

a) Forward difference : $\delta^+ = (Z - I)/k$

b) Backwards difference $\quad \delta^- = (I - Z^{-1})/k$

c) Center difference $\quad \delta^0 = (Z - Z^{-1})/2k$

d) 2nd order difference $\quad \delta^x = (Z - 2I + Z^{-1})/k^2$.

Def : The temporal averaging operators are the following:

a) Forward averaging $\mu^+ = (I+Z)/2$

b) Backwards averaging. $\mu^- = (Z^{-1}+I)/2$

c) Centered averaging $\mu^0 = (Z+Z^{-1})/2$.

Computing symbols of operators.

Most finite difference operators can be written in terms of $\delta_+, \delta_-, \delta_0, \delta_x, \mu_+, \mu_-, \mu_0, \delta^+, \delta^-, \delta^0, \delta^x, \mu^+, \mu^-, \mu^0$.

Once the symbols for those operators are known, then symbols can be computed by the following theorems:

Thm : Let $P_{k,h}, R_{k,h}$ be finite difference operators.

a) $S(s,\xi; \lambda_1 P_{k,h} + \lambda_2 R_{k,h}) = \lambda_1 S(s,\xi; P_{k,h}) + \lambda_2 S(s,\xi; R_{k,h})$.

b) $S(s,\xi; P_{k,h} R_{k,h}) = S(s,\xi; P_{k,h})\, S(s,\xi; R_{k,h})$

Thm : The symbols of K, Z, I are:

a) $S(s,\xi; I) = 1$.

b) $S(s,\xi; K) = e^{ih\xi}$

c) $S(s,\xi; Z) = e^{sk}$.

Thm : The symbols for the spatial difference operators are:

$$S(s,\xi;\delta_+) = \frac{e^{ih\xi}-1}{h} \qquad S(s,\xi;\delta_0) = \frac{e^{ih\xi}-e^{-ih\xi}}{2h}$$

$$S(s,\xi;\delta_-) = \frac{1-e^{-ih\xi}}{h} \qquad S(s,\xi;\delta_x) = \frac{e^{ih\xi}+e^{-ih\xi}-2}{h^2}.$$

Thm : The symbols for the spatial averaging operators are:

$$S(s,\xi;\mu_+) = \frac{1+e^{ih\xi}}{2} \qquad S(s,\xi;\mu_0) = \frac{e^{ih\xi}+e^{-ih\xi}}{2}.$$

$$S(s,\xi;\mu_-) = \frac{1+e^{-ih\xi}}{2}$$

Thm: The symbols for the temporal difference operators are:

$$\wp(s,\xi;\delta^+) = \frac{e^{sk}-1}{k} \qquad \wp(s,\xi;\delta^0) = \frac{e^{sk}-e^{-sk}}{2k}$$

$$\wp(s,\xi;\delta^-) = \frac{1-e^{-sk}}{k} \qquad \wp(s,\xi;\delta^\times) = \frac{e^{sk}+e^{-sk}-2}{k^2}$$

Thm: The symbols for the temporal averaging operators.

$$\wp(s,\xi;\mu^+) = \frac{1+e^{sk}}{2} \qquad \wp(s,\xi;\mu^0) = \frac{e^{sk}+e^{-sk}}{2}$$

$$\wp(s,\xi;\mu^-) = \frac{1+e^{-sk}}{2}$$

▼ Well-known schemes for certain PDEs.

① Hyperbolic model equation $\longrightarrow \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0$

a) Lax-Wendroff scheme: $\left[\delta^+ + a\delta_0 - \frac{a^2k}{2}\delta_x\right]v_m^n = \left[\mu^+ - \frac{ak}{2}\delta_0\right]f_m^n$

b) Crank-Nicholson scheme: $\left[\delta^+ + a\mu^+\delta_0\right]v_m^n = \mu^+ f_m^n$.

② Parabolic model equation. $\longrightarrow \dfrac{\partial u}{\partial t} = b\dfrac{\partial^2 u}{\partial x^2}$

a) Backward time-central space scheme:
$$\left[\delta^+ - b\,\delta_x Z\right]v_m^n = f_m^n$$
b) Crank-Nicholson scheme
$$\left[\delta^+ - b\mu^+\delta_x\right]v_m^n = \mu^+ f_m^n$$
c) Du Fort-Frankel scheme.
$$\left[\delta^0 - b\left(k - 2\mu^0 + k^{-1}\right)h^{-2}\right]v_m^n = f_m^n.$$

Analysis of finite difference schemes.

Analysis of a finite difference scheme should include the following:

1) Consistency and accuracy
2) Von Neumann stability analysis
3) Dispersion and dissipation analysis. $\Big\} \to$ to be developed.

We develop now the theory for 2,3.

Von Neumann stability analysis.

Def : Let $P_{k,h}$ be a finite difference operator with symbol $S(s,\xi;P_{k,h})$. The amplification polynomial of $P_{k,h}$ is given by

$$\boxed{\Phi(g,\vartheta;P_{k,h}) = S\left(\frac{\ln g}{k}, \vartheta h^{-1}\right).}$$

Remark : Given a scheme $P_{k,h} v_m^n = R_{k,h} f_m^n$, with Von Neumann stability analysis we see whether

$$v_m^n = g^n e^{im\vartheta}$$

is bounded or not, when it satisfies $P_{k,h} v_m^n = 0$.

Let $s = \frac{\ln g}{k} \Leftrightarrow \ln g = sk \Leftrightarrow g = e^{sk}$

and $\xi = \vartheta h^{-1} \Leftrightarrow \vartheta = \xi h$.

Then:

$$P_{k,h} v_m^n = P_{k,h}(g^n e^{im\vartheta}) = P_{k,h}(e^{skn} e^{im\xi h}) = S(s,\xi;P_{k,h}) e^{skn} e^{im\xi h} = 0$$

$\Leftrightarrow S(s,\xi;P_{k,h}) = 0 \Leftrightarrow \phi(g,\vartheta;P_{k,h}) = 0.$

In order for the solution to be bounded, the polynomial must be a simple Von Neumann.

Def : A scheme $P_{k,h} v_m^n = R_{k,h} f_m^n$ is Von-Neumann stable $\Leftrightarrow$ $\phi(g,\vartheta;P_{k,h})$ is a Von Neumann polynomial, $\forall \vartheta \in [-\pi,\pi]$.

Thm: Let $P_{k,h}, R_{k,h}$ be two ~~lineo~~ finite difference operators.

a) $\Phi(g,\vartheta; \lambda P_{k,h} + \mu R_{k,h}) = \lambda \Phi(g,\vartheta; P_{k,h}) + \mu \Phi(g,\vartheta; R_{k,h})$.

b) $\Phi(g,\vartheta; P_{k,h} R_{k,h}) = \Phi(g,\vartheta; P_{k,h}) \Phi(g,\vartheta; R_{k,h})$.

Thm: The amplification polynomials of I, K, Z are:

a) $\Phi(g,\vartheta; I) = 1$

b) $\Phi(g,\vartheta; K) = e^{i\vartheta}$

c) $\Phi(g,\vartheta; Z) = g$.

Thm: The amplification polynomials of the following operators are:

$$\Phi(g,\vartheta;\delta_+) = \frac{e^{i\vartheta}-1}{h} \qquad \Phi(g,\vartheta;\delta^+) = \frac{g-1}{h}$$

$$\Phi(g,\vartheta;\delta_-) = \frac{1-e^{-i\vartheta}}{h} \qquad \Phi(g,\vartheta;\delta^-) = \frac{1-g^{-1}}{h}$$

$$\Phi(g,\vartheta;\delta_0) = \frac{i\sin\vartheta}{h} \qquad \Phi(g,\vartheta;\delta^0) = \frac{g-g^{-1}}{h}$$

$$\Phi(g,\vartheta;\delta_x) = \frac{2(\cos\vartheta-1)}{h^2} \qquad \Phi(g,\vartheta;\delta^x) = \frac{g+g^{-1}-2}{h^2}$$

$$\Phi(g,\vartheta;\mu_+) = \frac{1+e^{i\vartheta}}{2} \qquad \Phi(g,\vartheta;\mu^+) = \frac{1+g}{2}$$

$$\Phi(g,\vartheta;\mu_-) = \frac{1+e^{-i\vartheta}}{2} \qquad \Phi(g,\vartheta;\mu^-) = \frac{1+g^{-1}}{2}$$

$$\Phi(g,\vartheta;\mu_0) = \frac{e^{i\vartheta}+e^{-i\vartheta}}{2} \qquad \Phi(g,\vartheta;\mu^0) = \frac{g+g^{-1}}{2}$$

Remark: Von Neumann stability is necessary but not sufficient for the stability of the scheme overall. Nevertheless, it is a good rule of thumb. Von Neumann stability usually fails due to:

a) Boundary conditions.

b) Non-linearities.

Stability and the Kreiss matrix theorem

Most finite difference schemes can be reduced to:

$$v_m^n = C^n v_m^0$$

where v_m^n are vectors containing multiple timesteps and C a matrix. The scheme will be stable iff C^n is bounded. If C is diagonalizable this is simple:

<u>Thm</u>: If C is diagonalizable then
C^n bounded $\Leftrightarrow \forall \lambda \in \lambda(C) : |\lambda| < 1 \vee (|\lambda| = 1 \wedge \lambda \text{ simple})$.

The most general known result is the Kreiss-matrix theorem. which uses the concept of pseudo-eigenvalues:

<u>Def</u>: Given $\varepsilon > 0$, $\lambda \in \mathbb{C}$ is an <u>ε-pseudoeigenvalue</u> of $A \Leftrightarrow \|(\lambda I - A)\| \leq \varepsilon$.
We write the set of all pseudo eigenvalues as:

$$\Lambda_\varepsilon(A) = \{\lambda \in \mathbb{C} : \|\lambda I - A\| \leq \varepsilon\}.$$

<u>Prop.</u>: $\lambda \in \lambda_\varepsilon(A) \Leftrightarrow \exists M \in \mathbb{C}^{n\times n} : \begin{cases} \lambda \in \lambda(A+M). \\ \|M\| \leq \varepsilon. \end{cases}$

<u>Thm</u>: (<u>Kreiss Matrix Theorem</u>) Let $F \subseteq \mathbb{C}^{n\times n}$. The following are equivalent:
a) $\exists C > 0 : \forall A \in F, \forall n \in \mathbb{N} : \|A^n\| \leq C$
b) $\exists C \geq 0 : \forall \lambda \in \lambda_\varepsilon(A), \forall \varepsilon > 0, |\lambda| \leq 1 + C\varepsilon.$
c) $\exists C > 0 : \forall A \in F, \forall z \in \mathbb{C}, |z| > 1 : \|(zI - A)^{-1}\| \leq C(|z|-1)^{-1}.$

Stability and the Kreiss matrix theorem

In general, given some manipulation, a finite difference scheme can be written as:

$$v_m^n = C^n v_m^0$$

The issue is whether or not C^n is bounded.
This is a simple matter if C is diagonalizable. Then the following theorem applies:

<u>Thm</u>: If C is diagonalizable then
C^n bounded $\Leftrightarrow \forall \lambda \in \lambda(C) : |\lambda| < 1 \vee (|\lambda| = 1 \wedge \lambda \text{ is simple})$

In general however, C may not be diagonalizable. In those cases we use the concept of pseudo-eigenvalues.

<u>Def</u>: Given $\varepsilon > 0$, $\lambda \in \mathbb{C}$ is an <u>ε-pseudo eigenvalue</u> of A $\Leftrightarrow \|(\lambda I - A)\| \leqslant \varepsilon$.
The set of all pseudoeigenvalues is written as

$$\Lambda_\varepsilon(A) = \{\lambda \in \mathbb{C} : \|\lambda I - A\| \leqslant \varepsilon\}.$$

<u>Thm</u>: $\lambda \in \Lambda_\varepsilon(A) \Leftrightarrow \exists E \in \mathbb{C}^{n \times n} : \begin{cases} \lambda \in \Lambda(A+E) \\ \|E\| \leqslant \varepsilon. \end{cases}$

The following theorem decides stability in the general case

<u>Thm</u>: <u>(Kreiss-Matrix theorem)</u>

a) A^n is bounded $\Leftrightarrow \exists C \in \mathbb{C} : \forall \lambda_\varepsilon \in \Lambda_\varepsilon(A) : |\lambda_\varepsilon| \leqslant 1 + C\varepsilon$

b) A^n is bounded $\Leftrightarrow \exists C \in \mathbb{C} : \forall z \in \mathbb{C} : \|zI - A\| \geqslant \underline{(|z|-1)}\, C.$

▼ Crank-Nicholson for the heat equation

The heat equation is: $\frac{\partial u}{\partial t} = b\frac{\partial^2 u}{\partial x^2} + f.$

The Crank-Nicholson scheme for this equation is:

$$[\delta^+ - b\mu^+\delta_x]\, v_m^n = \mu^+ f_m^n .$$

In expanded form:

$$\frac{v_m^{n+1} - v_m^n}{k} = \frac{1}{2} b \frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{h^2} + \frac{1}{2} b \frac{v_{m+1}^{n} - 2v_m^{n} + v_{m-1}^{n}}{h^2} + \frac{1}{2}\left(f_m^{n+1} + f_m^n\right).$$

Let $\mu = k/h^2$. Then we may rewrite this scheme as:

$$v_m^{n+1} - v_m^n = \frac{1}{2} b\mu \left(v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}\right) + \frac{1}{2} b\mu \left(v_{m+1}^{n} - 2v_m^{n} + v_{m-1}^{n}\right) + \frac{k}{2}\left(f_m^{n+1} + f_m^n\right)$$

$$\Leftrightarrow -\frac{b\mu}{2} v_{m+1}^{n+1} + \left(1 - \frac{b\mu}{2}\right) v_m^{n+1} - \frac{b\mu}{2} v_{m-1}^{n+1} = \frac{b\mu}{2}\left(v_{m+1}^{n} - 2v_m^{n} + v_{m-1}^{n}\right) + \frac{k}{2}\left(f_m^{n+1} + f_m^n\right).$$

● Accuracy of the scheme

$$S(s,\xi,P_{k,h}) = S(s,\xi;\delta^+ - b\mu^+\delta_x) = S(s,\xi;\delta^+) - b\, S(s,\xi;\mu^+)\, S(s,\xi;\delta_x) =$$

$$= \frac{e^{sk}-1}{k} - b\, \frac{e^{sk}+1}{2}\, \frac{e^{ih\xi} + e^{-ih\xi} - 2}{h^2}$$

$$S(s,\xi;R_{k,h}) = S(s,\xi;\mu^+) = \frac{e^{sk}+1}{2} .$$

$$S(s,\xi;P) = S(s,\xi;\partial_t - b\partial_x^2) = s - b(i\xi)^2 = s + b\xi^2 .$$

It follows that:

$$F(s,\xi) = S(s,\xi;P_{k,h}) - S(s,\xi;R_{k,h})\, S(s,\xi;P) =$$

$$= \frac{e^{sk}-1}{k} - b\, \frac{e^{sk}+1}{2}\, \frac{e^{ih\xi} + e^{-ih\xi} - 2}{h^2} - (s + b\xi^2)\, \frac{e^{sk}+1}{2} =$$

If $k = \mu h^2 \Rightarrow F(s,\xi) = \frac{1}{2}\left[\mu s^2 - \mu s(s + b\xi^2) + b\left(\mu s \xi^2 - \frac{\xi^4}{6}\right)\right] h^2 + O(h^4)$

$\Rightarrow$ the scheme is 2nd order accurate if $k = \mu h^2$.

Also if $k=\lambda h \Rightarrow F(s,\xi) = \left[\frac{\lambda^2 s^3}{6} - \frac{\lambda^2 s^2(s+b\xi^2)}{4} + \frac{b(2\lambda^2 s^2\xi^2 - \xi^4)}{8}\right]h^2 + O(h^4)$

$\Rightarrow$ the scheme is 2nd order accurate if $k=\lambda h$.

- <u>Von Neumann stability.</u>

$$s(s,\xi;P_{k,h}) = \frac{e^{sk}-1}{k} - b\,\frac{e^{sk}+1}{2}\,\frac{e^{ih\xi}+e^{-ih\xi}-2}{h^2}\ \cancel{\lambda(s+b\xi^2)\frac{e^{sk}+1}{2}} \Rightarrow$$

$$\Rightarrow \Phi(g,\vartheta;P_{k,h}) = s\left(\frac{\ln g}{k}, \vartheta h^{-1}; P_{k,h}\right) =$$

$$= \frac{g-1}{k} - b\,\frac{g+1}{2}\,\frac{e^{i\vartheta}+e^{-i\vartheta}-2}{h^2} \cancel{-\lambda\left(\frac{\ln g}{k}\right)} = 0 \Leftrightarrow$$

$$\Leftrightarrow 2(g-1) - b\mu\,(g+1)(e^{i\vartheta}+e^{-i\vartheta}-2) = 0 \Leftrightarrow$$

$$\Leftrightarrow \left[2 - b\mu(e^{i\vartheta}+e^{-i\vartheta}-2)\right]g - 2 - b\mu(e^{i\vartheta}+e^{-i\vartheta}-2) = 0 \Leftrightarrow$$

$$\Leftrightarrow g = \frac{2+b\mu(e^{i\vartheta}+e^{-i\vartheta}-2)}{2-b\mu(e^{i\vartheta}+e^{-i\vartheta}-2)} = \frac{2+b\mu(2\cos\vartheta-2)}{2-b\mu(2\cos\vartheta-2)} \leq 1 \Leftrightarrow$$

$$\Leftrightarrow 2+b\mu(2\cos\vartheta-2) \leq 2-b\mu(2\cos\vartheta-2) \Leftrightarrow$$

$$\Leftrightarrow 2b\mu(2\cos\vartheta-2) \leq 0 \Leftrightarrow \cos\vartheta \leq 1 \rightsquigarrow \text{true } \forall \vartheta\in\mathbb{R}.$$

Therefore the scheme is unconditionally stable

• Lax-Wendroff scheme for advection

The advection equation is $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = f$.
The Lax-Wendroff scheme is:

$$\left[\delta^+ + a\delta_0 - \frac{a^2 k}{2}\delta_x\right] v_m^n = \left[\mu^+ - \frac{ak}{2}\delta_0\right] f_m^n .$$

In expanded form, this is:

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} - \frac{a^2 k}{2}\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2} = \frac{f_m^{n+1} + f_m^n}{2} - \frac{ak}{4h}\left(f_{m+1}^n - f_{m-1}^n\right)$$

Let $k = \lambda h$. Then,

$$v_m^{n+1} = v_m^n - \frac{a\lambda}{2}\left(v_{m+1}^n - v_{m-1}^n\right) + \frac{a^2\lambda^2}{2}\left(v_{m+1}^n - 2v_m^n + v_{m-1}^n\right) + \frac{k}{2}\left(f_m^{n+1} + f_m^n\right) - \frac{ak\lambda}{4}\left(f_{m+1}^n - f_{m-1}^n\right)$$

• Accuracy.

$$\$(s,\xi; P_{k,h}) = \$(s,\xi; \delta^+ + a\delta_0 - \frac{a^2 k}{2}\delta_x) =$$
$$= \frac{e^{sk} - 1}{k} + a\frac{e^{ih\xi} - e^{-ih\xi}}{2h} - \frac{a^2 k}{2}\frac{e^{ih\xi} + e^{-ih\xi} - 2}{h^2}$$

$$\$(s,\xi; P) = \$(s,\xi; \partial_t + a\partial_x) = s + a(i\xi).$$

$$\$(s,\xi; R_{k,h}) = \$(s,\xi; \mu^+ - \frac{ak}{2}\delta_0) = \frac{1 + e^{sk}}{2} - \frac{ak}{2}\frac{e^{ih\xi} - e^{-ih\xi}}{2h}$$

therefore

$$F(s,\xi) = \$(s,\xi; P_{k,h}) - \$(s,\xi; P)\,\$(s,\xi; R_{k,h}) =$$

$$= \frac{e^{sk} - 1}{k} + a\frac{e^{ih\xi} - e^{-ih\xi}}{2h} - \frac{a^2 k}{2}\frac{e^{ih\xi} + e^{-ih\xi} - 2}{h^2} - (s + a(i\xi))\left[\frac{1 + e^{sk}}{2} - \frac{ak}{2}\frac{e^{ih\xi} - e^{-ih\xi}}{2h}\right] =$$

Finite difference schemes for PDEs.

Let $Pu = f$ be a PDE with $u(x,t)$ a function, $f(x,t)$ = a known function.
P = a differential operator.
Approximate $u(x,t)$ with $v_m^n = u(mh, nk)$, $m,n \in \mathbb{Z}$.
Want: approximate the PDE in terms of u_m^n.

Def: An s-step linear finite difference operator $P_{k,h}$ is defined by:

$$P_{k,h} v_m^n = \sum_{\nu=0}^{s} \sum_{\mu=-l}^{r} a_{\mu\nu} v_{j+\mu}^{n+1-\nu}$$

with $a_{00} \neq 0$, $a_{-l,\nu_1} \neq 0$, $\exists \nu_1$
$a_{r,\nu_2} \neq 0$, $\exists \nu_2$

b) If $a_{\mu 0} = 0$, $\forall \mu \neq 0 \Rightarrow P_{k,h}$ is explicit
If $a_{\mu 0} \neq 0$, $\exists \mu \neq 0 \Rightarrow P_{k,h}$ is implicit.

c) A finite difference scheme for $Pu = f$ is said to be an equation of the form $P_{k,h} v = R_{k,h} f$.

→ Desired properties: a) Consistency.
b) Accuracy
c) Stability.
d) Convergence ⊛

Def: (Consistency). Let $Pu = f$ be a PDE and $P_{k,h} v_m^n = R_{k,h} f_m^n$ be a scheme. We say that the scheme is consistent with the PDE $\Leftrightarrow$

$$\lim_{(k,h)\to(0,0)} [P_{k,h}\varphi - R_{k,h} P\varphi] = 0, \quad \forall \varphi(x,t) \text{ smooth}$$

Def: (Accuracy) Let $Pu = f$ be a PDE and $P_{k,h} v_m^n = R_{k,h} f_m^n$ be a consistent scheme. We say that the scheme is (p,q) order accurate $\Leftrightarrow$ $P_{k,h}\varphi - R_{k,h} P\varphi = O(k^p) + O(h^q)$, $\forall \varphi(x,t)$ smooth

b) Let $P_{k,h} v = R_{k,h} f$ with $k = \Lambda(h)$ be a ~~s~~ scheme consistent with $Pu = f$. We say that the scheme is r-order accurate $\Leftrightarrow$

$$P_{k,h}\varphi - R_{k,h}P\varphi = o(h^r), \ \forall \varphi(x,t) \text{ smooth}$$

Def: (Stability) Let $P_{k,h} v_m^n = R_{k,h} f_m^n$ be a scheme. We say that the scheme is stable $\Leftrightarrow$

$$\exists J \in \mathbb{N}^* : \forall T > 0, \exists C_T > 0 : \sum_{m=-\infty}^{+\infty} |v_m^n|^2 \leq C_T \sum_{j=0}^{J} \sum_{m=-\infty}^{+\infty} |v_m^j|^2.$$

b) Λ is the stability region $\Lambda = \{(k,h) \in \mathbb{R}_+^2 : P_{k,h} v_m^n = R_{k,h} f_m^n \text{ stable}\}$.

Remark: Corresponding to this concept is the concept of well-posedness

Def: The initial value problem for $Pu = f$ is well-posed $\Leftrightarrow$

$$\forall T > 0, \exists C_T > 0 : u(x,t) \text{ solution} \Rightarrow \int_{-\infty}^{+\infty} |u(x,t)|^2 dx \leq C_T \int_{-\infty}^{+\infty} |u(x,0)|^2 dx, \ \forall t \in (0,T).$$

Convergence

Def (Interpolation operator) $S : L^2(h\mathbb{Z}) \to L^2(\mathbb{R})$

Given $v \in L^2(h\mathbb{Z})$ define $\hat{v}(\xi)$ by

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{v}(\xi) d\xi.$$

Then $Sv \in L^2(\mathbb{R})$ is given by:

$$Sv = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{ix\xi} \hat{v}(\xi) d\xi.$$

Def (Convergence). Let u be the solution to $Pu = f$ and v the solution to $P_{k,h} v = R_{k,h} f$. The scheme converges to the PDE $\Leftrightarrow$

$$\lim_{(k,h)\to(0,0)} S v_m^n = u(mh, nk), \ \forall m, n.$$

Thm : (Lax-Richtmyer Equivalence Theorem)

a) Let $Pu=f$ be a well-posed PDE.
Let $P_{k,h}u=R_{k,h}f$ be a consistent and stable scheme
Then the scheme is convergent.

b) Let $Pu=f$ be a well-posed PDE
Let $P_{k,h}u=R_{k,h}f$ be a convergent scheme
Then the scheme is consistent and stable

Accuracy of finite difference schemes

Def : The symbol $p_{k,h}(s,\xi)$ of a finite diff. operator $P_{k,h}$ is defined as
$$p_{k,h}(s,\xi)=\frac{P_{k,h}(e^{skn}e^{imh\xi})}{e^{skn}e^{imh\xi}}$$

Def : The symbol of a differential operator P is defined by:
$$p(s,\xi)=\frac{P(e^{st}e^{i\xi x})}{e^{st}e^{i\xi x}}.$$

Thm : Let $Pu=f$ be a PDE and $P_{k,h}v=R_{k,h}f$ a consistent scheme.
The scheme is (p,q) order accurate $\Leftrightarrow$
$$p_{k,h}(s,\xi)-p(s,\xi)\,r_{k,h}(s,\xi)=O(k^p)+O(h^q).$$

Thm : Let $Pu=f$ be a PDE and $P_{k,h}v=R_{k,h}f$ with $k=\Lambda(h)$ a consistent scheme. The scheme is r-order accurate $\Leftrightarrow$
$$p_{k,h}(s,\xi)-p(s,\xi)\,r_{k,h}(s,\xi)=O(h^r).$$

Von-Neumann stability analysis.

Def: (Von-Neumann stability).
Let $P_{k,h}v=R_{k,h}f$ be a scheme. Let $p_{k,h}(s,\xi)$ be the symbol of $P_{k,h}$.
The amplification polynomial $\Phi(g,\vartheta)$ is defined by:
$$\Phi(g,\vartheta)=k\,p_{k,h}\left(\frac{\ln g}{k},\vartheta h^{-1}\right)$$

<u>Def</u>: We say that the scheme is stable $\iff$
$\forall \vartheta \in \mathbb{R}$: the roots g_n of $\Phi(g,\vartheta)=0$ are $|g_n| \lessgtr 1$. or
$|g_n|=1 \Rightarrow g_n$ simple root.

<u>Remark</u>: Von-Neumann stability is a good heuristic for stability overall.

▼ Difference operators.

<u>Def</u>: (shifting operators).

a) $Z v_m^n = v_m^{n+1}$ $\rightsquigarrow$ temporal shift operator

b) $K v_m^n = v_{m+1}^n$ $\rightsquigarrow$ spatial shift operator.

c) $I v_m^n = v_m^n$ $\rightsquigarrow$ identity operator.

<u>Def</u>: (~~Difference~~ Spatial operators)

a) $\delta_+ = \frac{1}{h}(K-I)$ $\rightsquigarrow$ forward difference $\delta_+ v_m^n = \frac{v_{m+1}^n - v_m^n}{h}$

b) $\delta_- = \frac{1}{h}(I-K^{-1})$ $\rightsquigarrow$ backward difference $\delta_- v_m^n = \frac{v_m^n - v_{m-1}^n}{h}$

c) $\delta_0 = \frac{1}{2h}(K-K^{-1})$ $\rightsquigarrow$ centered difference $\delta_0 v_m^n = \frac{v_{m+1}^n - v_{m-1}^n}{2h}$

d) $\delta_x^2 = \frac{1}{h^2}(K-2I+K^{-1})$ $\rightsquigarrow$ 2nd centered diff. $\delta^2 v_m^n = \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}$

e) $\mu_+ = \frac{1}{2}(I+K)$ $\rightsquigarrow$ forward averaging $\mu_+ v_m^n = \frac{v_{m+1}^n + v_m^n}{2}$

f) $\mu_- = \frac{1}{2}(I+K^{-1})$ $\rightsquigarrow$ backward averaging $\mu_- v_m^n = \frac{v_{m-1}^n + v_m^n}{2}$

g) $\mu_0 = \frac{1}{2}(K+K^{-1})$ $\rightsquigarrow$ centered averaging $\mu_0 v_m^n = \frac{v_{m+1}^n + v_{m-1}^n}{2}$.

<u>Def</u> (<u>Temporal operators</u>)

a) $\delta^+ = \frac{1}{h}(Z-I)$, $\delta^- = \frac{1}{h}(I-Z^{-1})$, $\delta^o = \frac{1}{2h}(Z-Z^{-1})$

b) $\delta^x = \frac{1}{h^2}(K-2I+K^{-1})$

c) $\mu^+ = \frac{1}{2}(I+K)$ $\mu^- = \frac{1}{2}(I+K^{-1})$ $\mu^o = \frac{1}{2}(K+K^{-1})$.

<u>notation</u> : The symbol of operator D is written $p_{K,h}(s,\xi;D)$.
The amplification polynomial $\Phi(g,\vartheta;D)$.

<u>Prop</u> : (<u>Symbols</u>).

a) $p_{K,h}(s,\xi;I) = 1$, $p_{K,h}(s,\xi;K) = e^{ih\xi}$, $p_{K,h}(s,\xi;Z) = e^{sk}$.

b) $\delta_o = \frac{\delta_+ + \delta_-}{2}$, $\delta_x = \delta_+\delta_- = \frac{\delta_+ - \delta_-}{h}$

$\delta^o = \frac{\delta^+ + \delta^-}{2}$, $\delta^x = \delta^+\delta^- = \frac{\delta^+ - \delta^-}{h}$.

c) $p_{k,h}(s,\xi;\delta_+) = e^{ih\xi} - 1$ $p_{K,h}(s,\xi;\delta^+) = e^{sk} - 1$.

$p_{K,h}(s,\xi;\delta_-) = 1 - e^{-ih\xi}$ $p_{K,h}(s,\xi;\delta^-) = 1 - e^{-sk}$.

$p_{K,h}(s,\xi;\delta_o) = i\sin(h\xi)$ $p_{K,h}(s,\xi;\delta^o) = \sinh(sk)$.

$p_{K,h}(s,\xi;\delta_x) = \frac{2\cos(h\xi)-2}{\cancel{h}}$ $p_{K,h}(s,\xi;\delta^x) = 2\cosh(sk) - 2$.

d) $p_{K,h}(s,\xi;\mu_+) = (e^{ih\xi}+1)/2$ $p_{K,h}(s,\xi;\mu^+) = (1+e^{sk})/2$

$p_{K,h}(s,\xi;\mu_-) = (1+e^{-ih\xi})/2$ $p_{K,h}(s,\xi;\mu^-) = (1+e^{-sk})/2$.

$p_{K,h}(s,\xi;\mu_o) = 2\cos(h\xi)$. $p_{K,h}(s,\xi;\mu^o) = \cosh(sk)$.

Prop : $\delta_0 = \dfrac{\delta_+ + \delta_-}{2}$, $\delta_x = \delta_+\delta_- = \dfrac{\delta_+ - \delta_-}{h}$

$\delta^0 = \dfrac{\delta^+ - \delta^-}{2}$, $\delta^x = \delta^+\delta^- = \dfrac{\delta^+ - \delta^-}{h}$.

Prop: (symbols to differential operators)

$$\boxed{\begin{aligned} p(s,\xi;\partial_x^n) &= (i\xi)^n \\ p(s,\xi;\partial_t^n) &= s^n \end{aligned}}$$

Prop : (symbols to difference operators)

a) $p(s,\xi;I) = 1$, $p(s,\xi;K) = e^{ih\xi}$, $p(s,\xi;Z) = e^{sk}$.

b) $p(s,\xi;\delta_+) = \dfrac{e^{ih\xi}-1}{h}$

$p(s,\xi;\delta_-) = \dfrac{1-e^{-ih\xi}}{h}$

$p(s,\xi;\delta_0) = \dfrac{e^{ih\xi}-e^{-ih\xi}}{2h} = \dfrac{i\sin(h\xi)}{h}$

$p(s,\xi;\delta_x) = \dfrac{e^{ih\xi}+e^{-ih\xi}-2}{h^2} = \dfrac{2\cos(h\xi)-2}{h^2}$

$p(s,\xi,\delta^+) = \dfrac{e^{sk}-1}{k}$

$p(s,\xi;\delta^-) = \dfrac{1-e^{-sk}}{k}$

$p(s,\xi;\delta^0) = \dfrac{e^{sk}-e^{-sk}}{2k} = \dfrac{\sinh(sk)}{k}$

$p(s,\xi;\delta^x) = \dfrac{e^{sk}+e^{-sk}-2}{k^2} = \dfrac{2\cosh(sk)-2}{k^2}$

c) $p(s,\xi;\mu_+) = \dfrac{e^{ih\xi}+1}{2}$

p
$p(s,\xi;\mu_-) = \dfrac{1+e^{-ih\xi}}{2}$

$p(s,\xi;\mu_0) = \dfrac{e^{ih\xi}+e^{-ih\xi}}{2} = \cos(h\xi)$

$p(s,\xi,\mu^+) = \dfrac{e^{sk}+1}{2}$

$p(s,\xi;\mu^-) = \dfrac{1+e^{-sk}}{2}$

$p(s,\xi;\mu^0) = \dfrac{e^{sk}+e^{-sk}}{2} = \cosh(sk)$.

<u>notation</u> : $\lambda = k/h$

<u>Prop</u> : (~~symbols~~ <u>Amplification polynomials</u>)

$$\Phi(g,\vartheta;D) = k\,p\left(\frac{\ln g}{k}, \vartheta h^{-1}\right).$$

a) $\Phi(g,\vartheta;I) = k$, $\Phi(g,\vartheta;K) = ke^{i\vartheta}$, $\Phi(g,\vartheta;Z) = kg$.

b) $\Phi(g,\vartheta;\delta_+) = \lambda(e^{i\vartheta}-1)$ $\qquad \Phi(g,\vartheta;\delta^+) = g-1$

$\Phi(g,\vartheta;\delta_-) = \lambda(1-e^{-i\vartheta})$ $\qquad \Phi(g,\vartheta;\delta^-) = 1-1/g$.

$\Phi(g,\vartheta;\delta_0) = i\lambda\sin\vartheta$ $\qquad \Phi(g,\vartheta;\delta^0) = \frac{1}{2}(g-1/g)$

$\Phi(g,\vartheta;\delta_x) = 2\lambda^2(\cos\vartheta-1)/k$ $\qquad \Phi(g,\vartheta;\delta^x) = \frac{1}{k}(g+1/g-2)$.

c) $\Phi(g,\vartheta;\mu_+) = k(e^{i\vartheta}+1)/2$ $\qquad \Phi(g,\vartheta;\mu^+) = k(g+1)/2$

$\Phi(g,\vartheta;\mu_-) = k(1+e^{-i\vartheta})/2$ $\qquad \Phi(g,\vartheta;\mu^-) = k(1+g^{-1})/2$

$\Phi(g,\vartheta;\mu_0) = k\cos\vartheta$ $\qquad \Phi(g,\vartheta;\mu^0) = k(g+g^{-1})/2$.

<u>Prop</u> : (combining symbols and amplification polynomials)

a) $p(s,\xi;aD_1+bD_2) = a\,p(s,\xi;D_1) + b\,p(s,\xi;D_2)$

$p(s,\xi;D_1D_2) = p(s,\xi;D_1)\,p(s,\xi;D_2)$.

b) $\Phi(g,\vartheta;aD_1+bD_2) = a\Phi(g,\vartheta;D_1) + b\Phi(g,\vartheta;D_2)$

$\Phi(g,\vartheta;D_1D_2) = \frac{\Phi(g,\vartheta;D_1)\,\Phi(g,\vartheta;D_2)}{k}$ $\qquad$!!! <u>caution</u>

~~$\Phi(g,\vartheta;D^{\mu}) =$~~

$$\Phi(g,\vartheta;D^{\mu}) = k^{1-\mu}\,\Phi^{\mu}(g,\vartheta;D)$$

$$\Phi\left(g,\vartheta;\prod_{j=1}^{\mu} D_j\right) = k^{1-\mu}\prod_{j=1}^{\mu}\Phi(g,\vartheta;D_j).$$

<u>Prop</u> : $\Phi(g,\vartheta;(ih/2)^{2r}\delta_x^{\,r}) = \sin^{2r}(\vartheta/2)$

$$\Phi(g,\vartheta;\delta_x) = -\frac{4\lambda^2}{k}\sin^2(\vartheta/2)$$

Schur & Von Neumann polynomials

Def: Let φ be a polynomial. with roots $\rho_1, \dots, \rho_n$.

a) φ Schur $\Leftrightarrow \forall k \in [n] : |\rho_k| < 1$.

b) φ von Neumann $\Leftrightarrow \forall k \in [n] : |\rho_k| \leq 1$.

c) φ simple von Neumann $\Leftrightarrow \begin{cases} \forall k \in [n] : |\rho_k| \leq 1 \\ |\rho_k| = 1 \Rightarrow \rho_k \text{ simple root.} \end{cases}$

d) φ conservative $\Leftrightarrow \forall k \in [n] : |\rho_k| = 1$.

Def: Let $\varphi_0(z)$ be a polynomial. Define:

a) $\varphi(z) = a_d z^d + \dots + a_0 \Rightarrow \varphi^*(z) = \bar{a}_0 z^d + \bar{a}_1 z^{d-1} + \dots + \bar{a}_d$.

b) $$\varphi_{j+1}(z) = \frac{\varphi_j^*(0)\,\varphi_j(z) - \varphi_j(0)\,\varphi_j^*(z)}{z}$$

Thm:

$$\begin{cases} \varphi_j \text{ Schur} \\ \deg \varphi_j = d \end{cases} \Leftrightarrow \begin{cases} \varphi_{j+1} \text{ Schur}, \ \deg \varphi_{j+1} = d-1. \\ |\varphi_j(0)| < |\varphi_j^*(0)|. \end{cases}$$

Thm

$$\begin{cases} \varphi_j \text{ simple von Neumann} \\ \deg \varphi_j = d \end{cases} \Leftrightarrow \begin{cases} \varphi \text{ simple von Neumann}, \ \deg \varphi_{j+1} = d-1 \\ |\varphi_j(0)| < |\varphi_j^*(0)|. \end{cases}$$

Thm

a) $\deg \varphi_{j+1} = \deg \varphi_j - 1$

b) φ_j Schur $\Leftrightarrow \varphi_{j+1}$ Schur $\wedge\ |\varphi_j(0)| < |\varphi_j^*(0)|$.

c) φ_j simple von Neumann $\Leftrightarrow \varphi_{j+1}$ simple von Neumann $\wedge\ |\varphi_j(0)| < |\varphi_j^*(0)|$.

d) φ_j Neumann $\Leftrightarrow \varphi_{j+1}$ von Neumann $\wedge\ |\varphi_j(0)| < |\varphi_j^*(0)|$

e) φ_j conservative $\Leftrightarrow \varphi_{j+1}$

Thm

a) $\deg \varphi_{j+1} = \deg \varphi_j - 1$.

b) φ_j Schur $\Leftrightarrow \begin{cases} \varphi_{j+1} \text{ Schur} \\ |\varphi_j(0)| < |\varphi_j^*(0)| \end{cases}$

c) φ_j von Neumann $\Leftrightarrow \begin{cases} \varphi_{j+1} \text{ von Neumann} \\ |\varphi_j(0)| < |\varphi_j^*(0)| \end{cases} \vee \begin{cases} \varphi_{j+1} = 0 \\ \varphi_j' \text{ von Neumann} \end{cases}$

d) φ_j simple von Neumann $\Leftrightarrow \begin{cases} \varphi_{j+1} \text{ von Neumann} \\ |\varphi_j(0)| < |\varphi_j^*(0)| \end{cases} \vee \begin{cases} \varphi_{j+1} = 0 \\ \varphi_j' \text{ ~~von Neumann~~ Schur} \end{cases}$

e) φ_j conservative $\Leftrightarrow \begin{cases} \varphi_{j+1} = 0 \\ \varphi_j' \text{ von Neumann.} \end{cases}$

f) φ_j simple conservative $\Leftrightarrow \begin{cases} \varphi_{j+1} = 0. \\ \varphi_j' \text{ Schur.} \end{cases}$

Using this theorem we can write down a condition on when a high order polynomial has one of these properties. Use in Von-Neumann stability analysis.