Complex Analysis

▼ Complex numbers

Def : The structure of complex numbers $(\mathbb{C},+,\cdot)$ is defined by

$$\mathbb{C} = \{(a,b) : a \in \mathbb{R}, b \in \mathbb{R}\}$$
$$(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2).$$
$$(x_1,y_1)\cdot(x_2,y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1).$$

Thm : $(\mathbb{C},+,\cdot)$ is a field with

$$-(x,y) = (-x,-y) \qquad 1 = (1,0)$$
$$(x,y)^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right). \qquad 0 = (0,0).$$

Def

a) $i = (0,1)$

b) $z = (a,b) \Leftrightarrow Re(z) = a \wedge Im(z) = b$

c) $z = (a,b) \Rightarrow |z| = \sqrt{a^2+b^2} \wedge \bar{z} = (a,-b)$.

Prop.

a) $z\bar{z} = |z|^2$

b) $|z_1+z_2| \leq |z_1|+|z_2|$.

▼ The Complex plane

Def : The set $\mathbb{C}$ is called the complex plane

Def : (subsets of S).

a) Let $z_0 \in \mathbb{C}$. $N_\varepsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ is called the ε-neighborhood of z_0.

b)

<u>Def</u> : (neighborhoods)
Let $z_0 \in \mathbb{C}$. The ε-neighborhood of $\mathbb{C}$ is

$$N_\varepsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}.$$

<u>Def</u> (limit/interior points)
Let $S \subseteq \mathbb{C}$ and $z_0 \in \mathbb{C}$.
a) z_0 is a <u>limit point</u> $\Leftrightarrow \forall \varepsilon > 0 : N_\varepsilon(z_0) \cap (S - \{z_0\}) \neq \emptyset$
b) z_0 is an <u>interior point</u> $\Leftrightarrow \exists \varepsilon > 0 : N_\varepsilon(z) \subseteq S$.

<u>notation</u> : $\text{int}\, S = \{z \in \mathbb{C} : z \text{ interior to } S\}$.
$\lim S = \{z \in \mathbb{C} : z \text{ limit point to } S\}$.

<u>Def</u> (open/closed sets)
Let $S \subseteq \mathbb{C}$.
a) S is an <u>open set</u> $\Leftrightarrow \text{int}\, S = S$.
b) S is a <u>closed set</u> $\Leftrightarrow \lim S \subseteq S$.

<u>Def</u> (boundary).
Let $S \subseteq \mathbb{C}$ and $z_0 \in \mathbb{C}$.
z_0 <u>boundary point</u> of $S \Leftrightarrow \forall \varepsilon > 0 : N_\varepsilon(z_0) \cap S \neq \emptyset \wedge N_\varepsilon(z_0) \cap (\mathbb{C} - S) \neq \emptyset$

<u>notation</u> : $\partial S = \{z \in \mathbb{C} : z \text{ boundary point of } S\}$.

<u>Def</u> : (bounded sets)
Let $S \subseteq \mathbb{C}$. S is <u>bounded</u> $\Leftrightarrow \exists M > 0 : \forall z \in S : |z| < M$.

<u>Prop</u> :
a) $\emptyset$, $\mathbb{C}$, $N_\varepsilon(z_0)$ are all open sets.
b) $\emptyset$, $\mathbb{C}$ are also closed sets
c) S is an open set $\Leftrightarrow \mathbb{C} - S$ is a closed set.

Curves and regions in complex space

Def: A simple Jordan arc is a set of points C of the form:

$$C = \{ z(t) = x(t) + iy(t) \mid t \in [0,1] \}$$

such that

a) $x(t), y(t)$ are continuous real-valued functions

b) $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$.

Def: A simple smooth arc is a set of points

$$C = \{ z(t) = x(t) + iy(t) \mid t \in [0,1] \}$$

such that

a) $x'(t) = dx/dt$, $y'(t) = dy/dt$ are continuous real-valued functions

b) $x'^2 + y'^2 \neq 0$, $\forall t \in [0,1]$.

Def: A simple closed Jordan arc is a set of points

$$C = \{ z(t) = x(t) + iy(t) \mid t \in [0,1] \}$$

such that

a) $x(t), y(t)$ are continuous real valued functions

b) $z(t_1) = z(t_2) \Leftrightarrow \begin{cases} t_1 = 0 \\ t_2 = 1 \end{cases} \vee \begin{cases} t_1 = 1 \\ t_2 = 0 \end{cases}$

Thm: (Jordan Curve theorem)

If $C \subseteq \mathbb{C}$ simple closed Jordan curves $\Rightarrow \exists R_1, R_2 \subseteq \mathbb{C}$ such that:

a) $R_1 \cap R_2 = \emptyset$

b) $R_1 \cup R_2 \cup C = \mathbb{C}$

c) R_1 is bounded

d) R_2 is not bounded.

Moreover, R_1, R_2 are unique!

notation : $R_1 = \text{int } C \rightsquigarrow$ interior of C

$R_2 = \text{ext } C \rightsquigarrow$ exterior of C.

Def : Let $S \subseteq \mathbb{C}$.

a) S is connected $\iff \forall z_1, z_2 \in S : \exists$ simple Jordan arc $C \subseteq S : z_1, z_2 \in C$.

b) S is a domain $\iff$ S is nonempty, connected and open.

c) S is simply connected $\iff \forall$ simple closed Jordan arc $C \subseteq D : \text{int}\, C \subseteq D$.

Functions of complex variable

Def : A mapping $f : \mathbb{C} \to \mathbb{C}$ is called a single-valued complex function.

Def : (limits)

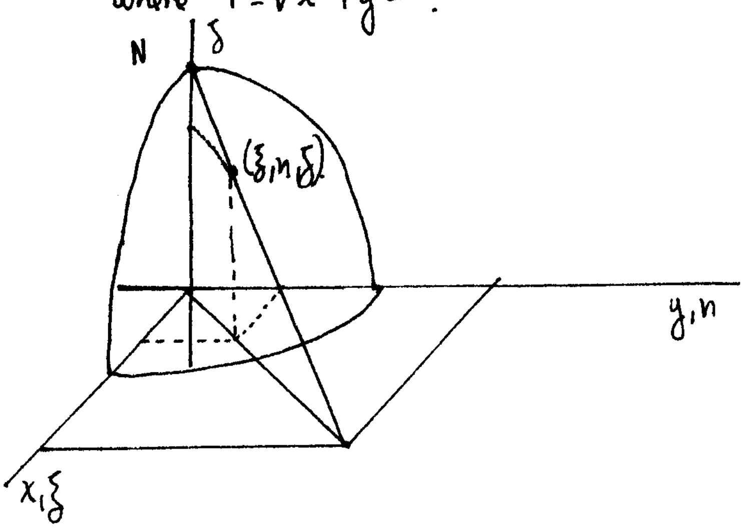
$$\lim_{z \to z_0} f(z) = l \iff \forall \varepsilon > 0, \exists \delta > 0 : z \in N_\delta(z_0) \Rightarrow f(z) \in N_\varepsilon(l).$$

To extend the definition of the limit to include the point at infinity we introduce the idea of stereographic projection.

Def : The stereographic projection is a map $\varphi : \mathbb{C} \to \mathbb{R}^3$ defined by:

$$\varphi(x+yi) = (\xi, \eta, \zeta) \iff \xi = \frac{2x}{r^2+1}, \quad \eta = \frac{2y}{r^2+1}, \quad \zeta = \frac{r^2-1}{r^2+1}$$

where $r = \sqrt{x^2+y^2}$.



Remark : Suppose we have a sphere

$$\begin{cases} \xi^2+\eta^2+\zeta^2 = 1 \\ \zeta > 0 \end{cases}$$

and let (x,y) be at ~~$(\xi,\eta,0)$~~. $(x,y,0)$.
The point (ξ,η,ζ) is the point where the line $(0,0,1) \to (x,y,0)$ intersects the sphere.
We call all the points on the sphere the extended complex plane for the reason that follows.

notation : The extended complex plane is written as $\overline{\mathbb{C}}$.

Prop: Let φ = the stereographic projection.
a) $\varphi(\mathbb{C}) = \mathbb{C} - \{(0,0,1)\}$.
b) $\forall z_1, z_2 \in \mathbb{C} : \varphi(z_1) = \varphi(z_2) \Leftrightarrow z_1 = z_2$.

Remark.
The set $\bar{\mathbb{C}}$ is an extension of $\mathbb{C}$ by one point: $(0,0,1)$.
We call that the point at infinity ∞ and extend complex arithmetic as follows:

$$\infty + z = \infty, \ \forall z \in \mathbb{C} \qquad \infty + \infty \text{, is not defined.}$$
$$\infty z = \infty, \ \forall z \in \mathbb{C} \qquad \infty \cdot \infty = \infty$$

Now to define limits at infinity we use the chordal metric.

Def (chordal metric)
Let $z_1, z_2 \in \bar{\mathbb{C}}$.
$\rho(z_1, z_2)$ = the distance of z_1, z_2 in $\mathbb{R}^3$ space.

Thm (evaluating the chordal metric)

a) $\forall z_1, z_2 \in \bar{\mathbb{C}} : \rho(z_1, z_2) = \dfrac{2|z_1 - z_2|}{\sqrt{1+|z_1|^2} \cdot \sqrt{1+|z_2|^2}}$

b) $\forall z_1 \in \bar{\mathbb{C}} : \rho(z_0, \infty) = \dfrac{2}{\sqrt{1+|z|^2}}$

Thm $\forall z_1, z_2 \in \bar{\mathbb{C}}$:
a) $\rho(z_1, z_2) = \rho(z_2, z_1)$
b) $\rho(z_1, z_2) \leq \rho(z_1, z_3) + \rho(z_3, z_2)$
c) $\rho(z_1, z_2) \geq 0$
d $\rho(z_1, z_2) = 0 \Leftrightarrow z_1 = z_2$.

Remark: The chordal metric is a notion of distance that includes the point at infinity. We can use it to define generalized neighborhoods.

Def : Let $\ell, z_0 \in \overline{\mathbb{C}}$ and $f : \mathbb{C} \to \mathbb{C}$.
$\lim_{z \to z_0} f(z) = \ell \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 : \forall z \in \mathbb{C} : \left(\rho(z, z_0) < \delta \Rightarrow \rho(f(z), \ell) < \varepsilon\right)$

Def : Let $f : \mathbb{C} \to \mathbb{C}$ and $z_0 \in \mathbb{C}$.

a) f continuous at $z_0 \Leftrightarrow \lim_{z \to z_0} f(z) = f(z_0)$.

b) f differentiable at $z_0 \Leftrightarrow \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ converges

c) f analytic at $z_0 \Leftrightarrow \exists \varepsilon > 0 : \forall z \in \{z \in \mathbb{C} : \rho(z, z_0) < \varepsilon\} : f$ differentiable at z.

Prop: f differentiable at $z_0 \Rightarrow f$ continuous at z_0.

notation : If f is analytic in a domain D, then we write
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

▼ Analyticity and Harmonic functions.

Thm : Let $f(x+yi) = u(x,y) + i v(x,y)$, $\forall (x,y) \in \mathbb{R}^2$.
If f analytic at $z_0 = x_0 + y_0 i \Rightarrow \begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0) \\ u_y(x_0, y_0) = -v_x(x_0, y_0). \end{cases}$

Remark : These conditions are called the Cauchy-Riemann conditions
Note that the inverse is not true. However, the following result is a usable "converse".

Thm : Let $f(x+yi) = u(x,y) + i v(x,y)$, $\forall (x,y) \in \mathbb{R}^2$
If $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ at $z = x_0 + y_0 i$ $\wedge \left(\exists \varepsilon > 0 : \forall z \in N_\varepsilon(x+yi) : u_x, u_y, v_x, v_y \text{ continuous at } z.\right)$
then $f(z)$ is differentiable at $z = x_0 + y_0 i$

Prop: If $u(x,y), v(x,y)$ are twice differentiable and
$f(x+yi) = u(x,y) + iv(x,y)$ is analytic in $x+yi \in D$
then $\nabla^2 u = \nabla^2 v = 0$ in D.

Proof

$$f \text{ analytic in } D \Rightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} u_{xx} = v_{yx} \\ u_{yy} = -v_{xy} = -v_{yx} \end{cases} \Rightarrow$$

$\Rightarrow \nabla^2 u = u_{xx} + u_{yy} = v_{yx} - v_{yx} = 0.$
By a similar argument $\nabla^2 v = 0$. D.

Def: A function $u(x,y)$ is called harmonic iff $\nabla^2 u = 0$.
Two functions $u(x,y), v(x,y)$ are conjugate harmonic iff
$f(x+yi) = u(x,y) + iv(x,y)$ is analytic.

▼ Transcendental functions

We begin by attempting to generalize the exponential function $f(x) = e^x, \forall x \in \mathbb{R}$.

Thm: Let $f: \mathbb{C} \to \mathbb{C}$ be a complex valued function such that
a) $f(z)$ analytic in $\mathbb{C}$
b) $f'(z) = f(z)$
c) $f(x) = e^x, \forall x \in \mathbb{C}$.
Then: $f(x+yi) = e^x(\cos y + i \sin y)$. , $\forall x,y \in \mathbb{R}$.

Def: We define the complex exponential as:
$$e^z = e^x(\cos y + i \sin y) \quad \text{where} \quad \begin{array}{l} x = Re(z) \\ y = Im(z) \end{array}$$

Theory of multivalued functions.

Def : Let $\mathcal{P}(\mathbb{C})$ be the set of all subsets of $\mathbb{C}$. A multivalued function is a mapping $f : \mathbb{C} \to \mathcal{P}(\mathbb{C})$.

Remark : To establish a theory we must extend all of our definitions to the multivalued case. Begin with the following:
If $\varphi : \mathbb{C} \to \mathbb{C}$ is a unary operator that applies on numbers, we extend it to functions by defining:

$$\hat{\varphi}(f)(x) = \varphi(f(x)) \ , \ \forall x \in \mathbb{C}.$$

Likewise if $\varphi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a binary operator, we extend it to:

$$\hat{\varphi}(f,g)(z) = \varphi(f(z), g(z)) \ , \ \forall x \in \mathbb{C}.$$

In this manner we have defined:

$$g = -f \Leftrightarrow g(z) = -f(z) \ , \forall z \in \mathbb{C}.$$

$$h = f+g \Leftrightarrow h(z) = f(z) + g(z). \ , \ \forall z \in \mathbb{C} \quad \text{etc.}$$

We now extend this definition as follows:

Def : Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a unary operator and $f : \mathbb{C} \to \mathcal{P}(\mathbb{C})$ a multivalued function. Then:

$$g = \hat{\varphi}(f) \Leftrightarrow g(z) = \varphi(f)(z) = \{\varphi(w) \mid w \in f(z)\}$$

Def : Let $\varphi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be a binary operator. and $f,g : \mathbb{C} \to \mathcal{P}(\mathbb{C})$ multivalued functions. Then

$$h = \varphi(f,g) \Leftrightarrow h(z) = \varphi(f,g)(z) = \{\varphi(w_1, w_2) \mid w_1 \in f(z) \wedge w_2 \in g(z)\}$$

examples

Let f, g be multivalued functions.

$$(f+g)(z) = \{w_1 + w_2 \mid w_1 \in f(z) \wedge w_2 \in g(z)\}$$

$$(fg)(z) = \{w_1 w_2 \mid w_1 \in f(z) \wedge w_2 \in g(z)\}.$$

$$(-f)(z) = \{-w \mid w \in f(z)\}.$$

Now we extend the definition of $f \circ g$.

<u>Def</u>: Let f, g be two multivalued functions. We say that $h = f \circ g$ is the <u>composition</u> of f and g iff

$$h(z) = (f \circ g)(z) = \bigcup_{w \in g(z)} f(w).$$

Some multivalued functions are inverses of single-valued functions.

<u>Def</u>: Let $f : \mathbb{C} \to \mathbb{C}$ be a single-valued function. The inverse f^{-1} is a multivalued function $f^{-1} : \mathbb{C} \to \mathcal{P}(\mathbb{C})$ defined by:

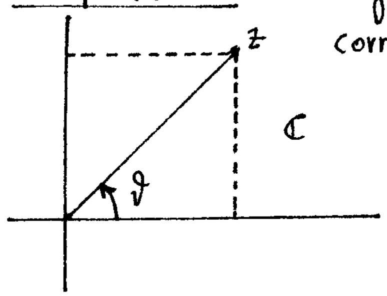
$$f^{-1}(z) = \{ w \in \mathbb{C} : f(w) = z \}$$

All multivalued functions of interest are composites of the argument function.

- <u>The argument function</u>

<u>Def</u>: $\boxed{\arg(z) = \{ \vartheta \in \mathbb{R} \mid |z| e^{i\vartheta} = z \}.}$

<u>interpretation</u>: $\arg z$ gives all the valid angles ϑ that correspond to the "polar coordinates" of z.



<u>Thm</u>: In the multivalued sense

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$
$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

~~Def~~

<u>Thm</u> : Let $z \in \mathbb{C}$ be given. If $\vartheta \in \arg z \Rightarrow \arg z = \{\vartheta + 2k\pi i \mid k \in \mathbb{Z}\}$.

<u>Corollary</u>:
The function $\operatorname{Arg} z = \arg z \cap [-\pi, \pi)$ is single-valued and it is called the <u>principle-value</u> function.

- <u>Argument-based functions and Riemann surfaces.</u>

<u>Def</u> : A multivalued function f is <u>argument-based</u> iff it can be composed with single-valued functions and the argument function $\arg(z)$.
<u>examples</u> : The following multivalued functions are argument based:
a) $\log z = \ln|z| + i\arg(z)$.
b) $z^w = \exp(w \log z)$.

<u>Def</u>: Let f be an argument-based function. The <u>Riemann surface</u> of f is given by:
$$G = \{(z,w) \in \mathbb{C}^2 \mid w \in f(z)\}.$$

<u>Def</u> : Let G be the Riemann surface of an argument based function. f.
a) A <u>Riemann sheet</u> G_1 is a subset $G_1 \subseteq G$ such that
$$\{(z,w_1),(z,w_2)\} \subset G_1 \Rightarrow w_1 = w_2.$$
b) The <u>projection of G_n</u> to the complex plane is defined as:
$$p(G_n) = \{z \in \mathbb{C} \mid \exists w \in \mathbb{C} : (z,w) \in G_n\}.$$
for all $G_n \subseteq G$.

<u>Def</u> : Let $f : \mathbb{C} \to \mathcal{P}(\mathbb{C})$ be a multivalued function with Riemann surface G. We say that a single valued function φ is a <u>Branch</u> of f iff the graph of f is a Riemann sheet of G.

<u>example</u>

The function $\varphi(z) = \operatorname{Arg} z$ is a branch of $f(z) = \arg z$.

<u>Thm</u>: Let G be the Riemann surface of an argument-based function f.

a) Then G can be partitioned as:

$$G = \bigcup_{j \in S} G_j$$

where S is a counting set and $\forall i,j \in S : i \neq j \Rightarrow G_i \cap G_j = \emptyset$.
such that G_j are Riemann sheets.

b) The cardinality of S is a property of f and it is the same for all Riemann sheet partitions of f.

c) Also, $|S|$ is countable $\leadsto$ <u>multivaluedness</u> of f.

d) Let φ_j be the branches of f corresponding to the sheets G_j.
If f is the composition of continuous single valued functions and arg then φ_j is continuous in $p(G_j - \partial G_j)$ where ∂G_j is the boundary of G_j in the G-manifold sense.

<u>Remark.</u>: $p(G_j - \partial G_j) = p(G_j) - p(\partial G_j)$

Usually branches are defined on a domain $p(G_j) = \mathbb{C}$ but they need to be cut by a branch cut $\leadsto p(\partial G_j)$ in order to be continuous in the entire domain.

The Riemann-sheet partitions of G describe the topology of G.

Usually, when we work with multivalued functions, we pick a branch and work on that one. Another approach is to work with contours on the G manifold.

- <u>Contours on the Riemann manifold</u>

~~<u>Thm</u>: Let $C = \{z(t) \mid t \in [0,1]\} \subseteq p(G)$ be a Jordan arc that lies on the projection of the Riemann manifold of G of an argument based function f. Then f has a branch φ such that $\varphi(z(t))$ is continuous.~~

Thm : Let f be an argument-based function which is the composite of continuous single-valued functions. ~~Then~~ Let G be the Riemann manifold of f. If $C=\{z(t)\in\mathbb{C} \mid t\in[0,1]\}\subseteq p(G)$ is a simple Jordan arc, then $\exists\, \varphi$ branch of f : $\varphi(z(t))$ continuous.

● Topology of the Riemann manifold.

Remark : To give meaning to analyticity, derivatives and integrals of multivalued functions we need to work directly with the topology of the Riemann manifold.
Let f be an argument based multivalued function.
Let $G\subseteq\mathbb{C}^2$ be the Riemann manifold of f and let G_j be a Riemann sheet partition of G :

$$G=\bigcup_{j\in S} G_j$$

We show now how to construct a manifold W in $\mathbb{C}\times S$ that will enable us to define analyticity etc.
Let $W_j=\{(z,j)\in\mathbb{C}\times S \mid z\in p(G_j-\partial G_j)\}$
and let W be the union of all W_j: $W=\bigcup_{j\in S} W_j$.
You can think of W_j as a collection of complex planes that have been cut by branch cuts.
The idea is that W is "homeomorphic" to $G-(\bigcup_{j\in S}\partial G_j)$.
In other words, if we exclude $\partial G_j, \forall j\in S$ from G, all other points of G can be mapped to W back and forth.
Let's put this in rigorous terms.
Def : Let M be some manifold and $\varphi: M\times M\longrightarrow\mathbb{R}$.
We say that φ is a metric of M iff:
a) $\varphi(z_1,z_2)=\varphi(z_2,z_1)$
b) $\varphi(z_1,z_2)\leqslant\varphi(z_1,z_3)+\varphi(z_3,z_2)$
c) $\varphi(z_1,z_2)\geqslant 0$
d) $\varphi(z_1,z_2)=0 \Leftrightarrow z_1=z_2$, $\forall z_1,z_2,z_3\in M$.

<u>Def</u>: Let (M_1, φ_1) and (M_2, φ_2) be two manifolds M_1, M_2 with metrics φ_1, φ_2. Let $\psi: M_1 \longrightarrow M_2$ be a map. and $z_0 \in M_1$.
We say that
$$\lim_{z \to z_0} \psi(z) = w \in M_2 \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 : \forall z \in M_1 : \varphi_1(z, z_0) < \delta \Rightarrow \varphi_2(\psi(z), \psi(z_0)) < \varepsilon.$$

<u>Def</u>: Let (M_1, φ_1) and (M_2, φ_2) be two manifolds and $\psi: M_1 \longrightarrow M_2$.
We say that

a) ψ <u>continuous</u> $\Leftrightarrow \forall z_0 \in M_1 : \lim_{z \to z_0} \psi(z) = \psi(z_0)$.

b) ψ is a <u>homeomorphism</u> $\Leftrightarrow \begin{cases} \psi \text{ continuous.} \\ \psi^{-1} \text{ single-valued and continuous.} \end{cases}$

<u>Remark.</u>
This is the most general definition of continuity in the most exotic manifolds. To do are:

a) Define a metric for G and W

b) Find the homeomorphism $\psi: G - \left(\bigcup_{j \in S} \partial G_j\right) \longrightarrow W$.

1) <u>Metric for G</u>

<u>Def</u>: Let $v_1, v_2 \in \mathbb{C}^2$. We define a metric in $\mathbb{C}^2$ as:
$$\Phi(v_1, v_2) = |v_1 - v_2|$$
where $\forall v = (z, w) \in \mathbb{C}^2 : |v| = \sqrt{|z|^2 + |w|^2}$.

<u>Def</u>: Let C be a simple Jordan arc <u>in $\mathbb{C}^2$</u>, given by
$$C = \{v(t) \in \mathbb{C}^2 \mid t \in [0,1]\}.$$
The <u>length</u> of C in $\mathbb{C}^2$ is given by $\ell(C) = \int_0^1 |v'(t)|\, dt$
where $v'(t)$ is the derivative of v.

<u>notation</u>: Let $L(v_1, v_2)$ be the set of all simple Jordan arcs that connect two points $v_1, v_2 \in \mathbb{C}^2$.
$$L(v_1, v_2) = \{\{v(t) \in \mathbb{C}^2 \mid t \in [0,1]\} \mid v(0) = v_1 \wedge v(1) = v_2\}.$$

Now we are ready to define a metric in G.

<u>Def</u> : Let G be the Riemann manifold of an argument based multivalued function f and $v_1, v_2 \in G$. Then we choose

$$\varphi(v_1, v_2) = \inf\{\ell(c) \mid c \in L(v_1, v_2) \wedge c \subseteq G\}.$$

<u>Thm</u> : φ is indeed a metric to G.

2) <u>Metric for W.</u>

<u>Thm</u> : Define $\psi : G - \bigcup_{j \in S} \partial G_j \longrightarrow W$ as follows:

$$\forall v = (z, w) \in G - \bigcup_{j \in S} \partial G_j \ : \ v \in G_j \Rightarrow \psi(v) = (z, j).$$

Then ψ is a "1-1" and "onto" map.

<u>Def</u> : Define a metric $\rho(w_1, w_2)$ for W as follows:

$$\rho(w_1, w_2) = \varphi(\psi^{-1}(v_1), \psi^{-1}(v_2)).$$

<u>Thm</u> : ρ is a metric for W and ψ is a homeomorphism from $(G - \bigcup_{j \in S} \partial G_j, \varphi)$ to (W, ρ).

- <u>Application on the argument function.</u>

<u>Prop.</u> : $\forall z \in \mathbb{C}$: $\boxed{\arg(z) = \{\mathrm{Arg}(z) + 2k\pi i \mid k \in \mathbb{Z}\}.}$

Note that $\varphi_{\vartheta_0}(z) = \arg(z) \cap [\vartheta_0, \vartheta_0 + 2\pi)$ is single valued.
Also note that for a given $\vartheta_0 \in \mathbb{R}$ if we define:

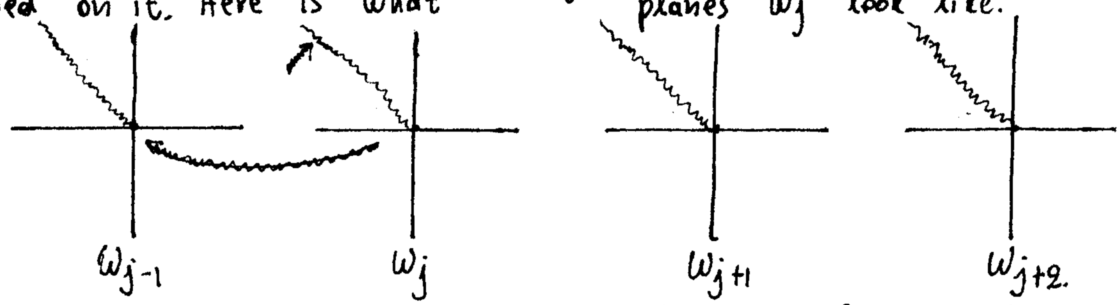
$$\varphi_k(z) = \arg(z) \cap [\vartheta_0 + 2\pi i k, \vartheta_0 + 2\pi i(k+1))$$

then $\arg(z) = \{\varphi_k(z) \mid k \in \mathbb{Z}\}$ so $\varphi_k(z)$ is a "complete" set of branches. If G_k is the graph of φ_k then G_k are a Riemann sheet partition of G.

The cardinality of the partitions is $|\mathbb{Z}|$ so we set $S = \mathbb{Z}$.
Let $W_j = \left(\mathbb{C} - \{\lambda e^{i\theta_0} \mid \lambda \in [0,+\infty]\}\right) \times \{j\}$. and

$$W = \bigcup_{j \in \mathbb{Z}} W_j .$$

Then W is homeomorphic with $G - \bigcup_{j \in S} \partial G_j$ and W has the norm ρ defined on it. Here is what planes W_j look like:



The red ink indicates the branch cuts, i.e. the areas of $\mathbb{C}$ that are taken out when defining W_j.
The green ink indicates where you go when you hit a branch cut clockwise. If you hit the branch cut counterclockwise follow ~~the~~ back the green arrow that comes to you.

Remark.
Since $\log z = \ln|z| + i\arg(z)$, the topology of the Riemann manifold of log is the same.

- Branch points.

Def : Let f be an argument based multivalued function and let W_j be a set of Riemann sheets. that corespond to a Riemann partition of G.

$z \in \overline{\mathbb{C}}$ branch point of $W_j \Leftrightarrow \exists \varepsilon > 0 : \forall$ closed contour C in $N_\varepsilon(z_0)$: $C \times \{j\}$ is not closed.

example
In the case of the argument function all sheets W_j have branch points at $z=0$ and $z=\infty$.

Remark
A branch point is a point in $\mathbb{C}$, not W_j. It is a property of W_j though. Sometimes, different sheets have different branch points.

▼ The root function.

Consider $f(z) = z^{1/K} = \exp\left(\frac{1}{K}\log z\right) = \exp\left(\frac{1}{K}\ln|z| + \frac{i}{K}\arg(z)\right) =$

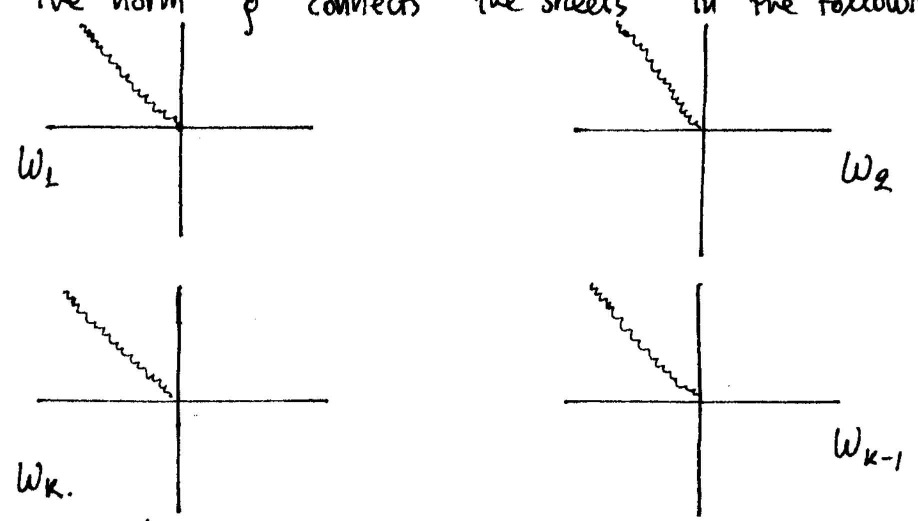
$$= |z|^{1/K} \exp\left(\frac{\arg(z)}{K} i\right).$$

The cardinality of the Riemann-sheet partitions is $|S| = K$ and

$$W_j = \left(\mathbb{C} - \{\lambda e^{i\vartheta_0} \mid \lambda \in [0,+\infty]\}\right) \times \{j\}, \quad \forall j \in [k].$$

$$W = \bigcup_{j \in [k]} W_j$$

and the norm ρ connects the sheets in the following way:



In all sheets W_j the points $z = 0, \infty$ are branch points.

▼ Examples of composite multivalued functions

example 1: $f(z) = (z^2+1)^{1/2}$.

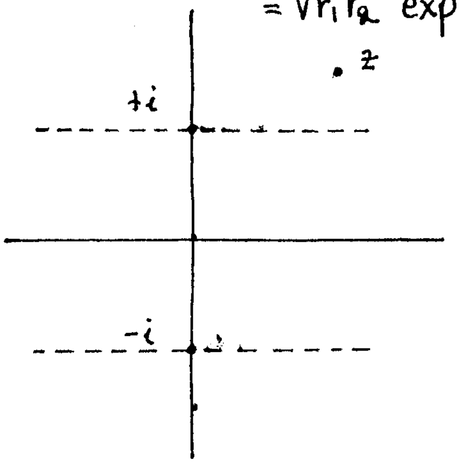
Let $f(z) = (z^2+1)^{1/2} = [(z+i)(z-i)]^{1/2} = (z+i)^{1/2}(z-i)^{1/2}$.

Write $z+i = r_1 \exp(i\vartheta_1)$ where $r_1 = |z+i|$, $\vartheta_1 = \arg(z+i)$

$z-i = r_2 \exp(i\vartheta_2)$. $r_2 = |z-i|$, $\vartheta_2 = \arg(z-i)$.

Then $f(z) = [r_1 \exp(i\vartheta_1)]^{1/2}[r_2 \exp(i\vartheta_2)] =$

$$= \sqrt{r_1 r_2}\, \exp\left[\frac{i(\vartheta_1+\vartheta_2)}{2}\right]$$



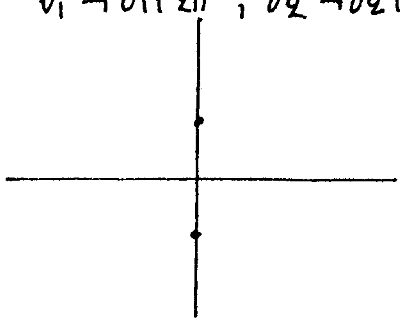
- $\forall$ closed contours C that donot include $\{\pm i\}$ $\vartheta_1 \xrightarrow{C} \vartheta_1$, $\vartheta_2 \xrightarrow{C} \vartheta_2$
  $\Rightarrow$ $\mathbb{C} - \{+i, -i\}$ are not branch points.

- Let C_1 be a curve that surrounds $+i$.
  $\vartheta_1 \xrightarrow{C_1} \vartheta_1$, $\vartheta_2 \xrightarrow{C_1} \vartheta_2 + 2\pi \Rightarrow$
  $\Rightarrow \dfrac{\vartheta_1+\vartheta_2}{2} \xrightarrow{C_1} \dfrac{\vartheta_1+\vartheta_2}{2} + \pi. \Rightarrow$

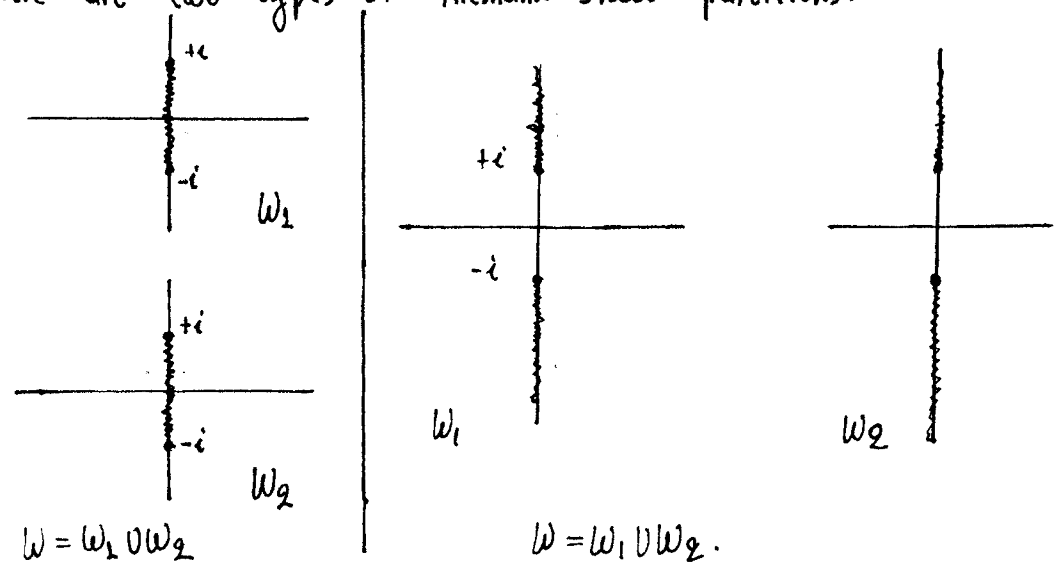
$\Rightarrow f(z) \xrightarrow{C} e^{i\pi} f(z) \Rightarrow +i$ is a branch point.

- Let C_2 be a curve that surrounds $-i$.
  $\vartheta_1 \to \vartheta_1 + 2\pi$, $\vartheta_2 \to \vartheta_2 \Rightarrow f(z) \to e^{i\pi} f(z) \Rightarrow -i$ is a branch point.

- Let C_3 be a curve that surrounds ∞.
  $\vartheta_1 \to \vartheta_1 + 2\pi$, $\vartheta_2 \to \vartheta_2 + 2\pi \Rightarrow f(z) \to f(z) \Rightarrow \infty$ is NOT a branch point!



There are two types of Riemann sheet partitions:



<u>example 2.</u> : $w(z) = [1 + (z^2+1)^{1/2}]^{1/2}$

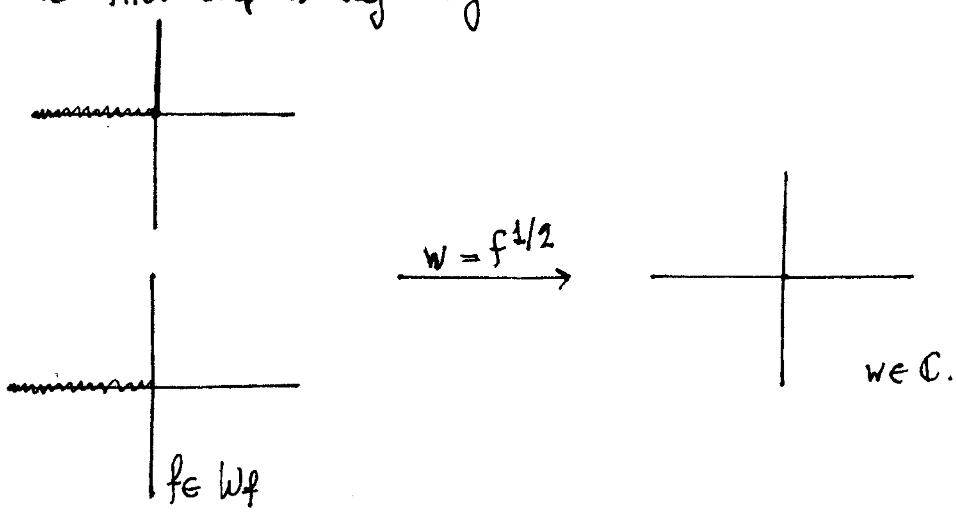
Let's look at the Riemann sheets.

w can be decomposed as: $w = f^{1/2}$, $f = 1+s$, $s = (z^2+1)^{1/2}$.

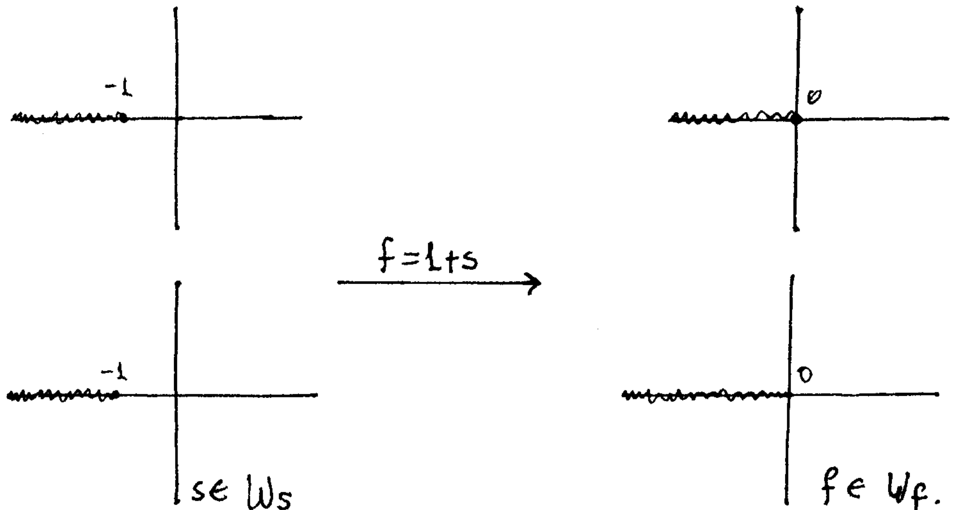
Naturally: $w \in \mathbb{C}$, $f \in W_f$, $s \in W_s$, $z \in W$.

To determine W, we need to determine W_f and W_s.

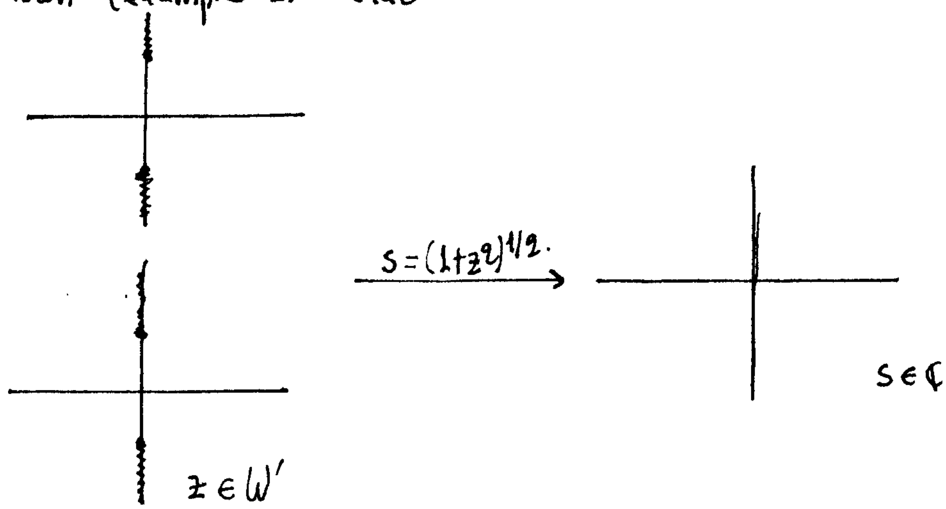
The first step is ~~eas~~ easy:



For the second step $f = f(s)$ is $f : W_s \to W_f$ therefore since f is single valued, W_s will have to be like:



Now the last step $s=(1+z^2)^{1/2}$ is tricky. If $s\in\mathbb{C}$ then we have shown (example 1) that:



Given this we need to modify W' so that the function can be defined with $s\in W_s$. To do this we apply the following methodology

Methodology.

- 1 Duplicate W' with it's current branch cuts as many times as there are planes in W_s.
- 2 Cut out even more points to make sure that no points in W get mapped on a branch cut of W_s.

b) If $z \in W_2 \cup W_4 \Rightarrow Re(s) = Re[(z^2+1)^{1/2}] \leq 0$, therefore given $x>0$:
$(z^2+1)^{1/2} = -1-x \Leftrightarrow z^2+1 = (1+x)^2 \Leftrightarrow z^2 = (1+x)^2 - 1 = 1+2x+x^2-1 = x(x+2) \Leftrightarrow$
$\Leftrightarrow z = \sqrt{x(x+2)} \quad \vee \quad z = -\sqrt{x(x+2)}$

therefore we must add cuts to W_2 and W_4.

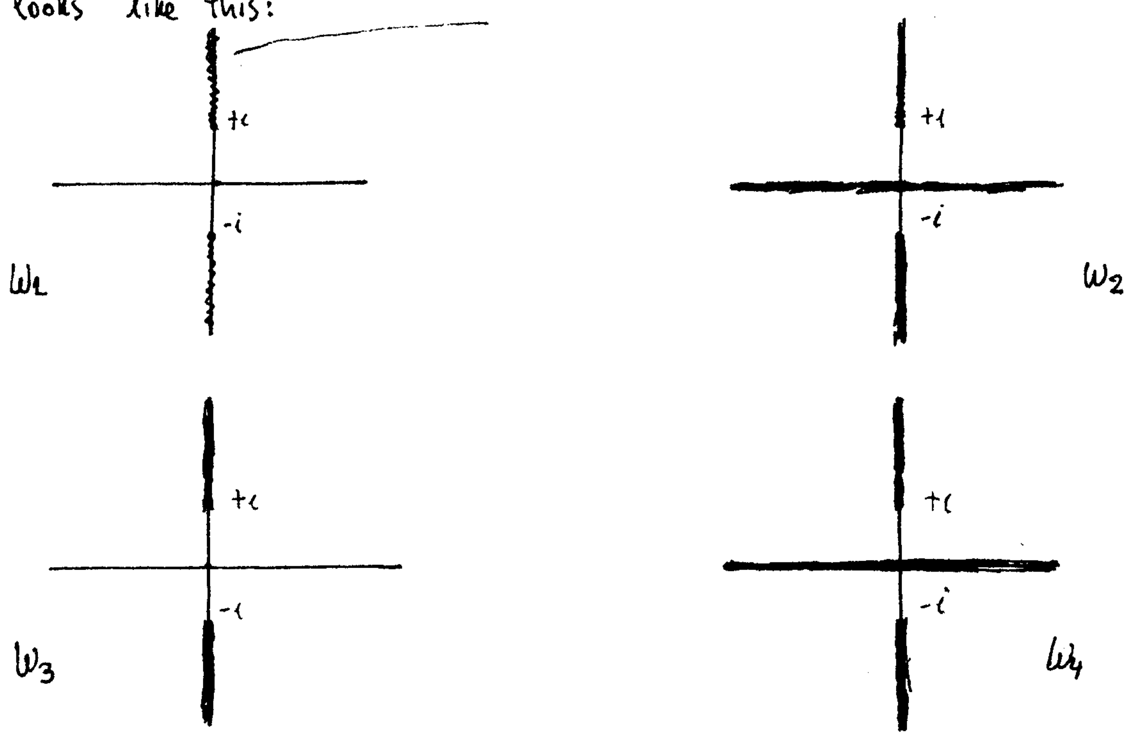
$x>0 \longrightarrow \cancel{z>0}\ z = \sqrt{x(x+2)} > 0$

$x>0 \longrightarrow \quad z = \sqrt{x(x+2)} < 0.$

so we must exclude the entire $\mathbb{R}$ from both W_2 and W_4.

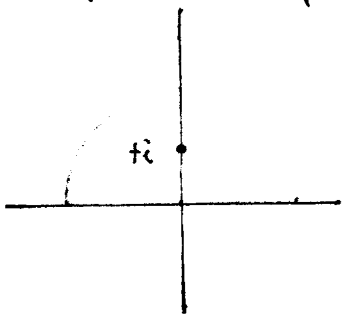
Adding such a branch cut begs the question: Where do we go when we cross it?

Note that we know full ~~wh~~ well where a point $s \in W_s$ goes when it crosses the branch cuts in W_s. If we project s to the image $z \in W$ we see that when we cross the real line in W_2, W_4 we flip between those two sets. This is to be expected since $W_1 \cup W_2$ and $W_3 \cup W_4$ have to be connected somehow. This extra information is added and W looks like this:



Examples of contour integration.

<u>example 1.</u> : Compute $I = \int_0^{+\infty} \frac{dx}{1+x^2}$.



Let $\Gamma_R = C_R \cup \gamma_R$ be the closed contour shown. Then

$$I = \int_0^{+\infty} \frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \lim_{R\to+\infty} \int_{C_R} \frac{dz}{1+z^2} =$$

$$= \lim_{R\to+\infty} \oint_{\Gamma_R} \frac{dz}{1+z^2} - \lim_{R\to+\infty} \oint_{\gamma_R} \frac{dz}{1+z^2} .$$

$$I_\gamma(R) = \int_{\gamma_R} \frac{dz}{1+z^2} = \frac{1}{2}\int_0^{\pi} \frac{iRe^{i\vartheta}}{1+R^2e^{2\vartheta i}}\, d\vartheta \Rightarrow$$

$$\Rightarrow \forall R>0 : |I_\vartheta(R)| \leqslant \frac{1}{2}\int_0^{\pi} \left| \frac{iRe^{i\vartheta}}{1+R^2e^{2i\vartheta}} \right| d\vartheta \leqslant \frac{1}{2}\int_0^{\pi} \frac{R\, d\vartheta}{|R^2-1|} = \frac{R}{2(R^2-1)}\pi \Rightarrow$$

$$\Rightarrow \lim_{R\to+\infty} \int_{\gamma_R} \frac{dz}{1+z^2} = 0 \Rightarrow$$

$$\Rightarrow I = \frac{1}{2}\lim_{R\to+\infty} \oint_{\Gamma_R} \frac{dz}{1+z^2} = 2\pi i \operatorname{Res}\left(\frac{1}{1+z^2} ; +i\right) = 2\pi i \lim_{z\to i} \frac{z-i}{z^2+1} =$$

$$= 2\pi i \lim_{z\to i} \frac{1}{z+i} = 2\pi i \frac{1}{i+i} = \pi/2. \quad \square$$

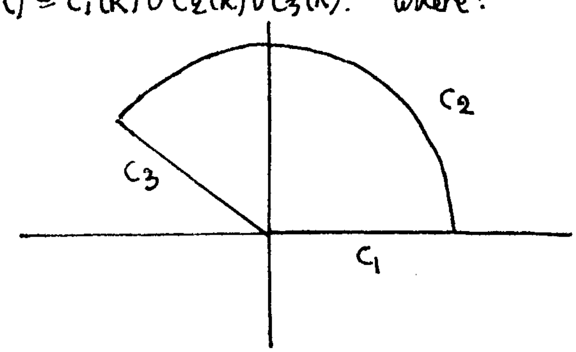
<u>example 2.</u> : Compute $I = \int_0^{+\infty} \frac{dx}{1+x^3}$.

Let C be the closed contour $C(R) = C_1(R) \cup C_2(R) \cup C_3(R)$. where:

$C_1(R) = \{ \lambda \mid \lambda \in [0,R] \}$.

$C_2(R) = \{ Re^{i\vartheta} \mid \vartheta \in [0, 2\pi/3] \}$

$C_3(R) = \{ \lambda e^{2\pi i/3} \mid \lambda \in [0,R] \}$



Let $f(z) = \frac{1}{1+z^3}$

$$I=\int_0^{+\infty}\frac{dx}{1+x^3}=\lim_{R\to+\infty}\left[\oint_{C(R)}f(z)dz-\int_{C_2(R)}f(z)dz-\int_{C_3(R)}f(z)dz\right].$$

$$\forall R>0:\left|\int_{C_2(R)}f(z)dz\right|=\left|\int_0^{2\pi/3}\frac{iRe^{i\vartheta}d\vartheta}{1+R^3e^{3\vartheta i}}\right|\leqslant\int_0^{2\pi/3}\frac{R}{|1+R^3e^{3\vartheta i}|}d\vartheta\leqslant\int_0^{2\pi/3}\frac{R}{R^3-1}d\vartheta=$$

$$=\frac{\pi R}{R^3-1}\Rightarrow\lim_{R\to+\infty}\int_{C_2(R)}f(z)dz=0\Rightarrow$$

$$\Rightarrow I=\lim_{R\to+\infty}\left[\oint_{C(R)}f(z)dz-\int_{C_3(R)}f(z)dz\right].$$

On $z\in C_3(R)\Rightarrow z=re^{2\pi i/3}$, $r\in[0,R]$ and $dz=e^{2\pi i/3}dr$, therefore

$$\int_{C_3(R)}f(z)dz=\int_{C_3(R)}\frac{dz}{1+z^3}=\int_0^{R}\frac{\cancel{r}e^{2\pi i/3}dr}{1+r^3(e^{2\pi i/3})^3}=e^{2\pi i/3}\int_0^R\frac{dr}{1+r^3}\Rightarrow$$

$$\Rightarrow\lim_{R\to+\infty}\int_{C_3(R)}f(z)dz=e^{2\pi i/3}\int_0^{+\infty}\frac{dr}{1+r^3}=e^{2\pi i/3}I.\Rightarrow$$

$$\Rightarrow I=\lim_{R\to+\infty}\oint_{C(R)}f(z)dz-e^{2\pi i/3}I\Rightarrow$$

$$\Rightarrow(1+e^{2\pi i/3})I=\lim_{R\to+\infty}\oint_{C(R)}f(z)dz=2\pi i\operatorname{Res}(f;e^{\pi i/3})=2\pi i\lim_{z\to e^{\pi i/3}}\left[(z-e^{\pi i/3})f(z)\right]=$$

$$=2\pi i\lim_{z\to e^{\pi i/3}}\frac{z-e^{\pi i/3}}{z^3+1}=2\pi i\lim_{z\to e^{\pi i/3}}\frac{1}{(z+1)(z-e^{-\pi i/3})}=$$

$$=2\pi i\,\cancel{\lim}\,\frac{}{(e^{\pi i/3}+1)(e^{\pi i/3}-e^{-\pi i/3})}\Rightarrow$$

$$\Rightarrow I=\frac{2\pi i}{(e^{\pi i/3}+1)(e^{\pi i/3}-e^{-\pi i/3})(1+e^{2\pi i/3})}=\frac{\pi}{\sin(\pi/3)}\,\frac{}{(1+}\ =$$

• Jordan's Lemma.

Thm: Let $f(z)$ be such that $\forall z \in \mathbb{C} : |z| = R \Rightarrow |f(z)| \leq M(R)$ with $\lim\limits_{R \to +\infty} M(R) = 0$.

Let $C(R) = \{ Re^{i\vartheta} \mid \vartheta \in [\vartheta_1, \vartheta_2] \}$.

a) If $\forall z \in C(R) : \mathrm{Im}(z) \geq 0 \Rightarrow \lim\limits_{R \to +\infty} \int_{C(R)} f(z) e^{imz} \, dz = 0$, $\forall m > 0$.

b) If $\forall z \in C(R) : \mathrm{Im}(z) \leq 0 \Rightarrow \lim\limits_{R \to +\infty} \int_{C(R)} f(z) e^{-imz} \, dz = 0$, $\forall m > 0$.

c) If $\forall z \in C(R) : \mathrm{Re}(z) \geq 0 \Rightarrow \lim\limits_{R \to +\infty} \int_{C(R)} f(z) e^{-mz} \, dz = 0$, $\forall m > 0$.

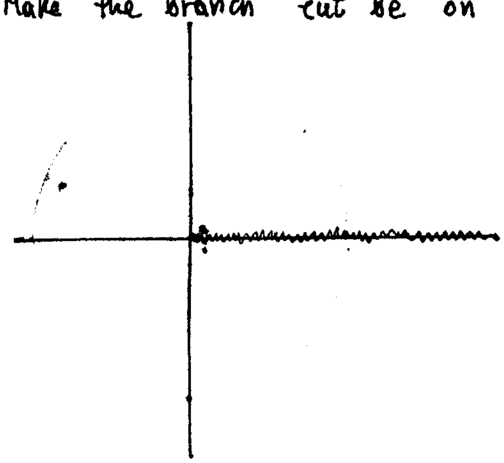
d) If $\forall z \in C(R) : \mathrm{Re}(z) \leq 0 \Rightarrow \lim\limits_{R \to +\infty} \int_{C(R)} f(z) e^{+mz} \, dz = 0$, $\forall m > 0$.

example 3: $I = \displaystyle\int_0^{+\infty} \frac{dx}{x^2 + 3x + 2}$

Note that we can not take $[0, +\infty) \to (-\infty, +\infty)$ as in example 1 ore we the technique of example 2. In this case if, in general, you need to do $I = \displaystyle\int_0^{+\infty} \frac{dx}{f(x)}$, take $g(z) = \dfrac{\log z}{f(z)}$

▸ In this case take $f(z) = \dfrac{\log z}{z^2 + 3z + 2}$

▸ Make the branch cut be on $[0, +\infty)$.



$C_1(R, \varepsilon) = \{ r + \varepsilon i \mid r \in [0, R] \}$

$C_2(R, \varepsilon) = \{ Re^{i\vartheta} \mid \vartheta \in (0, 2\pi) \}$

$C_3(R, \varepsilon) = \{ r - \varepsilon i \mid r \in [0, R] \}$

$C_4(R, \varepsilon) = \{ \varepsilon e^{i\vartheta} \mid \vartheta \in (0, 2\pi) \}$

On C_1: $\lim\limits_{(R,\varepsilon)\to(+\infty,0)} \int_{C_1} f(z)dz = \int_0^{+\infty} \frac{\ln x}{x^2+3x+2}\,dx$

On C_3: $\lim\limits_{(R,\varepsilon)\to(+\infty,0)} \int_{C_3} f(z)dz = -\int_0^{+\infty} \frac{\ln x + 2\pi i}{x^2+3x+2}\,dx$ $\Rightarrow$

$$\Rightarrow \lim_{(R,\varepsilon)\to(\infty,0)} \int_{C_1\cup C_3} f(z)\,dz = -2\pi i \int_0^{+\infty} \frac{dx}{x^2+3x+2} = -2\pi i I.$$

On C_2:

$$\forall R>0: \left|\int_{C_2} f(z)dz\right| = \left|\int_0^{2\pi} \frac{\log(Re^{i\vartheta})\, iRe^{i\vartheta}}{R^2e^{2\vartheta i}+3Re^{\vartheta i}+2}\,d\vartheta\right| \leq \int_0^{2\pi} \frac{(\ln R+i\vartheta)R}{R^2-3R-2}\,d\vartheta \to 0 \Rightarrow$$

$$\Rightarrow \lim_{(R,\varepsilon)\to(+\infty,0)} \int_{C_2} f(z)dz = 0.$$

On C_4:

$$\forall \varepsilon>0: \left|\int_{C_4} f(z)dz\right| = \left|\int_{3\pi/2}^{\pi/2} \frac{\log(\varepsilon e^{i\vartheta})\, i\varepsilon e^{i\vartheta}}{\varepsilon^2e^{2\vartheta i}+3\varepsilon e^{\vartheta i}+2}\,d\vartheta\right| \leq \int_{3\pi/2}^{\pi/2} \frac{(\ln\varepsilon+i\vartheta)}{|\varepsilon^2-3\varepsilon-2|}\,d\vartheta \to 0 \Rightarrow$$

$$\Rightarrow \lim_{(R,\varepsilon)\to(+\infty,0)} \int_{C_4} f(z)dz = 0$$

Therefore:

$$I = \int_0^{+\infty} \frac{dx}{x^2+3x+2} = \frac{-1}{2\pi i} \lim_{(R,\varepsilon)\to(\infty,0)} \int_{C_1\cup C_3} f(z)dz = \frac{-1}{2\pi i} \lim_{(R,\varepsilon)\to(\infty,0)} \oint_C f(z)dz$$

$$= -\frac{1}{2\pi i} 2\pi i\left[\operatorname{Res}(f;z_1)+\operatorname{Res}(f,z_2)\right] = -\left(\operatorname{Res}(f;z_1)+\operatorname{Res}(f;z_2)\right).$$

where $z_{1,2}$ are the roots of $z^2+3x+2=0$, $\Delta=9-8=1$ $\Rightarrow z_{1,2} = \frac{-3\pm1}{2} = \begin{cases} -2 \\ -1. \end{cases}$

therefore

$$I = -\operatorname{Res}(f;-1) - \operatorname{Res}(f;-2) =$$

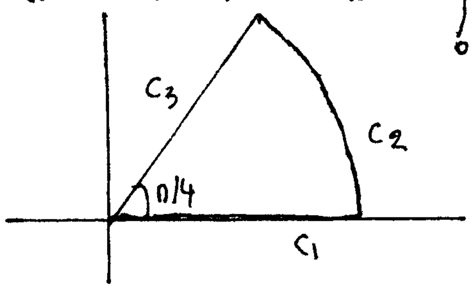
$$= -\lim_{z\to-1} \frac{(z+1)\log(z)}{z^2+3z+2} - \lim_{z\to-2} \frac{(z+2)\log(z)}{z^2+3z+2} =$$

$$= -\lim_{z\to-1} \frac{\log(z)}{z+2} - \lim_{z\to-2} \frac{\log(z)}{z+1} =$$

$$= -\frac{\log(-1)}{1} - \frac{\log(-2)}{-1} = -(0+\pi i) + (\ln 2+\pi i) = \ln 2.$$

<u>example 4.</u> : $F_1(a) = \int_0^{+\infty} \cos(ax^2)\,dx$, $F_2(a) = \int_0^{+\infty} \sin(ax^2)\,dx$.

Let $F(a) = F_1(a) + iF_2(a) = \int_0^{+\infty} e^{iaz^2}\,dz$.



Take $C_1 = \{r \mid r \in [0,R]\}$.
$C_2 = \{Re^{i\vartheta} \mid \vartheta \in [0,\pi/4]\}$
$C_3 = \{re^{\pi i/4} \mid r \in [0,R]\}$.

$$\lim_{R\to+\infty} \int_{C_1} e^{iaz^2}\,dz = \int_0^{+\infty} e^{iaz^2}\,dz = F(a).$$

$$\lim_{R\to+\infty} \int_{C_3} e^{iaz^2}\,dz = \int_{+\infty}^{0} e^{iar^2} e^{\pi i/2} e^{\pi i/4}\,dr = -\int_0^{+\infty} e^{-ar^2} e^{\pi i/4}\,dr =$$

$$= -e^{\pi i/4} \int_0^{+\infty} e^{-ar^2}\,dr = -e^{\pi i/4}\left(\frac{\pi}{4a}\right)^{1/2} = -\frac{e^{\pi i/4}}{2}\left(\frac{\pi}{a}\right)^{1/2}.$$

$$\forall R>0: \left|\int_{C_2} f(z)\,dz\right| = \int_0^{\pi/4} \left|e^{iaR^2(\cos\vartheta + i\sin\vartheta)^2}\, iRe^{i\vartheta}\right| d\vartheta \le \int_0^{\pi/4} e^{-2aR^2\sin\vartheta\cos\vartheta} R\,d\vartheta \to 0$$

as $R\to\infty$

therefore $\lim_{R\to+\infty} \int_{C_2} f(z)\,dz = 0$.

$$\text{So } \lim_{R\to+\infty} \oint_C f(z)\,dz = \lim_{R\to+\infty} \int_{C_1} f(z)\,dz + \lim_{R\to+\infty} \int_{C_2} f(z)\,dz = F(a) - \frac{e^{\pi i/4}}{2}\left(\frac{\pi}{a}\right)^{1/2} = 0$$

$$\to F(a) = \frac{+e^{\pi i/4}}{2}\left(\frac{\pi}{a}\right)^{1/2}.$$

Contour integration.

Def : Let $C = \{z(t) \in \mathbb{C} \mid t \in [0,1]\}$ be a contour ~~with~~ (either closed or non-closed) and let $f(z)$ be a continuous function in $\mathbb{C}$.
We define the contour integral of $f(z)$ on C to be given by:

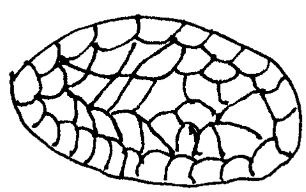
$$\int_C f(z)\,dz = \int_0^1 f(z(t))\,z'(t)\,dt$$

Cauchy Theorem

Thm : If C is a simple closed contour and $f(z)$ analytic in $A \subseteq \mathbb{C}$ then if $C \subseteq A$:

$$\oint_C f(z)\,dz = 0.$$

Proof outline.
Subdivide C into a mesh M which is a collection of contours that look like this:



Then $\displaystyle\oint_C f(z)\,dz = \sum_{C_j \in M} \oint_{C_j} f(z)\,dz$ because the interior edges of the mesh cancel.
Let $z_0 \in \operatorname{int} C$ be a point inside C.

f analytic at $z_0 \in \operatorname{int} C \Rightarrow \displaystyle\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0) \Rightarrow$

$\Rightarrow \forall \varepsilon > 0,\ \exists \delta > 0 : |z - z_0| < \delta \Rightarrow \left| \dfrac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \right| < \varepsilon \Rightarrow$

$\Rightarrow |f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon |z - z_0|$

Define $\varphi(z)$ such that: $f(z) = f(z_0) + f'(z_0)(z-z_0) + \varphi(z)$.
Then

$$|\varphi(z)| = |f(z) - f(z_0) - f'(z_0)(z-z_0)| \leqslant \varepsilon(z-z_0), \quad \forall z : |z-z_0| < \delta.$$

Given this:

$$\oint_C f(z)dz = \oint_C \left[f(z_0) + f'(z_0)(z-z_0) + \varphi(z) \right] dz =$$

$$= \oint_C f(z_0)\, dz + \oint_C f'(z_0)(z-z_0)\, dz + \oint_C \varphi(z)\, dz =$$

$$= f(z_0) \oint_C dz + f'(z_0) \oint_C (z-z_0) dz + \oint_C \varphi(z) dz$$

Notice that $\oint_C dz = 0$ and $\oint_C (z-z_0)\, dz = 0$.

It follows that

$$\oint_C f(z)dz = \oint_C \varphi(z) dz = \sum_{C_j \in M} \oint_{C_j} \varphi(z) dz \Rightarrow$$

$$\Rightarrow \left| \oint_C f(z) dz \right| = \left| \sum_{C_j \in M} \oint_{C_j} \varphi(z) dz \right| \leqslant \sum_{C_j \in M} \left| \oint_{C_j} \varphi(z) dz \right| \leqslant$$

$$\leqslant \sum_{C_j \in M} \oint_{C_j} |\varphi(z)|\, dz \leqslant \sum_{C_j \in M} \oint_{C_j} \varepsilon |z-z_0|\, dz \overset{|z-z_0|<\delta}{\leqslant}$$

$$\leqslant \sum_{C_j \in M} \varepsilon\delta \oint_{C_j} dz = \sum_{C_j \in M} \varepsilon\delta \cdot \ell(C_j).$$

Note that $\ell(C_j) \sim \delta$ and $|M| \sim \delta^{-2}$, therefore.

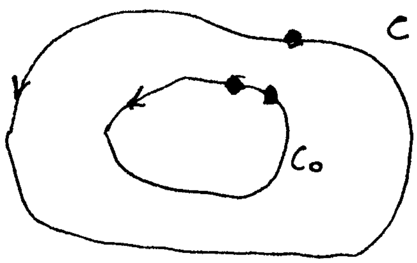
$$\left| \oint_C f(z) dz \right| \leqslant \varepsilon, \quad \forall \varepsilon > 0 \Rightarrow \oint_C f(z) dz = 0.$$

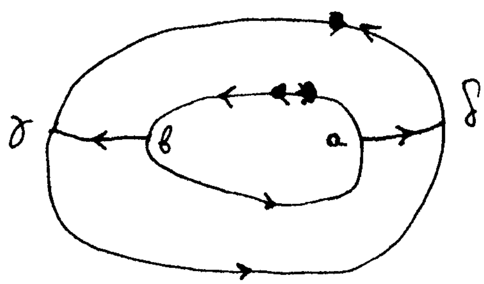
The deformation theorem.

Thm: Let C be a closed contour and C_0 another contour $C_0 \subseteq \text{int}\, C$. If in the region $A = \text{int}\, C - \text{int}\, C_0$, $f(z)$ is analytic, then

$$\oint_C f(z)\,dz = \oint_{C_0} f(z)\,dz. \quad \text{if } C \text{ have the same orientation.}$$

Proof





Let $a, b \in C_0$ and ~~c,d~~ $\gamma, \delta \in C$.

Define $C_{a\delta}$ = a contour from a to δ

$C_{b\gamma}$ = a contour from b to γ.

C_{ab} = a contour from a, moving counterclockwise to b

C_{ba} = a contour from b, moving counterclockwise to a

~~Then by Cauchy's thm:~~

Define $C_{\gamma\delta}$, $C_{\delta\gamma}$ similarly (counterclockwise).

Then by Cauchy's theorem:

$$\int_{C_{a\delta}} f(z)\,dz + \int_{C_{\delta\gamma}} f(z)\,dz - \int_{C_{b\gamma}} f(z)\,dz - \int_{C_{ab}} f(z)\,dz = 0 \qquad (1)$$

$$\int_{C_{b\gamma}} f(z)\,dz + \int_{C_{\gamma\delta}} f(z)\,dz - \int_{C_{a\delta}} f(z)\,dz - \int_{C_{ba}} f(z)\,dz = 0 \qquad (2)$$

Summing (1)+(2):

$$-\left[\int_{C_{ab}} f(z)dz + \int_{C_{\beta a}} f(z)dz\right] + \left[\int_{C_{\gamma\delta}} f(z)dz + \int_{C_{\delta\gamma}} f(z)dz\right] = 0 \Rightarrow$$

$$\Rightarrow \oint_C f(z) - \oint_{C_0} f(z)dz = 0 \Rightarrow \oint_C f(z)dz = \oint_{C_0} f(z)dz.$$

Corollary: Let C be a simple closed contour and let $z_0 \in \operatorname{int} C$. If C is counterclockwise, then

$$\oint_C \frac{1}{z-z_0}\,dz = 2\pi i$$

Proof-sketch: Deform C to a circle around z_0 and reduce the integral to a line integral

- ~~Functions defined by integration.~~

Thm

Cauchy integration formula

Thm: Let $f(z)$ be analytic within and on a simple closed contour C. Then if $z \in \operatorname{int} C$ and C counterclockwise:

$$\boxed{f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)\,d\zeta}{\zeta - z}}$$

Proof

$z \in \operatorname{int} C \Rightarrow \exists R > 0 : \forall \rho < R : C_0 = \{\zeta \in \mathbb{C} \mid |\zeta - z| = \rho\} \subset \operatorname{int} C$

Then, as we have shown:

$$f(z) = \frac{f(z)}{2\pi i} 2\pi i = \frac{f(z)}{2\pi i} \oint_{C_0} \frac{d\zeta}{\zeta - z}$$

and $$\frac{1}{2\pi i} \oint_C \frac{f(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi i} \oint_{C_0} \frac{f(z)d\zeta}{\zeta - z}.$$

It is sufficient to show that

$$\frac{1}{2\pi i}\left[\oint_{C_0} \frac{f(\zeta)d\zeta}{\zeta - z} - \oint_{C_0} \frac{f(z)d\zeta}{\zeta - z}\right] = 0.$$

f analytic at $z \Rightarrow f$ continuous at $z \Rightarrow \lim_{\zeta \to z} f(\zeta) = f(z) \Rightarrow$

$\Rightarrow \forall \varepsilon > 0 : \exists \delta > 0 : \forall \zeta \in \mathbb{C} : |\zeta - z| < \delta \Rightarrow |f(\zeta) - f(z)| < \varepsilon.$

Choose $\rho < \delta$. Then:

$$\left|\frac{1}{2\pi i}\left[\oint_{C_0} \frac{f(\zeta)d\zeta}{\zeta - z} - \oint \frac{f(z)d\zeta}{\zeta - z}\right]\right| = \left|\frac{1}{2\pi i}\oint_{C_0} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta\right| \leqslant$$

$$\leqslant \frac{1}{2\pi i}\oint_{C_0} \frac{|f(\zeta) - f(z)|}{|\zeta - z|}\,d\zeta \leqslant \frac{1}{2\pi i}\,\varepsilon \oint_{C_0} \frac{d\zeta}{|\zeta - z|} = \varepsilon \quad , \forall \varepsilon > 0 \Rightarrow$$

$$\Rightarrow \oint_{C_0} \frac{f(\zeta)d\zeta}{\zeta - z}\,d\zeta = \oint_{C_0} \frac{f(z)d\zeta}{\zeta - z} \Rightarrow f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)d\zeta}{\zeta - z} \quad \text{D.}$$

<u>Thm</u> : Let $f(z)$ be analytic within a simple closed contour C. Then if $z \in \operatorname{int} C$ and C is counterclockwise:

$$\boxed{f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta.}$$

<u>Proof</u>

$$f'(z) = \lim_{\xi \to z} \frac{f(\xi) - f(z)}{\xi - z} = \lim_{\xi \to z} \frac{1}{\xi - z} \oint_C \left[\frac{f(\zeta)}{\zeta - \xi} - \frac{f(\zeta)}{\zeta - z} \right] d\zeta =$$

$$= \lim_{\xi \to z} \left[\frac{1}{\xi - z} \oint_C \left(\frac{\zeta - z - \zeta + \xi}{(\zeta - \xi)(\zeta - z)} \right) f(\zeta)\, d\zeta \right] =$$

$$= \lim_{\xi \to z} \left[\frac{1}{\cancel{\xi - z}} \oint_C \frac{\cancel{\xi - z}}{(\zeta - \xi)(\zeta - z)} f(\zeta)\, d\zeta \right] =$$

$$\cancel{= \lim_{\xi \to z} \left[\frac{\quad}{\xi} \right.}$$

$$= \lim_{\xi \to z} \oint_C f(z) \frac{1}{(\zeta - \xi)(\zeta - z)}\, d\zeta = \oint_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta. \qquad \square$$

With an inductive argument we can generalize this result:

<u>Thm</u> : If $f(z)$ is analytic in $\operatorname{int} C$ of a simple closed contour C with C being counterclockwise, then:

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.}$$

<u>Corollary.</u> : If f analytic in $A \subseteq \mathbb{C} \Rightarrow f$ infinitely differentiable in A.

- **Liouville's theorem**

Def : Let $f: \mathbb{C} \to \mathbb{C}$. We say that

a) f is entire $\Leftrightarrow$ f analytic on $\mathbb{C}$.

b) f is bounded $\Leftrightarrow \exists M>0, \forall z \in \mathbb{C} : |f(z)| < M$.

Thm : (Liouville).

If $\begin{cases} f \text{ is entire} \\ f \text{ is bounded} \end{cases} \Rightarrow f$ is constant.

Proof

Let $C_r(z) = \{ z + re^{i\vartheta} \mid \vartheta \in [0, 2\pi] \}$ for a given $z \in \mathbb{C}$.

Then,

f entire $\Rightarrow$

$$\Rightarrow \forall r>0 : f'(z) = \frac{1}{2\pi i} \oint_{C_r(z)} \frac{f(\zeta)\, d\zeta}{(\zeta - z)^2} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z+re^{i\vartheta})}{r^2 e^{2\vartheta i}}\, ire^{i\vartheta}\, d\vartheta =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(z+re^{i\vartheta})}{r e^{2\vartheta i}}\, d\vartheta$$

Let $M(r) = \max\limits_{z_0 \in C_r(z)} |f(z_0)|$. Then

$$|f'(z)| \leq \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f(z+re^{i\vartheta})}{r e^{2\vartheta i}}\, d\vartheta \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(z+re^{i\vartheta})|}{r}\, d\vartheta \leq \frac{M(r)}{2\pi r}.$$

But f bounded $\Rightarrow \exists M>0 : \forall r>0 : M(r) < M \Rightarrow$

$\Rightarrow \forall r>0 : |f'(z)| \leq \dfrac{M}{2\pi r}, \forall z \in \mathbb{C} \Rightarrow f'(z) = 0, \forall z \in \mathbb{C} \Rightarrow f$ is constant □.

Series expansions and singularities.

Taylor series.

Thm : Let $f: \mathbb{C} \to \mathbb{C}$ and $z_0 \in \mathbb{C}$. If f is analytic in $D \subseteq \mathbb{C}$ and $z_0 \in D$, then

$$\exists \delta > 0 : \forall z \in D : |z - z_0| < \delta \Rightarrow f(z) = f(z_0) + \sum_{k=1}^{+\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k .$$

Proof

Let C be a circular contour centered at z_0 such that $\operatorname{int} C \subseteq D$.
Then:

$$\forall z \in \operatorname{int} C : f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta =$$

$$= \frac{1}{2\pi i} \oint_C f(\zeta) \frac{1}{\zeta - z_0} \frac{1}{(\zeta - z)/(\zeta - z_0)} d\zeta =$$

$$= \frac{1}{2\pi i} \oint_C f(\zeta) \frac{1}{\zeta - z_0} \frac{1}{(\zeta - z + z_0 - z_0)/(\zeta - z_0)} d\zeta =$$

$$= \frac{1}{2\pi i} \oint_C f(\zeta) \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} d\zeta =$$

$$= \frac{1}{2\pi i} \oint_C f(\zeta) \frac{1}{\zeta - z_0} \left[1 - \frac{z - z_0}{\zeta - z_0} \right]^{-1} d\zeta. \qquad (1)$$

Since $|z - z_0| < \delta$ and C a circle of radius δ:

$$|z - z_0| < \delta = |\zeta - z_0| \Rightarrow \frac{|z - z_0|}{|\zeta - z_0|} = \left| \frac{z - z_0}{\zeta - z_0} \right| < 1 \Rightarrow$$

$$\Rightarrow \left[1 - \frac{z - z_0}{\zeta - z_0} \right]^{-1} = \sum_{k=0}^{+\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^k \overset{(1)}{\Rightarrow}$$

$$\Rightarrow f(z) = \frac{1}{2\pi i} \oint_C f(\zeta) \frac{1}{\zeta - z_0} \sum_{k=0}^{+\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^k d\zeta =$$

$$
= \sum_{k=0}^{+\infty} \left[\frac{(z-z_0)^k}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta \right] =
$$

$$
= \sum_{k=0}^{+\infty} \left[\frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] =
$$

$$
= f(z_0) + \sum_{k=1}^{+\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k .
$$

▶ Note that we can make δ as large as the closes distance to the closest singularity. Therefore the radius of convergence of the expression is that distance.

<u>Def</u> :

a) A point $z_0 \in \mathbb{C}$ about which a function may be expanded is called a <u>regular point</u>. of f.

b) Otherewise we say that $z_0 \in \mathbb{C}$ is a <u>singular</u> point.

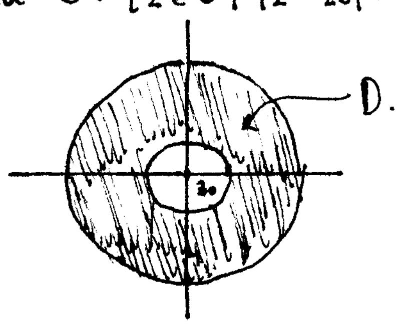
● <u>Laurent series.</u>

<u>Thm</u> : Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ and $D = \{ z \in \mathbb{C} : R_1 < |z - z_0| < R_2 \}$.
If f analytic in D, then

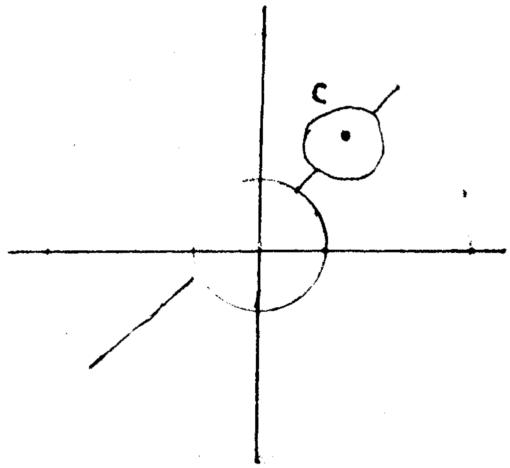
$$
f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k
$$

where
$$
a_k = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta .
$$

and $C = \{ z \in \mathbb{C} \mid |z - z_0| = R \}$ where $R \in (R_1, R_2)$.



Proof



By the Cauchy integrating formula:

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta =$$

$$= \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Note that

$$\forall \zeta \in C_1: \frac{1}{\zeta - z} = \frac{-1}{(z - z_0) - (\zeta - z_0)} = -\frac{1}{z - z_0}\left[1 - \frac{\zeta - z_0}{z - z_0}\right]^{-1} =$$

$$= -\frac{1}{z - z_0}\sum_{k=0}^{+\infty}\left(\frac{\zeta - z_0}{z - z_0}\right)^k = -\frac{1}{z - z_0}\sum_{k=-\infty}^{0}\frac{(z - z_0)^k}{(\zeta - z_0)^k} =$$

$$= \sum_{k=-\infty}^{0}\frac{-(z - z_0)^{k-1}}{(\zeta - z_0)^k} = -\sum_{k=-\infty}^{-1}\frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}}$$

and

$$\forall \zeta \in C_2: \frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0}\left[1 - \frac{z - z_0}{\zeta - z_0}\right]^{-1} =$$

$$= \sum_{k=0}^{+\infty}\frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}}.$$

therefore

$$f(z) = \frac{-1}{2\pi i}\oint_{C_1} f(\zeta)\left[-\sum_{k=-\infty}^{-1}\frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}}\right]d\zeta + \frac{1}{2\pi i}\oint_{C_2} f(\zeta)\left[\sum_{k=0}^{+\infty}\frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}}\right]d\zeta =$$

$$= \sum_{k=-\infty}^{-1} (z-z_0)^k \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{(\zeta-z_0)^{k+1}} d\zeta + \sum_{k=0}^{+\infty} (z-z_0)^k \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{(\zeta-z_0)^{k+1}} d\zeta =$$
$$= \sum_{k=-\infty}^{+\infty} a_k (z-z_0)^k. \qquad \square.$$

▼ Singular points.

Def : Let $f : \mathbb{C} \to \mathbb{C}$. We say that f has an isolated singularity at $z_0 \in \mathbb{C}$ iff

$$\begin{cases} \exists \varepsilon > 0 : f \text{ analytic at } N_\varepsilon(z_0) - \{z_0\} \\ f \text{ not analytic at } z_0. \end{cases}$$

Thm : If f has an isolated singularity at $z_0 \in \mathbb{C} \Rightarrow$
$\Rightarrow \exists R > 0 : \forall z \in \{z \in \mathbb{C} \mid 0 < |z - z_0| < R\} : f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_0)^k$.

Def : Let f be a function that has an isolated singularity at $z_0 \in \mathbb{C}$ and Laurent expansion

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_0)^k.$$

The principal part of f is given by:

$$\varphi(z) = \sum_{k=-\infty}^{-1} a_k (z-z_0)^k.$$

Def : Let $f(z)$ have an isolated singularity $z_0 \in \mathbb{C}$ and principal part $\varphi(z)$.

a) z_0 is a removable singularity $\Leftrightarrow \varphi(z) = 0$.

b) z_0 is a pole of order n $\Leftrightarrow \varphi(z) = \sum_{k=1}^{n} \frac{a_k}{(z-z_0)^k}$

c) z_0 is an essential singularity $\Leftrightarrow \varphi(z) = \sum_{k=1}^{+\infty} \frac{a_k}{(z-z_0)^k}$.

Thm : Let f be a function with an isolated singularity $z_0 \in \mathbb{C}$.

a) z_0 removable singularity $\Rightarrow \lim_{z \to z_0} f(z) = a_0 = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)} d\zeta$.

b) z_0 pole of order $n \Rightarrow \begin{cases} \lim_{z \to z_0} [(z-z_0)^n f(z)] \text{ converges} \\ \lim_{z \to z_0} [(z-z_0)^k f(z)] \text{ diverges}, \forall k < n. \end{cases}$

Residue theory

<u>Def</u>: Let $f(z)$ be a function with an isolated singularity $z_0 \in \mathbb{C}$ and Laurent expansion

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_0)^k \quad , \forall |z-z_0| < R.$$

The <u>residue of z_0</u> is given by $\operatorname{Res}(z_0; f) = a_{-1}$.

<u>Thm</u>: Suppose that f is analytic in $D - \{z_0\}$, where $z_0 \in D$. Then.

$$\oint_{\partial D} f(z)\,dz = 2\pi i \operatorname{Res}(f; z_0).$$

<u>Proof</u>

Let C be a disk around $z_0 \in D$ in which $f(z)$ has the Laurent expansion

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_0)^k \quad , \forall z \in \operatorname{int} C.$$

~~Then $\oint_C f(z)dz = \oint_C [$~~

Then $\displaystyle\oint_{\partial D} f(z)\,dz = \oint_C f(z)\,dz = \oint_C \left[\sum_{k=-\infty}^{+\infty} a_k (z-z_0)^k\right] dz =$

$$= \sum_{k=-\infty}^{+\infty} a_k \oint_C (z-z_0)^k\,dz \quad (1).$$

By Cauchy's theorem $\forall k \geq 0$: $\oint_C (z-z_0)^k\,dz = 0$.

~~Using Cauchy's integration formula, if $\varphi(z) = 1$, then.~~

$$\cancel{\oint_C (z-z_0)^k dz = \oint_C \varphi(z)(z-z_0^k)dz = \varphi^{(k-1)}(z_0) \quad , \forall k < 0.}$$

For $k = -1$, $\displaystyle\oint_C \frac{dz}{z-z_0} = 2\pi i$

For $k<-1$, define $\varphi(z)=1$, $\forall z\in\mathbb{C}$. Then,

$$\oint_C \frac{dz}{(z-z_0)^k} = \varphi^{(k-1)}(z_0) = 0 \quad , \forall k>1.$$

therefore. $\displaystyle\oint_{\partial D} f(z)dz = \sum_{k=-\infty}^{+\infty} a_k \oint_C (z-z_0)^k dz = 2\pi i a_{-1} = 2\pi i \operatorname{Res}(f; z_0).$ □

Corollary. : Suppose that f is analytic in $D-S$ except for a countable set $z_0\in S$ of ~~singularities~~. isolated singularities.

Then

$$\oint_{\partial D} f(z)dz = 2\pi i \sum_{z_0\in S} \operatorname{Res}(f; z_0)$$

Remark. : In general, to find the Residue we need to obtain the Laurent expansion. For example, for essential singularities.
Sometimes, for poles the following results might be useful:

Thm : If $f(z)$ has a pole of order m at $z_0\in\mathbb{C}$, then

$$\operatorname{Res}(f; z_0) = \frac{1}{(m-1)!}\lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

Thm (De L'Hospital's rule)

$$\begin{cases} \lim\limits_{z\to z_0} f(z) = 0 \\ \lim\limits_{z\to z_0} g(z) = 0 \end{cases} \Rightarrow \lim_{z\to z_0}\frac{f(z)}{g(z)} = \lim_{z\to z_0}\frac{f'(z)}{g'(z)}.$$

Fourier Transforms.

$$\hat{f}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-izt}\, dt \iff f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(z) e^{+izt}\, dz.$$

Properties of Fourier transforms.

1) $F[c_1 f_1 + c_2 f_2] = c_1 F[f_1] + c_2 F[f_2]$.
2) $F[f(t-\tau)] = e^{-i\tau z} F[f(t)]$.
3) $F[f(kt)] = \frac{1}{|k|} F[f]\left(\frac{z}{k}\right)$.
4) $F[f^{(k)}] = (iz)^k F[f]$
5) $\frac{d^n}{dz^n} F[f] = (-i)^n F[t^n f(t)]$.
6) $h(t) = \int^t f(\tau) d\tau \Rightarrow F[h] = \frac{F[f]}{iz}$.

Convolution theorem.

$$\boxed{f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-\xi) g(\xi)\, d\xi.}$$

Thm: $F[f*g] = F[f]F[g]$ or:

$$\int_{-\infty}^{+\infty} F(z) G(z) e^{+izx}\, dz = \int_{-\infty}^{+\infty} f(x-\xi) g(\xi)\, d\xi.$$

where $F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-izt}\, dt.$

$G(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) e^{-izt}\, dt.$

- **Parseval's identity.**

$$\int_{-\infty}^{+\infty} f(t)g^*(t)\,dt = \int_{-\infty}^{+\infty} F(z)\,G^*(z)\,dz$$

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \int_{-\infty}^{+\infty} |F(z)|^2 dz$$

Laplace transform

$$\hat{f}(s) = \int_0^{+\infty} f(t)\, e^{-st} dt \Leftrightarrow f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \hat{f}(s)\, e^{st} ds.$$

- **Properties of the Laplace transform.**

1) $L[af + bg] = aL[f] + bL[g]$
2) $L[f'] = sL[f] - f(0^+)$.
3) $g(t) = \int^t f(\tau)d\tau \Rightarrow L[g] = \frac{1}{s}L[f]$.
4) $L[e^{ct}f(t)] = L[f](s-c)$.

- **Convolution theorem**

$$h(t) = f * g = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau$$

$$L[f*g] = L[f]L[g].$$

Laplace transform inversion

1) $L^{-1}(1/s^c) = u(t)\,t^{c-1}/\Gamma(c)$
2) $L^{-1}(1/(s-a)) = u(t)\,e^{at}$
3) $L^{-1}(1/(s-a)^k) = u(t)\,t^{k-1}\,e^{at}/\Gamma(k)$
4) $L^{-1}(1/(s^2+w^2)) = u(t)\,\sin wt/w$
5) $L^{-1}(s/(s^2+w^2)) = u(t)\,\cos wt$
6) $L^{-1}(1/(s^2-w^2)) = u(t)\,\sinh wt/w$
7) $L^{-1}(s/(s^2-w^2)) = \cancel{\cos}\ \cosh(wt)$
8) $L^{-1}(1/\sqrt{z^2+k^2}) = J_0(kt)$.
9) $L^{-1}(\tan^{-1}(w/z)) = \sin wt/t$

Dirichlet problem on a disk of radius 1.

$$u(r,\alpha) = \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \frac{1-r^2}{1-2r\cos(\alpha-\theta)+r^2} \, d\theta.$$

Conformal mapping.

The problem:

$$\nabla^2 \varphi = 0 \qquad \text{on } D$$

$$A(t)\varphi + B(t)\frac{\partial \varphi}{\partial n} = c(t). \qquad \text{on } \partial D.$$

Bilinear transformation.

$$\frac{(u-u_1)(u_2-u_3)}{(u-u_2)(u_1-u_3)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_2)(z_1-z_3)}.$$

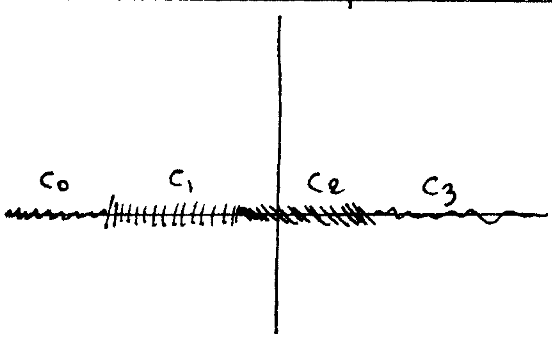
$z \to \infty \to 1 - 1/z$

$\Omega(z) = \varphi(z) + i\psi(z)$

$\nabla^2 \Omega(t) = 0.$

Ω analytic

A solution to Laplace Dirichlet



$\varphi(\xi, 0) = c_0$

$\varphi(\xi, 0) = c_\lambda \,, \quad \xi_\lambda < \xi < \xi_{\lambda+1}$

$\varphi(\xi, 0) = c_n \,, \quad \xi > \xi_n.$

$$\chi(\xi) = c_n - \frac{i}{\pi} \sum_{\lambda=0}^{n-1} (c_\lambda - c_{\lambda+1}) \ln(\xi - \xi_{\lambda+1})$$

Uniqueness of the solution to the Dirichlet problem

$\nabla^2 \varphi = 0$ in D.

$\varphi = f$ in ∂D

Suppose the solution is not unique

Let φ_1, φ_2 be solutions $\psi = \varphi_1 - \varphi_2$.

Then
$$\begin{cases} \nabla^2 \psi = 0 & \text{on } D \\ \psi = 0 & \text{on } \partial D. \end{cases}$$

Use the identity:

$$\nabla \cdot (\mathcal{J} \nabla \psi) = \mathcal{J} \nabla^2 \psi + \nabla \psi \cdot \nabla \mathcal{J}$$

For $\mathcal{J} = \psi$:

$$\psi \nabla^2 \psi = \nabla \cdot (\psi \nabla \psi) - \nabla \psi \cdot \nabla \psi.$$

$$\int_D \psi \nabla^2 \psi \, dV = \int_D \nabla \cdot (\psi \nabla \psi) \, dV - \int_D \nabla \psi \cdot \nabla \psi \, dV$$

$$\nabla^2 \psi = 0 \Rightarrow \int_D \psi \nabla^2 \psi \, dV = 0 \Rightarrow$$

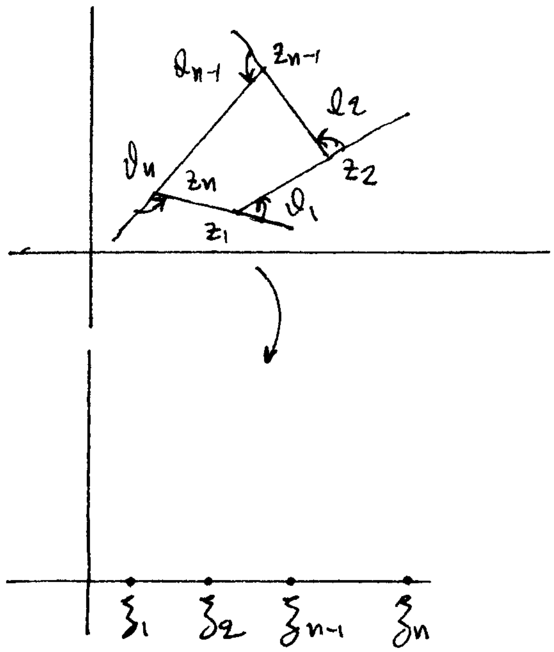
$$\Rightarrow \int_D \nabla \cdot (\psi \nabla \psi) \, dV = \int_D \nabla \psi \cdot \nabla \psi \, dV.$$

Also $\displaystyle\int_D \nabla \cdot (\psi \nabla \psi) dV = \int_{\partial D} (\psi \nabla \psi) \cdot n \, dS = 0.$

so $\displaystyle\int_D \nabla \psi \cdot \nabla \psi \, dV = 0 \Rightarrow \nabla \psi = 0$ (bc $|\nabla \psi|^2 \geq 0$, $\forall x \in D$).

$\Rightarrow \psi = \text{const.}$
But $\psi = 0$ at ∂D $\Big\} \Rightarrow \psi = 0 \Rightarrow \varphi_1 = \varphi_2.$

Schwarz - Christoffel transformation.



$$\sum_{i=1}^{n} \theta_i = 2\pi.$$

Map $z = f(\zeta)$.

Let γ_j be the turnangles.

$$f'(\zeta) = a(\zeta-\xi_1)^{k_1}(\zeta-\xi_2)^{k_2}\cdots(\zeta-\xi_n)^{k_n}, \quad k_j = -\gamma_j/\pi.$$

$$z = f(\zeta) = a\int_{\zeta_0}^{\zeta} \prod_{j=1}^{n}(\zeta-\xi_j)^{-\gamma_j/\pi}\, d\zeta + b.$$

$\zeta_0 \to$ arbitrary.