Fundamentals of incompressible turbulence

In the following notes we review the fundamentals of incompressible turbulence.

▼ The governing equations

In the Eulerian representation, the state of a fluid is specified by a vector-field u_a that gives the fluid velocity at a point (x, y, z) at time t.

For incompressible turbulence the governing equations for u_a are the Navier-Stokes equations:

$$\boxed{\begin{aligned} &\frac{\partial u_a}{\partial t} + u_b \partial_b u_a = -\partial_a \varphi + \nu \nabla^2 u_a \\ &\partial_a u_a = 0 \end{aligned}}$$

where
φ = the potential pressure field
$\partial_a = \partial/\partial x_a$ = spatial derivative
$\nabla^2 = \partial_a \partial_a$ = the Laplacian
ν = the viscosity

The second equation is called the incompressibility condition. The pressure field φ is such that the incompressibility condition is maintained for all times.

In theoretical turbulence we assume an unbound problem on $\mathbb{R}^3$ (i.e. we are very far from any boundary), so we generate turbulence by adding a stirring force at large scales. We then have the forced Navier-Stokes equations:

$$\boxed{\begin{aligned} &\frac{\partial u_a}{\partial t} + u_b \partial_b u_a = -\partial_a \varphi + \nu \nabla^2 u_a + f_a \\ &\partial_a u_a = 0 \end{aligned}}$$

Non-dimensionalization

The Navier-Stokes equations can be non-dimensionalized by setting:

$$u \longrightarrow u/\mathcal{V}, \quad \varphi \longrightarrow \varphi/\mathcal{V}^2, \quad \nu \longrightarrow \nu/L^2$$
$$f \longrightarrow f/(\mathcal{V}^2/L)$$

where $\mathcal{V}$ and L are a characteristic velocity and length. We define the Reynolds number, a non-dimensional quantity, in terms of $\mathcal{V}, L$ and ν via:

$$\boxed{Re = \mathcal{V}L/\nu}$$

Then the non-dimensional Navier-Stokes equations are

$$\boxed{\begin{aligned} &\frac{\partial u_a}{\partial t} + u_\beta \partial_\beta u_a = -\partial_a \varphi + \frac{1}{Re} \nabla^2 u_a + f_a \\ &\partial_a u_a = 0 \end{aligned}}$$

The irregularity of the Navier-Stokes equations

In principle, the full solution of the incompressible fluid dynamics problem is a functional $u_a(\vec{r}, t, [f]; \nu)$ that gives the history of the velocity field from the history of forcing $[f]$, parameterized by the viscosity ν. The flow for $\nu = 0$ is called Euler-flow. Because ν appears on the highest-order derivative, the functional u_a is irregular in the limit $\nu \to 0$. This means that:

$$\boxed{\lim_{\nu \to 0^+} u_a(\vec{r}, t, [f]; \nu) \neq u_a(\vec{r}, t, [f]; 0)}$$

Turbulence arises when $0 < \nu \ll 1$, or equivalently, in the limit $Re \longrightarrow +\infty$, and it is substantially different from Euler flow.

Quadratic form of the Navier-Stokes equations

We will now show how to eliminate φ from the Navier-Stokes equations and write them in quadratic form:
First we show two preliminary results:

Lemma: $\boxed{u_\beta \partial_\beta u_\alpha = \partial_\beta (u_\alpha u_\beta)}$

Proof

$$u_\beta \partial_\beta u_\alpha = u_\beta \partial_\beta u_\alpha + u_\alpha \partial_\beta u_\beta = \partial_\beta (u_\alpha u_\beta)$$

because $\partial_\beta u_\beta = 0$. $\square$

Lemma: $\boxed{\nabla^2 \varphi = -\partial_{\alpha\beta} (u_\alpha u_\beta)}$

Proof

Recall that $\partial_\alpha \varphi = \nu \nabla^2 u_\alpha - \partial_t u_\alpha - u_\beta \partial_\beta u_\alpha$
Then it follows that

$$\begin{aligned}
\nabla^2 \varphi &= \partial_\alpha (\partial_\alpha \varphi) = \partial_\alpha [\nu \nabla^2 u_\alpha - \partial_t u_\alpha - u_\beta \partial_\beta u_\alpha] = \\
&= \nu \nabla^2 (\partial_\alpha u_\alpha) - \partial_t (\partial_\alpha u_\alpha) - \partial_\alpha (u_\beta \partial_\beta u_\alpha) = \\
&= -\partial_\alpha (u_\beta \partial_\beta u_\alpha) = -\partial_\alpha (\partial_\beta (u_\alpha u_\beta)) = \\
&= -\partial_{\alpha\beta} (u_\alpha u_\beta) \qquad \square
\end{aligned}$$

When we study the theory of turbulence we assume that the boundary is very far away. This leads to an appropriate model of <u>turbulence decay</u>. If we want to model stationary turbulence we must either introduce the boundary conditions or introduce a forcing term. Ignoring the boundary makes it easy to invert the Laplacian operator.

Define the Green's function $G(\vec{r}, \vec{r}_0)$ via:

$$\nabla^2 G(\vec{r}, \vec{r}_0) = \delta(\vec{r}, - \vec{r}_0).$$

The solution for G may diverge; it should be thought of as a generalized function. Then it can be shown that

$$\nabla^2 \varphi = f \Rightarrow \varphi(\vec{r}) = \int d\vec{r}_0 \; G(\vec{r}, \vec{r}_0) f(\vec{r}_0)$$

Thus we may think of $G(\vec{r}, \vec{r}_0)$ as the representation of a linear operator which is the operating inverse of ∇^2 and which we denote as ∇^{-2}. Now we can derive the quadratic form of the Navier-Stokes equations:

<u>Theorem</u>: Assume no boundary. Then u_a can be given by:

$$\boxed{\frac{\partial u_a}{\partial t} = \mathcal{M}_{a\beta\gamma}[u_\beta u_\gamma] - \nu \nabla^2 u_a}$$

where $\mathcal{M}$ is an operator tensor given by:

$$\boxed{\begin{aligned} \mathcal{M}_{a\beta\gamma} &= (-1/2)\left[\partial_\beta \Delta_{a\gamma} + \partial_\gamma \Delta_{a\beta}\right] \\ \Delta_{a\beta} &= \delta_{a\beta} - \partial_{a\beta} \nabla^{-2} \end{aligned}}$$

<u>Proof</u>

Since $\nabla^2 \varphi = - \partial_{a\beta}(u_a u_\beta) \Rightarrow$

$$\Rightarrow \varphi = - \nabla^{-2} \partial_{a\beta}(u_a u_\beta) = - \partial_{a\beta} \nabla^{-2}(u_a u_\beta)$$

It is easy to see that $\partial_{a\beta}$ and ∇^{-2} commute by doing the differentiation and integration in Fourier space, where both operators become algebraic. For convenience we rewrite:

$$\varphi = - \partial_{\beta\gamma} \nabla^{-2}(u_\beta u_\gamma)$$

It follows that:

$$\partial_t u_a = -u_\beta \partial_\beta u_a - \partial_a \varphi + \nu \nabla^2 u_a =$$
$$= -\partial_\beta (u_a u_\beta) - \partial_a [-\partial_{\beta\gamma} \nabla^{-2} (u_\beta u_\gamma)] + \nu \nabla^2 u_a =$$
$$= -\partial_\beta (\delta_{a\gamma} u_\beta u_\gamma) - \partial_\beta [-\partial_{a\gamma} \nabla^{-2} (u_\beta u_\gamma)] + \nu \nabla^2 u_a =$$
$$= -\partial_\beta (\delta_{a\gamma} - \partial_{a\gamma} \nabla^{-2})(u_\beta u_\gamma) + \nu \nabla^2 u_a =$$
$$= -\partial_\beta \mathcal{P}_{a\gamma} (u_\beta u_\gamma) + \nu \nabla^2 u_a$$

Since $\partial_\beta \mathcal{P}_{a\gamma} (u_\beta u_\gamma) = \partial_\gamma \mathcal{P}_{a\beta} (u_\gamma u_\beta)$
$= \partial_\gamma \mathcal{P}_{a\beta} (u_\beta u_\gamma)$.

It follows that

$$\mathcal{M}_{a\beta\gamma} (u_\beta u_\gamma) = -\frac{1}{2} \partial_\beta \mathcal{P}_{a\gamma} (u_\beta u_\gamma) - \frac{1}{2} \partial_\gamma \mathcal{P}_{a\beta} (u_\beta u_\gamma) =$$

$$= -\partial_\beta \mathcal{P}_{a\gamma} (u_\beta u_\gamma).$$ so

we obtain our main result:

$$\partial_t u_a = \mathcal{M}_{a\beta\gamma} [u_\beta u_\gamma] - \nu \nabla^2 u_a \qquad \square$$

Let $\mathcal{F}$ be the forward Fourier transform operator and $\mathcal{F}^{-1}$ its inverse, the backward Fourier transform. Then our operators $\mathcal{M}_{a\beta\gamma}$ and $\mathcal{P}_{a\beta}$ can be "diagonalized" as:

$$\mathcal{P}_{a\beta} = \mathcal{F}^{-1} P_{a\beta} \mathcal{F}$$
$$\mathcal{M}_{a\beta\gamma} = \mathcal{F}^{-1} M_{a\beta\gamma} \mathcal{F}$$

where $P_{a\beta}$ and $M_{a\beta\gamma}$ are algebraic operators given by:

$$\boxed{\begin{aligned} P_{a\beta}(\vec{k}) &= \delta_{a\beta} - k_a k_\beta / \|\vec{k}\|^2 \\ M_{a\beta\gamma}(\vec{k}) &= (2i)^{-1} [k_\beta P_{a\gamma}(\vec{k}) + k_\gamma P_{a\beta}(\vec{k})] \end{aligned}}$$

These operators can help us write the quadratic form in Fouriers space. Let $\hat{u}_a \equiv \mathcal{F} u_a$. Then we have:

$$\boxed{\frac{\partial \hat{u}_a(\vec{k})}{\partial t} = M_{a\beta\gamma}(\vec{k}) \iint d\vec{p}\, d\vec{q}\; \hat{u}_\beta(\vec{p}) \hat{u}_\gamma(\vec{q}) \delta(\vec{p}+\vec{q}-\vec{k}) + \nu k^2 \hat{u}_a(\vec{k}).}$$

Existence and uniqueness results.

The quadratic form of the Navier-Stokes equations highlights that it is a time-evolution problem where we propagate an initial condition forward in time. It is not obvious however that there will exist a unique smooth $u_a(\vec{r},t)$ given a smooth initial condition $u_a(0,t)$. We now review without proof, what has been shown about the Navier-Stokes equations via functional analysis on this issue:

A) When $n=2$ (2d turbulence), then smooth initial condition and smooth forcing yield a smooth velocity field for all times, which is unique.

B) When $n=3$ (3d turbulence), then the following has been proven so far:

   a) If a smooth solution exists for all time, then it is unique.

   b) A smooth solution exists on a certain time interval. $(0,T_*)$

   c) A weak solution exists for all time but it may not be unique.

(see Dubois-Jauberteau-Temam 1999)

Let Ω denote the phase-space of fluid-flow, i.e. all vector functions u_a that satisfy $\partial_a u_a = 0$ and the boundary conditions. On the basis of physics we conjecture that there is a mapping

$$g(t): \Omega \longrightarrow \Omega \quad , \quad t \in \mathbb{R}$$

which advects u_a at a given time forward by time t:

$$u_a(t+t_0) = g(t)\, u_a(t_0)$$

thus for every $u_a \in \Omega$, g defines a trajectory in phase-space. Although this assumption has been formally proven only when $n=2$, we will take it for granted for $n=3$ too.

Vorticity dynamics

Vorticity plays a very important role in the study of turbulence, so we will review the main results about it.

The vorticity field ω_a is defined as the curl of the velocity field u_a:

$$\omega_a = \varepsilon_{ab\gamma}\partial_b u_\gamma$$

where $\varepsilon_{ab\gamma}$ is defined as:

$$\varepsilon_{ab\gamma} = \begin{cases} 1, & \text{if } (a\,b\,\gamma) \text{ is an even permutation} \\ -1, & \text{if } (a\,b\,\gamma) \text{ is an odd permutation} \\ 0, & \text{if } a=b \vee b=\gamma \vee \gamma=a \end{cases}$$

It follows that the components of ω_a are:

$$\omega_1 = \partial_2 u_3 - \partial_3 u_2$$
$$\omega_2 = \partial_3 u_1 - \partial_1 u_3$$
$$\omega_3 = \partial_1 u_2 - \partial_2 u_1$$

The following identities about ε are well-known from tensor algebra:

$$\varepsilon_{ab\gamma} x_a x_b = \varepsilon_{ab\gamma} x_b x_\gamma = 0$$
$$\varepsilon_{ab\gamma} = \varepsilon_{b\gamma a} = \varepsilon_{\gamma ab}$$
$$\varepsilon_{ab\gamma} = -\varepsilon_{\gamma ba}$$
$$\varepsilon_{ab\gamma}\varepsilon_{\gamma kl} = \delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}$$

The vorticity is related with the ~~solid~~ local rate of <u>solid rotation</u> of the fluid. To bring that out we decompose the gradient tensor to a symmetric and skew-symmetric component:

$$\partial_a u_b = s_{ab} + r_{ab}$$
$$r_{ab} \equiv (1/2)(\partial_a u_b - \partial_b u_a) \longrightarrow \text{rotation tensor}$$
$$s_{ab} \equiv (1/2)(\partial_a u_b + \partial_b u_a) \longrightarrow \text{strain tensor}$$

<u>Proposition</u> :

$\omega_a = \varepsilon_{ab\gamma} r_{b\gamma}$	$r_{ab} = \frac{1}{2}\varepsilon_{ab\gamma}\omega_\gamma$

<u>Proof</u>

First note that:

$$\varepsilon_{a\beta\gamma} s_{\beta\gamma} = (1/2)\, \varepsilon_{a\beta\gamma} (\partial_\beta u_\gamma + \partial_\gamma u_\beta) =$$
$$= (1/2)(\varepsilon_{a\beta\gamma} + \varepsilon_{a\gamma\beta})\, \partial_\beta u_\gamma = (1/2)\cdot 0 \cdot (-) = 0$$

so:

$$\omega_a = \varepsilon_{a\beta\gamma} \partial_\beta u_\gamma = \varepsilon_{a\beta\gamma} (r_{\beta\gamma} + s_{\beta\gamma}) =$$
$$= \varepsilon_{a\beta\gamma} r_{\beta\gamma} + \varepsilon_{a\beta\gamma} s_{\beta\gamma} = \varepsilon_{a\beta\gamma} r_{\beta\gamma}$$

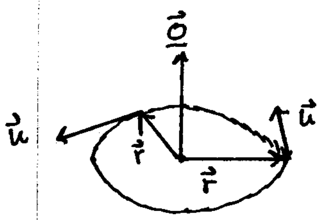
To obtain the converse relation:

$$\varepsilon_{a\beta\gamma} \omega_\gamma = \varepsilon_{a\beta\gamma} \varepsilon_{\gamma\kappa l} r_{\kappa l} = (\delta_{a\kappa}\delta_{\beta l} - \delta_{al}\delta_{\beta\kappa}) r_{\kappa l} =$$
$$= \delta_{a\kappa}\delta_{\beta l} r_{\kappa l} - \delta_{al}\delta_{\beta\kappa} r_{\kappa l} =$$
$$= r_{a\beta} - r_{\beta a} = r_{a\beta} + r_{a\beta} = 2 r_{a\beta} \Rightarrow r_{a\beta} = (1/2)\, \varepsilon_{a\beta\gamma} \omega_\gamma \quad \square$$

Now suppose that u_a is such that the fluid undergoes <u>only</u> a solid rotation:

$$\vec{u} = \vec{\Omega} \times \vec{r} \Rightarrow u_a = \varepsilon_{a\beta\gamma} \Omega_\beta x_\gamma$$

where $\vec{\Omega}$ is the angular velocity vector.



We now show that the corresponding vorticity is uniformly distributed and given by $\omega_a = 2\Omega_a$:

<u>Proposition</u> :

$$\boxed{\begin{matrix} u_a = \varepsilon_{a\beta\gamma} \Omega_\beta x_\gamma \\ \Omega_a = \text{constant} \end{matrix} \Bigg\} \Rightarrow \omega_a = 2\Omega_a = \text{constant}}$$

<u>Proof</u>

$$\omega_a = \varepsilon_{a\beta\gamma} \partial_\beta u_\gamma = \varepsilon_{a\beta\gamma} \partial_\beta (\varepsilon_{\gamma\kappa l} \Omega_\kappa x_l) =$$
$$= \varepsilon_{a\beta\gamma} \varepsilon_{\gamma\kappa l} \Omega_\kappa \partial_\beta x_l = \varepsilon_{a\beta\gamma} \varepsilon_{\gamma\kappa l} \delta_{\beta l} \Omega_\kappa =$$
$$= (\delta_{a\kappa}\delta_{\beta l} - \delta_{al}\delta_{\beta\kappa}) \delta_{\beta l} \Omega_\kappa =$$
$$= \delta_{a\kappa} (\delta_{\beta l}\delta_{\beta l}) \Omega_\kappa - \delta_{al}\delta_{\beta l}\delta_{\beta\kappa} \Omega_\kappa =$$
$$= 3\delta_{a\kappa} \Omega_\kappa - \delta_{al}\delta_{\beta l} \Omega_\beta = 3\Omega_a - \delta_{al} \Omega_l =$$
$$= 3\Omega_a - \Omega_a = 2\Omega_a \quad \square$$

So vorticity is twice the local rate of rotation of the fluid. We now derive the dynamical equation for w_a:

<u>Theorem</u> : $\boxed{\dfrac{Dw_a}{Dt} = w_b \partial_b u_a + \nu \nabla^2 w_a}$

<u>Proof</u>

Recall that $\partial_t u_a + u_b \partial_b u_a = -\partial_a \varphi + \nu \nabla^2 u_a$
Define $K \equiv (1/2) u_b u_b$. Now we rewrite the quadratic term:

$$u_b \partial_b u_a = -u_b(\partial_a u_b - \partial_b u_a) + u_b \partial_a u_b =$$
$$= -2u_b r_{ab} + \partial_a K = -2u_b (1/2) \varepsilon_{ab\gamma} w_\gamma + \partial_a K =$$
$$= -\varepsilon_{ab\gamma} u_b w_\gamma + \partial_a K.$$

so we obtain:

$$\partial_t u_a = \varepsilon_{ab\gamma} u_b w_\gamma - \partial_a (K + \varphi) + \nu \nabla^2 u_a$$

It follows that:

$$\partial_t w_a = \partial_t (\varepsilon_{ab\gamma} \partial_b u_\gamma) = \varepsilon_{ab\gamma} \partial_b (\partial_t u_\gamma) =$$
$$= \varepsilon_{ab\gamma} \partial_b [\varepsilon_{\gamma \kappa \ell} u_\kappa w_\ell - \partial_a (K+\varphi) + \nu \nabla^2 u_\gamma]$$
$$= \varepsilon_{ab\gamma} \varepsilon_{\gamma\kappa\ell} \partial_b (u_\kappa w_\ell) - \varepsilon_{ab\gamma} \partial_a \partial_b (K+\varphi) + \nu \nabla^2 (\varepsilon_{ab\gamma} \partial_b u_\gamma) =$$
$$= \varepsilon_{ab\gamma} \varepsilon_{\gamma\kappa\ell} \partial_b (u_\kappa w_\ell) + \nu \nabla^2 w_a$$

because $\varepsilon_{ab\gamma} \partial_b \partial_\gamma = 0$. Now we simplify the non-linear term:

$$\varepsilon_{ab\gamma} \varepsilon_{\gamma\kappa\ell} \partial_b (u_\kappa w_\ell) = (\delta_{a\kappa} \delta_{b\ell} - \delta_{a\ell} \delta_{b\kappa}) \partial_b (u_\kappa w_\ell) =$$
$$= \delta_{b\ell} \delta_{a\kappa} \partial_b (u_\kappa w_\ell) - \delta_{b\kappa} \delta_{a\ell} \partial_b (u_\kappa w_\ell) =$$
$$= \delta_{b\ell} \partial_b (u_a w_\ell) - \delta_{b\kappa} \partial_b (u_\kappa w_a) =$$
$$= \partial_b (u_a w_b) - \partial_b (u_b w_a)$$
$$= (\partial_b u_a) w_b + u_a \cancel{(\partial_b w_b)} - \cancel{(\partial_b u_b)} w_a - u_b \partial_b w_a$$

However $\partial_b u_b = 0$, by incompressibility.
Also $\partial_a w_a = \partial_a (\varepsilon_{ab\gamma} \partial_b u_\gamma) = \varepsilon_{ab\gamma} \partial_a \partial_b u_\gamma = 0$, so

$$\partial_t w_a = w_b (\partial_b u_a) - u_b \partial_b w_a + \nu \nabla^2 w_a$$

It follows that

$$\frac{Dw_a}{Dt} = \partial_t w_a + u_b \partial_b w_a = w_b (\partial_b u_a) + \nu \nabla^2 w_a \qquad \square$$

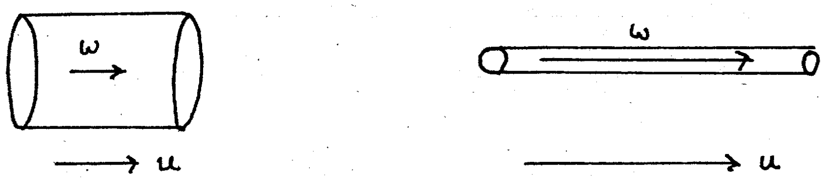
The physical picture suggested by the vorticity equation is ~~as~~ the following:

a) Vortices are advected by the velocity field. Note that a vortex is a blob of intense vorticity.

b) A vortex is amplified (stretched) when it moves into a region where the velocity field increases ($\partial_b u_a > 0$) and damped otherwise.

c) Vortices dissipate their energy into heat via the $\nu \nabla^2 \omega_a$ term.

Note that a vortex that amplifies as it moves, stretches to preserve ~~vorticity~~ its angular momentum.



A very accurate picture of turbulence is that large eddies, generated by forcing, subsequently stretch generating smaller eddies. When the eddies are small enough, the $\nu \nabla^2 \omega_a$ dominates the dynamics and dissipate the small eddies into heat.

Why are vortices stable? The reason is that the production term $\omega_b \partial_b u_a$ depends only on the strain tensor s_{ab} and not on the rotation tensor r_{ab}. Inside a vortex r_{ab} dominates whereas s_{ab} is negligible. To bring that out:

<u>Proposition</u> : $\boxed{\omega_b \partial_b u_a = \omega_b s_{ab}}$

<u>Proof</u>

$$\begin{aligned}
\omega_b \partial_b u_a &= \omega_b (r_{ba} + s_{ba}) = -\omega_b r_{ab} + s_{ab}\omega_b = \\
&= -\omega_b (1/2)\varepsilon_{ab\gamma}\omega_\gamma + \omega_b s_{ab} = \\
&= -(1/2)\varepsilon_{ab\gamma}\omega_b\omega_\gamma + \omega_b s_{ab} = \omega_b s_{ab}.
\end{aligned}$$

To show that s_{ab} is negligible inside a vortex consider the extreme case of a solid vortex:

Proposition: $\boxed{\left.\begin{array}{l} u_a = \varepsilon_{ab\gamma}\Omega_b x_\gamma \\ \Omega_a = \text{constant} \end{array}\right\} \Rightarrow \quad s_{ab} = 0}$

Proof

$$\partial_a u_b = \partial_a(\varepsilon_{bkl}\Omega_k x_l) = \varepsilon_{bkl}\Omega_k \partial_a x_l =$$
$$= \varepsilon_{bkl}\Omega_k \delta_{al} = \varepsilon_{bka}\Omega_k = \varepsilon_{abk}\Omega_k$$

therefore:

$$2s_{ab} = \partial_a u_b + \partial_b u_a = \varepsilon_{abk}\Omega_k + \varepsilon_{bak}\Omega_k =$$
$$= (\varepsilon_{abk} + \varepsilon_{bak})\Omega_k = 0 \qquad \square$$

Of course, s_{ab} will vary somewhat within the vortex. These variations will not destroy the coherence of the vortex. Instead they will turn it into a bundle of smaller eddies, triggering a cascade.

This physical image of vortex dynamics has inspired both the Kolmogorov theory of turbulence and the more recent theory of She-Leveque, and it is very important to keep in mind. The SL theory in particular uses the geometry of vortex bundles as its point of departure whereas Kolmogorov considers more vaguely the notion of large eddies yielding small eddies.

Another observation about vorticity that is useful in calculations is that it inherits solenoidality from the velocity field u_a. So:

$$\boxed{\begin{array}{l} \partial_a w_a = 0 \quad \text{and} \\ w_a \partial_a \varphi = \partial_a(w_a \varphi) \end{array}}$$

for any field φ.

Global conservation laws and the NV equations

Let $f(\vec{r})$ be an arbitrary field that vanishes at infinity. We denote its <u>total</u> as:

$$\langle f \rangle = \int_{\mathbb{R}^3} f(\vec{r})\, d\vec{r}$$

Let u_α be such a velocity field and $w_\alpha = \varepsilon_{\alpha\beta\gamma}\partial_\beta u_\gamma$ the corresponding vorticity field. We are interested in the following quantities:

$$\boxed{E = \left\langle \frac{1}{2} u_\alpha u_\alpha \right\rangle} \longrightarrow \text{total energy}$$

$$\boxed{H = \left\langle \frac{1}{2} u_\alpha w_\alpha \right\rangle} \longrightarrow \text{total helicity.}$$

which are functions only of time. We will show that they evolve according to the following conservation laws:

$$\boxed{\begin{aligned} \frac{dE}{dt} &= -\nu \frac{\langle S_{\alpha\beta} S_{\alpha\beta} \rangle}{2} + \langle f_\alpha u_\alpha \rangle \\ \frac{dH}{dt} &= -\nu \langle w_\alpha \varepsilon_{\alpha\beta\gamma} \partial_\beta w_\gamma \rangle + \langle f_\alpha w_\alpha \rangle \end{aligned}}$$

First we derive some preliminary results:

Properties of the totaling operator

Let f, g be fields that vanish at infinity. Then

<u>Proposition</u> : $\boxed{\langle \partial_\alpha f \rangle = 0}$

<u>Proof</u>

Rewrite the volume differential as $d\vec{r} = dx_a \, dS$ where dS is a product of all differentials except for dx_a. It follows that

$$\langle \partial_a f \rangle = \int d\vec{r} \, \partial_a f(\vec{r}) = \int dS \left[\int_{-\infty}^{+\infty} dx_a \, \partial_a f \right] =$$

$$= \int dS \left[\lim_{x_a \to +\infty} f - \lim_{x_a \to -\infty} f \right] = \int dS \, [0 - 0] = 0 \qquad \square$$

<u>Corollary</u>:
$$\boxed{\begin{array}{l} \langle f \partial_a g \rangle = -\langle g \partial_a f \rangle \\ \langle f \nabla^2 g \rangle = -\langle (\partial_a f)(\partial_a g) \rangle \end{array}}$$

<u>Proof</u>

a) $\langle f \partial_a g \rangle + \langle g \partial_a f \rangle = \langle f \partial_a g + g \partial_a f \rangle =$
$= \langle \partial_a (fg) \rangle \overset{*}{=} 0 \Rightarrow$
$\Rightarrow \langle f \partial_a g \rangle = -\langle g \partial_a f \rangle$

b) $\langle f \nabla^2 g \rangle = \langle f \partial_a \partial_a g \rangle = \langle f \partial_a (\partial_a g) \rangle \overset{*}{=}$
$= -\langle (\partial_a f)(\partial_a g) \rangle \qquad \square$

For the helicity conservation law we use the following additional corollaries:

<u>Corollary</u>:
$$\boxed{\begin{array}{l} \langle f_a \varepsilon_{a\beta\gamma} \partial_\beta g_\gamma \rangle = \langle g_a \varepsilon_{a\beta\gamma} \partial_\beta f_\gamma \rangle \\ \partial_a g_a = 0 \Rightarrow \langle f_a \nabla^2 g_a \rangle = -\langle (\varepsilon_{a\beta\gamma} \partial_\beta f_\gamma)(\varepsilon_{a\kappa\lambda} \partial_\kappa g_\lambda) \rangle \end{array}}$$

<u>Proof</u>

a) $\langle f_a \varepsilon_{a\beta\gamma} \partial_\beta g_\gamma \rangle = \varepsilon_{a\beta\gamma} \langle f_a \partial_\beta g_\gamma \rangle = -\varepsilon_{a\beta\gamma} \langle g_\gamma \partial_\beta f_a \rangle =$
$= -\varepsilon_{\gamma\beta a} \langle g_a \partial_\beta f_\gamma \rangle =$

$$= \varepsilon_{\alpha\beta\gamma} \langle g_\alpha \partial_\beta f_\gamma \rangle = \langle g_\alpha \varepsilon_{\alpha\beta\gamma} \partial_\beta f_\gamma \rangle$$

b) We start from the right-hand-side and work our way to the left-hand-side:

$$\begin{aligned}
\langle (\varepsilon_{\alpha\beta\gamma} \partial_\beta f_\gamma)(\varepsilon_{\alpha\kappa\ell} \partial_\kappa g_\ell) \rangle &= \varepsilon_{\alpha\beta\gamma} \varepsilon_{\alpha\kappa\ell} \langle (\partial_\beta f_\gamma)(\partial_\kappa g_\ell) \rangle = \\
&= -\varepsilon_{\gamma\beta\alpha} \varepsilon_{\alpha\kappa\ell} \langle (\partial_\beta f_\gamma)(\partial_\kappa g_\ell) \rangle = \\
&= -(\delta_{\gamma\kappa}\delta_{\beta\ell} - \delta_{\gamma\ell}\delta_{\beta\kappa}) \langle (\partial_\beta f_\gamma)(\partial_\kappa g_\ell) \rangle = \\
&= -\delta_{\gamma\kappa} \langle (\partial_\beta f_\gamma)(\partial_\kappa g_\beta) \rangle + \langle (\partial_\beta f_\gamma)(\partial_\beta g_\gamma) \rangle \\
&= +\delta_{\gamma\kappa} \langle f_\gamma \partial_\beta \partial_\kappa g_\beta \rangle - \langle f_\alpha \nabla^2 g_\alpha \rangle \\
&= \delta_{\gamma\kappa} \langle f_\gamma \partial_\kappa (\partial_\beta g_\beta) \rangle - \langle f_\alpha \nabla^2 g_\alpha \rangle = \\
&= 0 - \langle f_\alpha \nabla^2 g_\alpha \rangle = -\langle f_\alpha \nabla^2 g_\alpha \rangle . \qquad \square
\end{aligned}$$

Derivation of the conservation laws

Theorem : $\boxed{\dfrac{dE}{dt} = -\nu \langle \omega_\alpha \omega_\alpha \rangle + \langle f_\alpha u_\alpha \rangle}$

Proof

Note that :

$$\frac{\partial}{\partial t}\left(\frac{u_\alpha u_\alpha}{2}\right) = u_\alpha \frac{\partial u_\alpha}{\partial t} = u_\alpha \left[-u_\beta \partial_\beta u_\alpha - \partial_\alpha \varphi + \nu \nabla^2 u_\alpha + f_\alpha \right] =$$

$$\begin{aligned}
&= -u_\alpha \partial_\beta (u_\alpha u_\beta) - u_\alpha \partial_\alpha \varphi + \nu u_\alpha \nabla^2 u_\alpha + f_\alpha u_\alpha = \\
&= -\partial_\beta (u_\alpha u_\alpha u_\beta) - \partial_\alpha (u_\alpha \varphi) + \nu u_\alpha \nabla^2 u_\alpha + f_\alpha u_\alpha
\end{aligned}$$

therefore :

$$\frac{dE}{dt} = \left\langle \frac{\partial}{\partial t}\left(\frac{u_\alpha u_\alpha}{2}\right) \right\rangle =$$

$$\begin{aligned}
&= -\langle \partial_\beta (u_\alpha u_\alpha u_\beta) \rangle - \langle \partial_\alpha (u_\alpha \varphi) \rangle + \nu \langle u_\alpha \nabla^2 u_\alpha \rangle + \langle f_\alpha u_\alpha \rangle \\
&= \nu \langle u_\alpha \nabla^2 u_\alpha \rangle + \langle f_\alpha u_\alpha \rangle = \\
&= -\nu \langle (\varepsilon_{\alpha\beta\gamma} \partial_\beta u_\gamma)(\varepsilon_{\alpha\kappa\ell} \partial_\kappa u_\ell) + \langle f_\alpha u_\alpha \rangle \\
&= -\nu \langle \omega_\alpha \omega_\alpha \rangle + \langle f_\alpha u_\alpha \rangle \qquad \square
\end{aligned}$$

The energy conservation law can be equivalently rewritten in terms of the strain tensor s_{ab} using the following relation:

Proposition: $\boxed{\langle w_a w_a \rangle = \frac{1}{2} \langle s_{ab} s_{ab} \rangle}$

Proof

$$\langle s_{ab} s_{ab} \rangle = \langle (\partial_a u_b + \partial_b u_a)(\partial_a u_b + \partial_b u_a) \rangle$$
$$= \langle (\partial_a u_b)(\partial_a u_b) \rangle + \langle (\partial_b u_a)(\partial_b u_a) \rangle + 2 \langle (\partial_a u_b)(\partial_b u_a) \rangle$$

Note that:

$$\langle (\partial_a u_b)(\partial_b u_a) \rangle = - \langle u_b \partial_a \partial_b u_a \rangle =$$
$$= - \langle u_b \partial_b (\partial_a u_a) \rangle = 0$$

and

$$\langle (\partial_a u_b)(\partial_a u_b) \rangle = - \langle u_b \nabla^2 u_b \rangle =$$
$$= + \langle (\varepsilon_{ab\gamma} \partial_b u_\gamma)(\varepsilon_{a\kappa l} \partial_\kappa u_l) \rangle$$
$$= \langle w_a w_a \rangle$$

Similarly $\langle (\partial_b u_a)(\partial_b u_a) \rangle = \langle w_a w_a \rangle$, therefore

$\langle s_{ab} s_{ab} \rangle = \langle w_a w_a \rangle + \langle w_a w_a \rangle + 0 \Rightarrow \langle w_a w_a \rangle = \frac{1}{2} \langle s_{ab} s_{ab} \rangle$. □

Now we show conservation of helicity:

Theorem: $\boxed{\frac{dH}{dt} = - \nu \langle w_a \varepsilon_{ab\gamma} \partial_b w_\gamma \rangle + \langle f_a w_a \rangle}$

Proof

Recall that the governing equation of the vorticity, with forcing, is:

$$\frac{\partial w_a}{\partial t} = (w_b \partial_b u_a - u_b \partial_b w_a) + \nu \nabla^2 w_a + \varepsilon_{ab\gamma} \partial_b f_\gamma$$

Also note that:

$$\frac{dH}{dt} = \left\langle \frac{\partial}{\partial t}\left(\frac{u_a w_a}{2}\right)\right\rangle = \frac{1}{2}\left\langle u_a \frac{\partial w_a}{\partial t}\right\rangle + \frac{1}{2}\left\langle w_a \frac{\partial u_a}{\partial t}\right\rangle =$$

$$= \frac{1}{2}\left\langle u_a \frac{\partial w_a}{\partial t}\right\rangle + \frac{1}{2}\left\langle u_a \varepsilon_{a\beta\gamma}\partial_\beta \frac{\partial u_\gamma}{\partial t}\right\rangle =$$

$$= \frac{1}{2}\left\langle u_a \frac{\partial w_a}{\partial t}\right\rangle + \frac{1}{2}\left\langle u_a \frac{\partial w_a}{\partial t}\right\rangle = \left\langle u_a \frac{\partial w_a}{\partial t}\right\rangle =$$

$$= \langle u_a w_\beta \partial_\beta u_a - u_a u_\beta \partial_\beta w_a\rangle + \nu\langle u_a \nabla^2 w_a\rangle + \langle u_a \varepsilon_{a\beta\gamma}\partial_\beta f_\gamma\rangle$$

Recall from the derivation on p. 9 that

$$w_\beta \partial_\beta u_a - u_\beta \partial_\beta w_a = \varepsilon_{a\beta\gamma}\varepsilon_{\gamma\kappa l}\partial_\beta (u_\kappa w_l) =$$

$$= \varepsilon_{a\beta\gamma}\partial_\beta(\varepsilon_{\gamma\kappa l} u_\kappa w_l)$$

It follows that the non-linear term contribution is:

$$NL = \langle u_a w_\beta \partial_\beta u_a - u_a u_\beta \partial_\beta w_a\rangle =$$

$$= \langle u_a \varepsilon_{a\beta\gamma}\partial_\beta(\varepsilon_{\gamma\kappa l} u_\kappa w_l)\rangle =$$

$$= \langle(\varepsilon_{a\beta\gamma}\partial_\beta u_\gamma)(\varepsilon_{a\kappa l} u_\kappa w_l)\rangle =$$

$$= \langle w_a \varepsilon_{a\kappa l} u_\kappa w_l\rangle =$$

$$= \langle u_\kappa(\varepsilon_{a\kappa l} w_a w_l)\rangle = 0$$

because $\varepsilon_{a\kappa l} w_a w_l = -\varepsilon_{a\kappa a l} w_a w_l = 0$

Also:

$$\langle u_a \nabla^2 w_a\rangle = -\langle(\varepsilon_{a\beta\gamma}\partial_\beta u_\gamma)(\varepsilon_{a\kappa l}\partial_\kappa w_l)\rangle =$$

$$= -\langle w_a \varepsilon_{a\kappa l}\partial_\kappa w_l\rangle = -\langle \varepsilon_{a\beta\gamma} w_a \partial_\beta w_\gamma\rangle$$

and

$$\langle u_a \varepsilon_{a\beta\gamma}\partial_\beta f_\gamma\rangle = \langle f_a \varepsilon_{a\beta\gamma}\partial_\beta u_\gamma\rangle = \langle f_a w_a\rangle$$

It follows that

$$\frac{dH}{dt} = -\nu\langle w_a \varepsilon_{a\beta\gamma}\partial_\beta w_\gamma\rangle + \langle f_a w_a\rangle \qquad \square$$

Related local definitions

Our analysis of the global conservation laws motivates the introduction of the following local quantities:

a) The <u>local energy dissipation</u> $\varepsilon(\vec{r},t)$ is defined by:

$$\boxed{\varepsilon = \frac{S_{\alpha\beta} S_{\alpha\beta}}{2}}$$

and it measures the energy lost locally due to viscous straining, as opposed to advection.

b) The <u>local energy input</u> $\varepsilon_{in}(\vec{r},t)$ is defined by:

$$\boxed{\varepsilon_{in} = f_\alpha u_\alpha}$$

and it measures the energy injected into the fluid by forcing, but not the energy advected from elsewhere.

c) The <u>local helicity dissipation</u> $h(\vec{r},t)$ has similar interpretation, and it is defined by:

$$\boxed{h = w_\alpha \varepsilon_{\alpha\beta\gamma} \partial_\beta w_\gamma}$$

Note, incidentally that this quantity is the helicity of w_α; Frisch has proposed the term <u>vortical helicity</u>.

d) The <u>local helicity injection</u> $h_{in}(\vec{r},t)$ is given by

$$\boxed{h_{in} = f_\alpha w_\alpha}$$

and it has interpretation similar to $h_{in}(\vec{r},t)$.

Scale budget equations for conservation laws

Our global conservation laws indicate a balance between the forcing term, which injects energy, and the viscous term which dissipates it. The purpose of the scale budget equations is to show that the quadratic term $M_{\alpha\beta\gamma}(u_\beta u_\gamma)$ redistributes energy between various scales. To set these equations up, we introduce scaling filters:

Scale filtering.

Let $\varphi(\vec{r})$ be some arbitrary spatial field. Then we define:

a) The <u>low-pass-filtering</u> of φ by:

$$\boxed{\varphi^<(\vec{r};K) = \int d\vec{r}_0\, d\vec{k}_0\, \frac{H(K-\|\vec{k}_0\|)}{(2\pi)^n} \exp(i\vec{k}_0\cdot(\vec{r}-\vec{r}_0))\,\varphi(\vec{r}_0)}$$

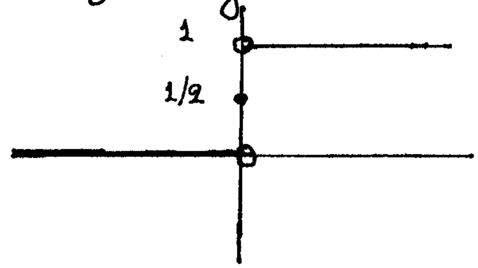
This involves transforming $\varphi(\vec{r})$ to Fourier space, removing all wavenumbers $\|\vec{k}_0\| > K$ and then transforming back to real space.

b) The <u>high-pass-filtering</u> of φ by:

$$\boxed{\varphi^>(\vec{r},K) = \int d\vec{r}_0\, d\vec{k}_0\, \frac{H(\|\vec{k}_0\|-K)}{(2\pi)^n} \exp(i\vec{k}_0\cdot(\vec{r}-\vec{r}_0))\,\varphi(\vec{r}_0)}$$

This involves transforming $\varphi(\vec{r})$ to Fourier space, removing all wavenumbers $\|\vec{k}_0\| < K$ and then transforming back to real space.

where n is the dimensionality of the problem (usually $n=3$) and where $H(x)$ is the Heavyside function given by



$$H(x) = \begin{cases} 1 & , x > 0 \\ 1/2 & , x = 0 \\ 0 & , x < 0 \end{cases}$$

Equivalently we may represent the filtering operators with generalized functions $P^{<}(\vec{r}, \vec{r}_0; k)$ and $P^{<}(\vec{r}, \vec{r}_0, k)$ such that

$$\boxed{\begin{aligned} \varphi^{<}(\vec{r}; k) &= \int d\vec{r}_0 \; P_K^{<}(\vec{r}, \vec{r}_0) \, \varphi(\vec{r}_0) \\ \varphi^{>}(\vec{r}; k) &= \int d\vec{r}_0 \; P_K^{>}(\vec{r}, \vec{r}_0) \, \varphi(\vec{r}_0) \end{aligned}}$$

where $P_K^{<}$ and $P_K^{>}$ are given by:

$$\boxed{\begin{aligned} P_K^{<}(\vec{r}, \vec{r}_0) &= \int d\vec{k}_0 \, \frac{H(k - \|\vec{k}_0\|)}{(2\pi)^n} \exp[i \vec{k}_0 \cdot (\vec{r} - \vec{r}_0)] \\ P_K^{>}(\vec{r}, \vec{r}_0) &= \int d\vec{k}_0 \, \frac{H(\|\vec{k}_0\| - k)}{(2\pi)^n} \exp[i \vec{k}_0 \cdot (\vec{r} - \vec{r}_0)] \end{aligned}}$$

Finally, we may denote filtering using operator notation:

$$\boxed{\varphi^{<}(k) = P_K^{<} \varphi} \qquad \boxed{\varphi^{>}(k) = P_K^{>} \varphi}$$

→ <u>Properties of scale filtering</u>

a) Since $P_K^{<}(\vec{r}, \vec{r}_0) + P_K^{>}(\vec{r}, \vec{r}_0) = \delta(\vec{r} - \vec{r}_0)$, it follows that

$$\boxed{\varphi^{<}(\vec{r}; k) + \varphi^{>}(\vec{r}; k) = \varphi(\vec{r}), \; \forall k > 0}$$

b) Both $P_K^>$ and $P_K^<$ are <u>projector operators</u>; they satisfy:

$$P_K^> P_K^> = P_K^> \quad \text{and} \quad P_K^< P_K^< = P_K^<$$

c) $P_K^<$ and $P_K^>$ commute with ∂_a and ∇^2.

d) Let $\hat{\varphi} = \mathcal{F}\varphi$ be the Fourier transform of φ, and let $\hat{\varphi}_K^< = \mathcal{F} P_K^< \varphi$ and $\hat{\varphi}_K^> = \mathcal{F} P_K^> \varphi$.
Then:

$$\boxed{\begin{aligned} \hat{\varphi}_K^<(\vec{k}_0) &= H(K - \|\vec{k}_0\|)\, \hat{\varphi}(\vec{k}_0) \\ \hat{\varphi}_K^>(\vec{k}_0) &= H(\|\vec{k}_0\| - K)\, \hat{\varphi}(\vec{k}_0) \end{aligned}}$$

We may combine this result with Parseval's theorem which says that

$$\boxed{\langle \varphi \psi \rangle = \int d\vec{k}\ \hat{\varphi}(\vec{k})\, \hat{\psi}(\vec{k})}$$

to show the following:

$$\boxed{\begin{aligned} &\text{i)}\ \langle \varphi (P_K^< \psi) \rangle = \langle (P_K^< \varphi) \psi \rangle \\ &\quad\ \ \langle \varphi (P_K^> \psi) \rangle = \langle (P_K^> \varphi) \psi \rangle \\ &\text{ii)}\ \langle (P_K^< \varphi)(P_K^> \psi) \rangle = 0 \end{aligned}}$$

(self-adjoint property)

(orthogonality property)

Energy and helicity spectra

Let $\mathbf{E}(K)$ be the total energy due to scales $< K$ and $\mathcal{H}(K)$ the total helicity due to the same scales. Then:

$$\boxed{\begin{aligned} \mathcal{E}(K) &= \frac{\langle u_a^<(K)\, u_a^<(K) \rangle}{2} \\ \mathcal{H}(K) &= \frac{\langle u_a^<(K)\, \omega_a^<(K) \rangle}{2} \end{aligned}}$$

We call $\mathcal{E}(K)$ and $\mathcal{H}(K)$ the cumulative energy/helicity spectra.

The energy and helicity spectra are defined by differentiating the corresponding cumulative spectra wrt k:

$$\boxed{E(k) = \frac{d\mathcal{E}(k)}{dk}} \quad \boxed{H(k) = \frac{d\mathcal{H}(k)}{dk}}$$

Note that the total energy and total helicity and the corresponding spectrums are related by:

$$\boxed{E(t) = \int_0^{+\infty} dk\, E(t,k)} \quad \boxed{H(t) = \int_0^{+\infty} H(t,k)\,dk}$$

→ <u>The energy budget equation</u>

Our main result is the budget equation for the cumulative energy spectrum:

$$\boxed{\frac{\partial \mathcal{E}(k)}{\partial t} + \Pi(k) = -2\nu\, \Omega(k) + \mathcal{F}(k)}$$

where:

$$\mathcal{F}(k) = \langle f_a^{<}(k)\, u_a^{<}(k) \rangle \longrightarrow \text{cumulative energy injection}$$
$$\Omega(k) = (1/2)\langle \omega_a^{<}(k)\, \omega_a^{<}(k) \rangle \longrightarrow \text{cumulative enstrophy.}$$

and:

$$\boxed{\Pi = \langle u_a^{<} u_b^{<} \partial_b u_a^{>} \rangle + \langle u_a^{<} u_b^{>} \partial_b u_a^{>} \rangle}$$

is the <u>energy flux</u> through wavenumber k.

<u>Proof</u>

$$\frac{\partial \mathcal{E}(k)}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\langle u_a^{<} u_a^{<} \rangle}{2}\right) = \left\langle u_a^{<} \frac{\partial u_a^{<}}{\partial t} \right\rangle$$

To evaluate the time derivative, we apply $P_k^{<}$ on both sides of the Navier-Stokes equations:

$$\frac{\partial u_a^<}{\partial t} + P^<[u_b \partial_b u_a] = -\partial \varphi^< + \nu \nabla^2 u_a^< + f_a^<$$

$$\partial_a u_a^< = 0$$

It follows that:

$$\frac{\partial E}{\partial t} = -\langle u_a^< P^<[u_b \partial_b u_a] \rangle - \langle u_a^< \partial_a \varphi^< \rangle + \nu \langle u_a^< \nabla^2 u_a^< \rangle + \langle f_a^< u_a^< \rangle$$

Now, we examine the RHS term by term:

$$\langle u_a^< \partial_a \varphi^< \rangle = -\langle (\partial_a u_a^<) \varphi^< \rangle = -0 = 0$$

$$\begin{aligned} \langle u_a^< \nabla^2 u_a^< \rangle &= -\langle (\varepsilon_{ab\gamma} \partial_b u_\gamma^<)(\varepsilon_{a\kappa l} \partial_\kappa u_l^<) \rangle = \\ &= -\langle P^<(\varepsilon_{ab\gamma} \partial_b u_\gamma) P^<(\varepsilon_{a\kappa l} \partial_\kappa u_l) \rangle = \\ &= -\langle (P^< \omega_a)(P^< \omega_a) \rangle = -\langle \omega_a^< \omega_a^< \rangle = -2\Omega \end{aligned}$$

The energy flux Π is given by the non-linear term:

$$\begin{aligned} \Pi &= \langle u_a^< P^<[u_b \partial_b u_a] \rangle = \langle (P^< u_a^<)[u_b \partial_b u_a] \rangle = \\ &= \langle u_a^< [(u_b^< + u_b^>) \partial_b (u_a^< + u_a^>)] \rangle = \\ &= \langle u_a^< u_b^< \partial_b u_a^< \rangle + \langle u_a^< u_b^> \partial_b u_a^< \rangle + \langle u_a^< u_b^< \partial_b u_a^> \rangle + \langle u_a^< u_b^> \partial_b u_a^> \rangle \end{aligned}$$

Let Π_0 denote the first two terms. It can be shown that $\Pi_0 = 0$ as follows:

$$\begin{aligned} \Pi_0 &= \langle u_a^< u_b^< \partial_b u_a^< \rangle + \langle u_a^< u_b^> \partial_b u_a^< \rangle = \\ &= \langle u_a^< u_b \partial_b u_a^< \rangle = \langle u_a^< \partial_b (u_b u_a^<) \rangle = \\ &= -\langle (\partial_b u_a^<)(u_b u_a^<) \rangle = -\langle u_a^< u_b \partial_b u_a^< \rangle = -\Pi_0 \Rightarrow \\ &\Rightarrow 2\Pi_0 = 0 \Rightarrow \underline{\Pi_0 = 0} \end{aligned}$$

therefore $\Pi = \langle u_a^< u_b^< \partial_b u_a^> \rangle + \langle u_a^< u_b^> \partial_b u_a^> \rangle$ □

The helicity budget equation

~~Our main~~ We know proceed to derive a similar equation for the helicity cumulative spectrum:

$$\boxed{\frac{\partial \mathcal{H}(k)}{\partial t} + \mathcal{P}(k) = -2\nu W(k) + \eta(k)}$$

where:

$$\boxed{\begin{aligned} \eta(k) &= \langle f_a^< w_a^< \rangle \\ W(k) &= \langle w_a^< z_a^< \rangle \\ z_a &= \varepsilon_{ab\gamma}\partial_b w_\gamma \end{aligned}}$$

$\eta(k)$ ↔ cumulative helicity injection
$W(k)$ ↔ vortical helicity cumulative spectrum
z_a ↔ vortical vorticity

and:

$$\boxed{\mathcal{P}(k) = \langle \varepsilon_{ab\gamma} u_a w_b^> w_\gamma^< \rangle}$$

is the helicity flux through wavenumber k.

Proof

The first step is to follow closely the reduction of the time derivative from p. 16:

$$\frac{\partial \mathcal{H}(k)}{\partial t} = \left\langle \frac{\partial}{\partial t}\left(\frac{u_a^< w_a^<}{2}\right)\right\rangle = \frac{1}{2}\left\langle u_a^< \frac{\partial w_a^<}{\partial t}\right\rangle + \frac{1}{2}\left\langle w_a^< \frac{\partial u_a^<}{\partial t}\right\rangle =$$

$$= \frac{1}{2}\left\langle u_a^< \frac{\partial w_a^<}{\partial t}\right\rangle + \frac{1}{2}\left\langle u_a^< \frac{\partial w_a^<}{\partial t}\right\rangle$$

$$= \left\langle u_a^< \frac{\partial w_a^<}{\partial t}\right\rangle$$

If we apply the $P^<$ operator on both sides of the dynamical equation of vorticity, we obtain:

$$\frac{\partial w_a^<}{\partial t} = P^<(w_b \partial_b u_a - u_b \partial_b w_a) + \nu \nabla^2 w_a^< + \varepsilon_{ab\gamma}\partial_b f_\gamma^<$$

We substitute this to our expression for $\partial \mathcal{H}/\partial t$ above to obtain the main contributing terms:

$$\frac{\partial \mathcal{H}(t)}{\partial t} = \left\langle u_a^< P^< (w_b \partial_b u_a - u_b \partial_b w_a) \right\rangle + \nu \left\langle u_a^< \nabla^2 w_a^< \right\rangle + \varepsilon_{ab\gamma} \left\langle u_a^< \partial_b f_\gamma^< \right\rangle$$

Recall from p. 16 that

$$w_b \partial_b u_a - u_b \partial_b w_a = \varepsilon_{ab\gamma} \partial_b (\varepsilon_{\gamma\kappa l} u_\kappa w_l)$$

It follows that

$$\begin{aligned}
-\mathcal{P}(k) &= \langle u_a^< P^< (w_b \partial_b u_a - u_b \partial_b w_a) \rangle = \\
&= \langle (P^< u_a^<)(\varepsilon_{ab\gamma} \partial_b (\varepsilon_{\gamma\kappa l} u_\kappa w_l)) \rangle = \\
&= \langle u_a^< (\varepsilon_{ab\gamma} \partial_b (\varepsilon_{\gamma\kappa l} u_\kappa w_l)) \rangle = \\
&= \langle (\varepsilon_{ab\gamma} \partial_b u_\gamma^<)(\varepsilon_{a\kappa l} u_\kappa w_l) \rangle = \\
&= \langle \varepsilon_{a\kappa l} w_a^< u_\kappa w_l \rangle = -\langle \varepsilon_{ab\gamma} u_a w_b w_\gamma^< \rangle \\
&= -\langle \varepsilon_{ab\gamma} u_a (w_b^< + w_b^>) w_\gamma^< \rangle = \\
&= -\langle \varepsilon_{ab\gamma} u_a w_b^< w_\gamma^< \rangle - \langle \varepsilon_{ab\gamma} u_a w_b^> w_\gamma^< \rangle \\
&= -\langle \varepsilon_{ab\gamma} u_a w_b^> w_\gamma^< \rangle
\end{aligned}$$

and, using an argument similar to p. 22

$$\begin{aligned}
\langle u_a^< \nabla^2 w_a^< \rangle &= -\langle (\varepsilon_{ab\gamma} \partial_b u_\gamma^<)(\varepsilon_{a\kappa l} \partial_\kappa w_l^<) \rangle = \\
&= -\langle P^< (\varepsilon_{ab\gamma} \partial_b u_\gamma) P^< (\varepsilon_{a\kappa l} \partial_\kappa w_l) \rangle = \\
&= -\langle w_a^< z_a^< \rangle = -2W(k)
\end{aligned}$$

Finally, following p. 16 we have:

$$\begin{aligned}
\varepsilon_{ab\gamma} \langle u_a^< \partial_b f_\gamma^< \rangle &= \langle u_a^< (\varepsilon_{ab\gamma} \partial_b f_\gamma^<) \rangle = \\
&= \langle (\varepsilon_{ab\gamma} \partial_b u_\gamma^<) f_a^< \rangle = \langle f_a^< w_a^< \rangle
\end{aligned}$$

consequently:

$$\frac{\partial \mathcal{H}(k)}{\partial t} + \underbrace{\varepsilon_{ab\gamma} \langle u_a w_b^> w_\gamma^< \rangle}_{\mathcal{P}(k)} = -2\nu \underbrace{\langle w_a^< z_a^< \rangle}_{W(k)} + \underbrace{\langle f_a^< w_a^< \rangle}_{\eta(k)} \quad \square$$

Transfer equations for conservation laws

The transfer equations are obtained by differentiating the scale budget equations with respect to k. The resulting equations describe the evolution of the energy and helicity spectra, instead of the cumulative spectra. In order to simplify the transfer equations to their definitive form, we need to re-examine the scale filtering operators more carefully:

Further results on filtering operators

Let n denote the dimensionality of our problem. Usually, we take $n=3$. Then:

Proposition: The derivatives of $P_k^{<}$ and $P_k^{>}$ with respect to k are given by

$$\boxed{\frac{\partial P_k^{<}(\vec{r},\vec{r}_0)}{\partial k} = S_k(\vec{r},\vec{r}_0)}$$

$$\boxed{\frac{\partial P_k^{<}(\vec{r},\vec{r}_0)}{\partial k} = -S_k(\vec{r},\vec{r}_0)}$$

where $S_k(\vec{r},\vec{r}_0)$ is given by:

$$\boxed{S_k(\vec{r},\vec{r}_0) = \frac{k^{n-1}}{(2\pi)^n}\int_{SO(n)} d\Omega(A)\, \exp[i(kA\vec{e})\cdot(\vec{r}-\vec{r}_0)]}$$

Proof

$$\frac{\partial P_k^{<}(\vec{r},\vec{r}_0)}{\partial k} = \frac{\partial}{\partial k}\int d\vec{k}_0\, \frac{H(k-\|\vec{k}_0\|)}{(2\pi)^n}\exp[i\vec{k}_0\cdot(\vec{r}-\vec{r}_0)] =$$

$$= \frac{1}{(2\pi)^n} \int d\vec{k}_o \, \frac{\partial}{\partial k} \left[H(k - \|\vec{k}_o\|) \right] \exp\left[i\vec{k}_o \cdot (\vec{r} - \vec{r}_o) \right] =$$

$$= \frac{1}{(2\pi)^n} \int_0^{+\infty} dk_o \int_{SO(3)} d\Omega(A) \; k_o^{n-1} \, \delta(k - k_o) \exp\left[i(k_o A\vec{e}) \cdot (\vec{r} - \vec{r}_o) \right] =$$

$$= \frac{1}{(2\pi)^n} \int_{SO(3)} d\Omega(A) \; k^{n-1} \exp\left[i(kA\vec{e}) \cdot (\vec{r} - \vec{r}_o) \right]$$

$$\equiv S_k(\vec{r}, \vec{r}_o)$$

It is easy to see via chain rule that

$$\frac{\partial P_k^{>}(\vec{r}, \vec{r}_o)}{\partial k} = - \frac{\partial P_k^{<}(\vec{r}, \vec{r}_o)}{\partial k} = - S_k(\vec{r}, \vec{r}_o) \qquad \square$$

The effect of the $\mathcal{S}_k$ operator is to remove all the wavenumbers except for a spherical shell and amplify that shell by a spherical delta peak. We show that in the next proposition:

<u>Proposition</u> : If $\psi = S_k \varphi$ and $\hat{\varphi} = \mathcal{F}\varphi$, $\hat{\psi} = \mathcal{F}\psi$ are the corresponding Fourier transforms, then:

$$\boxed{\begin{array}{l} \psi(\vec{r}) = \displaystyle\int_{SO(3)} d\Omega(A) \left[k^{n-1} \hat{\varphi}(kA\vec{e}) \right] \exp\left[i(kA\vec{e}) \cdot \vec{r} \right] \\ \hline \hat{\psi}(\vec{k}_o) = \varphi(\vec{k}_o) \, \delta(k - \|\vec{k}_o\|) \end{array}}$$

<u>Proof</u>

$$\psi(\vec{r}) = \int d\vec{r}_o \; S_k(\vec{r}, \vec{r}_o) \, \varphi(\vec{r}_o) =$$

$$= \int d\vec{r}_o \int d\Omega(A) \frac{k^{n-1}}{(2\pi)^n} \exp\left[i(kA\vec{e}) \cdot (\vec{r} - \vec{r}_o) \right] \varphi(\vec{r}_o) =$$

$$= \int d\Omega(A) \; k^{n-1} \left[\frac{1}{(2\pi)^n} \int \varphi(\vec{r}_o) \exp\left[i(kA\vec{e}) \cdot \vec{r}_o \right] d\vec{r}_o \right] \exp\left[i(kA\vec{e}) \cdot \vec{r} \right]$$

$$= \int d\underline{\Omega}(A) \; \left[k^{n-1} \, \hat{\varphi}(kA\vec{e}) \right] \exp\left[i(kA\vec{e}) \cdot \vec{r} \right] \qquad (1)$$

To show the 2nd relation note that

$$\psi(\vec{r}) = \int_0^{+\infty} dk_0 \int_{SO(3)} d\underline{\Omega}(A) \; \delta(k-k_0) \; k_0^{n-1} \; \hat{\varphi}(k_0 A\vec{e}) \exp\left[i(k_0 A\vec{e}) \cdot \vec{r} \right]$$

$$= \int d\vec{k}_0 \;\; \hat{\varphi}(\vec{k}_0) \, \delta(k - \|\vec{k}_0\|) \exp\left[i\, \vec{k}_0 \cdot \vec{r} \right]$$

$$= \int d\vec{k}_0 \;\; \hat{\psi}(\vec{k}_0) \exp\left[i \vec{k}_0 \cdot \vec{r} \right]$$

consequently $\qquad \hat{\psi}(\vec{k}_0) = \hat{\varphi}(\vec{k}_0) \, \delta(k - \|\vec{k}_0\|) \qquad \square$

Now we develop some very useful identities:

Theorem : $\boxed{\int d\vec{r} \;\; S_k(\vec{r}, \vec{r}_1) P_k^{<}(\vec{r}, \vec{r}_2) = \frac{1}{(2\pi)^n} \int d\vec{k}_0 \; \frac{\delta(k - \|\vec{k}_0\|)}{2} \exp\left[i\vec{k}_0 \cdot (\vec{r}_2 - \vec{r}_1) \right]}$

Proof

$$I = \int d\vec{r} \;\; S_k(\vec{r}, \vec{r}_1) \, P_k^{<}(\vec{r}, \vec{r}_2) =$$

$$= \int d\vec{r} \left[\frac{1}{(2\pi)^n} \int d\vec{k}_1 \;\; \delta(k - \|\vec{k}_1\|) \exp\left[i\vec{k}_1 \cdot (\vec{r} - \vec{r}_1) \right] \right]$$

$$\times \left[\frac{1}{(2\pi)^n} \int d\vec{k}_2 \;\; H(k - \|\vec{k}_2\|) \exp\left[i\vec{k}_2 \cdot (\vec{r} - \vec{r}_2) \right] \right]$$

$$= \frac{1}{(4\pi^2)^n} \int d\vec{k}_1 d\vec{k}_2 \;\; \delta(k - \|\vec{k}_1\|) H(k - \|\vec{k}_2\|) \left[\int d\vec{r} \exp\left[(\vec{k}_1 + \vec{k}_2) \cdot \vec{r} \right] \right]$$

$$\times \exp\left[-i \left(\vec{k}_1 \cdot \vec{r}_1 + \vec{k}_2 \cdot \vec{r}_2 \right) \right]$$

$$= \frac{1}{(4\pi^2)^n} \int d\vec{k}_1 d\vec{k}_2 \;\; \delta(k - \|k_1\|) \, H(k - \|\vec{k}_2\|) \, \delta(\vec{k}_1 + \vec{k}_2) \, (2\pi)^n \exp\left[-i \left(\vec{k}_1 \cdot \vec{r}_1 + \vec{k}_2 \cdot \vec{r}_2 \right) \right]$$

$$= \frac{1}{(2\pi)^n} \int d\vec{k}_1 \;\; \delta(k - \|\vec{k}_1\|) \, H(k - \|\vec{k}_1\|) \exp\left[i\, \vec{k}_1 \cdot (\vec{r}_2 - \vec{r}_1) \right]$$

$$= \frac{1}{(2\pi)^n} \int d\vec{k}_1 \, \frac{\delta(k - \|\vec{k}_1\|)}{2} \exp[i\vec{k}_1 \cdot (\vec{r}_2 - \vec{r}_1)]$$

where, in the last step, we use the definition of the Heavyside function: $H(0) = 1/2$. □

Note that if we compare the outcome with the definition of S_k, it follows that:

$$\boxed{\int d\vec{r} \; S_k(\vec{r}, \vec{r}_1) P_k^<(\vec{r}, \vec{r}_2) = \frac{S_k(r_2, r_1)}{2}}$$

Also note that this result can still be obtained if we replace $P_k^<$ with $P_k^>$:

$$\boxed{\int d\vec{r} \; S_k(\vec{r}, \vec{r}_1) P_k^>(\vec{r}, \vec{r}_2) = \frac{S_k(r_2, r_1)}{2}}$$

The main application of these identities is in evaluating products of filtered fields, integrated over space, and differentiated with respect to k.

<u>Theorem</u>: Let a, b be two fields with $\hat{a} = \mathcal{F}a$ and $\hat{b} = \mathcal{F}b$ their Fourier transforms. Then:

$$\boxed{\frac{d}{dk} \langle a^< b^< \rangle = \int d\vec{k}_0 \; \delta(k - \|\vec{k}_0\|) \, \frac{\hat{a}^*(\vec{k}_0)\hat{b}(\vec{k}_0) + \hat{a}(\vec{k}_0)\hat{b}^*(\vec{k}_0)}{2}}$$

<u>Proof</u>

$$\frac{d}{dk} \langle a^< b^< \rangle = \left\langle \frac{d}{dk}[a^< b^<] \right\rangle = \left\langle \frac{da^<}{dk} b^< \right\rangle + \left\langle a^< \frac{db^<}{dk} \right\rangle$$

Without loss of generality, we consider only the first term of the above expression.

$$1\text{st} = \left\langle \frac{da^<}{dk} b^< \right\rangle = \int d\vec{r}\; \frac{da^<(\vec{r})}{dk}\, b^<(\vec{r}) =$$

$$= \int d\vec{r}\, d\vec{r}_1 d\vec{r}_2\;\; S_K(\vec{r},\vec{r}_1) P_K^<(\vec{r},\vec{r}_2)\; a(\vec{r}_1) b(\vec{r}_2) =$$

$$= \int d\vec{r}_1 d\vec{r}_2 \left[\int d\vec{r}\;\; S_K(\vec{r},\vec{r}_1) P_K^<(\vec{r},\vec{r}_2) \right] a(\vec{r}_1)\, b(\vec{r}_2) =$$

$$= \int d\vec{r}_1 d\vec{r}_2\; \frac{S_K(\vec{r}_1,\vec{r}_2)}{2}\; a(\vec{r}_1)\, b(\vec{r}_2) =$$

$$= \int d\vec{r}_1 d\vec{r}_2 d\vec{k}_0\; \frac{1}{(2\pi)^n}\; \frac{\delta(k-\|\vec{k}_0\|)}{2}\; \exp\left[i\vec{k}_0 \cdot (\vec{r}_2 - \vec{r}_1) \right] a(\vec{r}_1)\, b(\vec{r}_2) =$$

$$= \int d\vec{k}_0\; \delta(k-\|\vec{k}_0\|)\; \frac{\hat{a}(\vec{k}_0)\, \hat{b}^*(\vec{k}_0)}{2}$$

Similarly, the 2nd term is :

$$2\text{nd} = \int d\vec{k}_0\;\; \delta(k-\|\vec{k}_0\|)\; \frac{\hat{a}^*(\vec{k}_0)\, \hat{b}(\vec{k}_0)}{2}$$

and the theorem follows. □

This integral can be simplified into a spherical integral:

$$\boxed{\frac{d}{dk} \left\langle a^< b^< \right\rangle = \int_{SO(n)} d\Omega(A)\; \frac{\hat{a}^*(kA\vec{e})\, \hat{b}(kA\vec{e}) + \hat{a}(kA\vec{e})\, \hat{b}^*(kA\vec{e})}{2}\, k^{n-1}}$$

When $a = b$, it can be simplified even further:

$$\boxed{\frac{d}{dk} \left\langle a^< a^< \right\rangle = \int_{SO(n)} d\Omega(A) k^{n-1} \left| \hat{a}(kA\vec{e}) \right|^2}$$

which is the physically intuitive idea of adding together a spherical shell of Fourier-mode amplitudes squared.

Theory of triadic interactions

Now we generalise this approach to triadic terms. For motivation note that the flux terms in the budget equations of conservation laws have the form $\langle a b^< c^> \rangle$ which indicates interactions between large scales and small scales. The derivatives of such terms with respect to k are given by:

<u>Proposition</u> :

$$\boxed{\frac{d}{dk}\langle a b^< c^> \rangle = \int d\vec{r}_0\, d\vec{r}_1\, d\vec{r}_2\; \tau(k, r_0, r_1, r_2)\, a(r_0)\, b(r_1)\, c(r_2)}$$
$$\boxed{\tau(k, \vec{r}_0, \vec{r}_1, \vec{r}_2) = S_k(\vec{r}_0, \vec{r}_1)\, P_k^>(\vec{r}_0, \vec{r}_2) - S_k(\vec{r}_0, \vec{r}_2)\, P_k^<(\vec{r}_0, \vec{r}_1)}$$

<u>Proof</u>

Note that a is independent of k, therefore:

$$\frac{d}{dk}\langle a b^< c^> \rangle = \left\langle a \frac{db^<}{dk} c^> \right\rangle + \left\langle a b^< \frac{dc^>}{dk} \right\rangle -$$

Now, we consider each term separately:

$$1\text{st} = \left\langle a \frac{db^<}{dk} c^> \right\rangle = \int d\vec{r}\; a(\vec{r})\, \frac{db^<(\vec{r})}{dk}\, c^>(\vec{r}) =$$
$$= \int d\vec{r}\, d\vec{r}_1\, d\vec{r}_2\;\; a(\vec{r})\, S_k(\vec{r}, \vec{r}_1)\, b(\vec{r}_1)\, P_k^>(\vec{r}, \vec{r}_2)\, c(\vec{r}_2)$$
$$= \int d\vec{r}_0\, d\vec{r}_1\, d\vec{r}_2\; [S_k(r_0, r_1)\, P_k^>(r_0, r_2)]\, a(\vec{r}_0)\, b(\vec{r}_1)\, c(\vec{r}_2)$$

and

$$2\text{nd} = \left\langle a b^< \frac{dc^>}{dk} \right\rangle = \int d\vec{r}\;\; a(\vec{r})\, b^<(\vec{r})\, \frac{dc^>(\vec{r})}{dk} =$$
$$= \int d\vec{r}\, d\vec{r}_1\, d\vec{r}_2\;\; a(\vec{r})\, P_k^<(\vec{r}, \vec{r}_1)\, b(\vec{r}_1)\, [-S_k(\vec{r}, \vec{r}_2)]\, c(\vec{r}_2) =$$
$$= \int d\vec{r}_0\, d\vec{r}_1\, d\vec{r}_2\; [-S_k(\vec{r}_0, \vec{r}_2)\, P_k^<(\vec{r}_0, \vec{r}_1)]\, a(\vec{r}_0)\, b(\vec{r}_1)\, c(\vec{r}_2) \qquad \square$$

The next step is to write this expression in terms of the Fourier components of a, b, c.

<u>Theorem</u>: Let a, b, c be fields with Fourier ~~comp~~ transforms $\hat{a} = \mathcal{F}a$, $\hat{b} = \mathcal{F}b$, and $\hat{c} = \mathcal{F}c$. Then

$$\boxed{\frac{d}{dk}\langle a b^{<} c^{>}\rangle = \int d\vec{k}_0 d\vec{k}_1 d\vec{k}_2 \; \hat{\tau}(k,\vec{k}_0,\vec{k}_1,\vec{k}_2)\, \hat{a}(\vec{k}_0)\hat{b}(\vec{k}_1)\hat{c}(\vec{k}_2)}$$

where $\tau(k,\vec{k}_0,\vec{k}_1,\vec{k}_2)$ represents the triadic interactions kernel and it is given by:

$$\boxed{\tau(k,\vec{k}_0,\vec{k}_1,\vec{k}_2) = (2\pi)^n \delta(\vec{k}_0+\vec{k}_1+\vec{k}_2)\left[\delta(k-\|\vec{k}_1\|)H(\|\vec{k}_2\|-k) - \delta(k-\|\vec{k}_2\|)H(k-\|\vec{k}_1\|)\right]}$$

<u>Proof</u>

We start by considering the real integral split into two terms:

$$I_1 = \int d\vec{r}_0 d\vec{r}_1 d\vec{r}_2 \; S_k(\vec{r}_0,\vec{r}_1)\, P_k^{>}(\vec{r}_0,\vec{r}_2)\, a(\vec{r}_0)\, b(\vec{r}_1)\, c(\vec{r}_2)$$

$$I_2 = \int d\vec{r}_0 d\vec{r}_1 d\vec{r}_2 \; S_k(\vec{r}_0,\vec{r}_2)\, P_k^{<}(\vec{r}_0,\vec{r}_1)\, a(\vec{r}_0)\, b(\vec{r}_1)\, c(\vec{r}_2)$$

For the first integral:

$$\begin{aligned} I_1 = \int d\vec{r}_0 d\vec{r}_1 d\vec{r}_2 \; b(\vec{r}_1)\, c(\vec{r}_2) & \left[\int d\vec{k}_0 \; \hat{a}(\vec{k}_0) \exp[i\vec{k}_0\cdot\vec{r}_0]\right] \\ & \times \left[\frac{1}{(2\pi)^n}\int d\vec{k}_1 \; \delta(k-\|\vec{k}_1\|) \exp[i k_1\cdot(\vec{r}_0-\vec{r}_1)]\right] \\ & \times \left[\frac{1}{(2\pi)^n}\int d\vec{k}_2 \; H(\|\vec{k}_2\|-k) \exp[i k_2\cdot(\vec{r}_0-\vec{r}_2)]\right] \end{aligned}$$

$$= \int d\vec{k}_0 d\vec{k}_1 d\vec{k}_2 \left[\hat{a}(\vec{k}_0)\, \delta(k - \|\vec{k}_1\|)\, H(\|\vec{k}_2\| - k) \right]$$

$$\times \left[\int d\vec{r}_0 \exp[i(\vec{k}_0 + \vec{k}_1 + \vec{k}_2) \cdot \vec{r}_0] \right]$$

$$\times \left[\frac{1}{(2\pi)^n} \int d\vec{r}_1\, b(\vec{r}_1) \exp[-i \vec{k}_1 \cdot \vec{r}_1] \right]$$

$$\times \left[\frac{1}{(2\pi)^n} \int d\vec{r}_2\, b(\vec{r}_2) \exp[-i \vec{k}_2 \cdot \vec{r}_2] \right]$$

$$= \int d\vec{k}_0 d\vec{k}_1 d\vec{k}_2\ \hat{a}(\vec{k}_0) \hat{b}(\vec{k}_1) \hat{c}(\vec{k}_2) \underline{\delta(k - \|\vec{k}_1\|) H(\|\vec{k}_2\| - k)}\, (2\pi)^n \delta(\vec{k}_0 + \vec{k}_1 + \vec{k}_2)$$

I_2 can be evaluated similarly: all factors are retained except for the underlined part where $k_1 \leftrightarrow k_2$ are exchanged and the argument of the Heavyside function is reflected:

$$I_2 = \int dk_0 dk_1 dk_2\ \hat{a}(\vec{k}_0) \hat{b}(\vec{k}_1) \hat{c}(\vec{k}_2) \delta(k - \|\vec{k}_2\|) H(k - \|\vec{k}_1\|) (2\pi)^n \delta(\vec{k}_0 + \vec{k}_1 + \vec{k}_2)$$

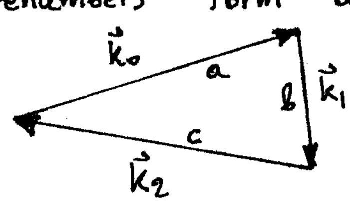
Since $(d/dk) \langle a b^< c^> \rangle = I_1 - I_2$, it follows that the triadic interactions kernel is given by:

$$\tau(k, \vec{k}_0, \vec{k}_1, \vec{k}_2) = (2\pi)^n \delta(\vec{k}_0 + \vec{k}_1 + \vec{k}_2) [\delta(k - \|\vec{k}_1\|) H(\|\vec{k}_2\| - k) - \delta(k - \|\vec{k}_2\|) H(k - \|\vec{k}_1\|)]$$

$\square$

The triadic interactions kernel is interesting because it contains information about the nature of the interactions that transfer conserved quantities between modes:

a) The only wavenumbers that interact are the ones that satisfy the relation $\vec{k}_0 + \vec{k}_1 + \vec{k}_2 = 0$; this means that the wavenumbers form a triangle:



We indicate on the diagram if the corresponding mode is an a-mode, b-mode, or c-mode.

b) The term $\tau(k, \vec{k}_0, \vec{k}_1, \vec{k}_2)\, \hat{a}(\vec{k}_0)\, \hat{b}(\vec{k}_1)\, \hat{c}(\vec{k}_2)$ represents how much "energy" is created at the shell k of the spectrum by the interaction between $\vec{k}_0, \vec{k}_1, \vec{k}_2$ modes. If we integrate this term over $k \in (0, +\infty)$ then we may find if a particular group of interactions creates or destroys "energy". Of course, the global conservation law tells us that <u>all</u> the interactions together conserve "energy". We will now prove a stronger result: we will show that every group $(\vec{k}_0, \vec{k}_1, \vec{k}_2)$ by itself is conservative.

<u>Theorem</u>: $\boxed{\int_0^{+\infty} \tau(k, \vec{k}_1, \vec{k}_2, \vec{k}_3)\, dk = 0}$

<u>Proof</u>

$$I = \int_0^{+\infty} dk\, \tau(k, \vec{k}_0, \vec{k}_1, \vec{k}_2) =$$

$$= (2\pi)^n \delta(\vec{k}_0 + \vec{k}_1 + \vec{k}_2) \left\{ \int_0^{+\infty} dk\, \delta(k - \|\vec{k}_1\|) H(\|k_2\| - k) - \int_0^{+\infty} dk\, \delta(k - \|\vec{k}_2\|) H(k - \|\vec{k}_1\|) \right\} =$$

$$= (2\pi)^n \delta(\vec{k}_0 + \vec{k}_1 + \vec{k}_2) \left[H(\|\vec{k}_2\| - \|\vec{k}_1\|) - H(\|\vec{k}_2\| - \|\vec{k}_1\|) \right] = 0 \quad \square$$

So, every triad interaction conserves "energy", and this is equivalent to the $\langle ab^{<} c^{>} \rangle$ form of the flux terms. This result is known as the <u>detailed conservation theorem</u>, and it is a generalization of a similar result by Kraichnan. The generalization comes in by being able to say all this without dissussing what a, b, c are and how they are related. In this sense, this result is wider because it is based on fewer assumptions. However, it is also narrower, because it says <u>less</u> than Kraichnan's formulation. We now make the connection to Kraichnan's formalism in the next remarks. We will reconsider Kraichnan's approach later.

c) If we maintain the naive viewpoint that a, b, c are unrelated, then we may ask how much "energy" is transfered from a-shell k_0, b-shell k_1, and c-shell k_2 to the "energy"-shell k. Lets denote that as $T_0(k|k_0,k_1,k_2)$. By definition, it follows that:

$$T_0(k|k_0,k_1,k_2) = \int d\Omega(A_0)d\Omega(A_1)d\Omega(A_2)\,\tau(k,k_0A_0\vec{e},k_1A_1\vec{e},k_2A_2\vec{e}) \times \hat{a}(k_0A_0\vec{e})\,\hat{b}(k_1A_1\vec{e})\,\hat{c}(k_2A_2\vec{e})$$

If we carry out this integral, using theorem p.31, we find that there are two types of interactions that may contribute to T:

$$\boxed{T_0(k|k_0,k_1,k_2) = T_c(k_0,k_1,k_2)[\delta(k-k_1)H(k_2-k) - \delta(k-k_2)H(k-k_1)]}$$

where $T_c(k_0,k_1,k_2)$ is the triple convolution of the k_0 a-shell, k_1 b-shell, and k_2 c-shell:

$$\boxed{T_c(k_0,k_1,k_2) = \int \prod_{a=0}^{2} d\Omega(A_a)\,\delta(k_0A_0\vec{e}+k_1A_1\vec{e}+k_2A_2\vec{e}) \times \hat{a}(k_0A_0\vec{e})\,\hat{b}(k_1A_1\vec{e})\,\hat{c}(k_2A_2\vec{e})}$$

d) Now we bring in a specialization. Recall that the flux terms that we have encountered so far are:

$$\Pi(k) = \langle u_a^{<} u_b \partial_b u_a^{>} \rangle \quad \text{(energy flux)}$$
$$\mathcal{P}(k) = \langle \varepsilon_{ab\gamma} u_a w_b^{>} w_\gamma^{<} \rangle \quad \text{(helicity flux)}$$

In Fourier space the $\hat{a}, \hat{b}, \hat{c}$ modes are all multiples of $\hat{u}_a(\vec{k})$. The resulting energy and helicity spectra can be shown to also tie back to $\hat{u}(k)$. So, all the interacting shells are really u-shells, and the shell of the spectrum of the conserved quantity is also, when it comes down to it, also a u-shell. This is also true for conservation of enstrophy in 2d. It is in fact true for any conserved quantity which is a quadratic of the same field in Fourier space.

In this context, the distinction between (k_0, k_1, k_2), (k_1, k_2, k_0), and (k_2, k_0, k_1) interactions is artificial, therefore we should be looking at

$$\boxed{\begin{aligned} T(k|k_0,k_1,k_2) \equiv\ & T_0(k|k_0,k_1,k_2) + T_0(k|k_1,k_2,k_0) + \\ & + T_0(k|k_2,k_0,k_1) + T_0(k|k_2,k_1,k_0) + \\ & + T_0(k|k_1,k_0,k_2) + T_0(k|k_0,k_2,k_1) \end{aligned}}$$

This expression represents the creation or destruction of "energy" at shell k by the interactions between shells (k_0, k_1, k_2) in any permutation. Now we can finally bring out the connection between this formalism and kraichonan's.

We employ our analytic expression for T_0 and collect the common delta function terms:

$$T(k|k_0,k_1,k_2) = a(k|k_0,k_1,k_2)\,\delta(k-k_0) + b(k|k_0,k_1,k_2)\,\delta(k-k_1) + c(k|k_0,k_1,k_2)\,\delta(k-k_2)$$

This expression says that k-shell gets "energy" iff $k=k_0$ or $k=k_1$ or $k=k_2$. The corresponding amounts are given by the weight in front of the corresponding delta function. We can gauge out these amounts by integrating every term separately over $k \in (0, +\infty)$; we naturally expect to obtain the amount of "energy" transfered to each shell by the other two shells.

To demonstrate this claim rigorously, we introduce the following notation:

$$T_0(k|k_0,k_1,k_2) \longrightarrow T_0(k|012)$$
$$T_c(k_0,k_1,k_2) \longrightarrow T_c(012)$$
$$T(k|k_0,k_1,k_2) \longrightarrow T(k|012)$$

and similarly for a, b, c. We also use:

$$\delta(k-k_0) \longrightarrow \delta(k0)$$
$$H(k-k_0) \longrightarrow H(k0)\,,\quad H(k_0-k) \longrightarrow H(0k)$$

Now, we will prove the existence of a function $T(k|p,q)$ that represents the transfer of "energy" from shells p, q to the shell k. We will use similarly abbreviated notation for this function too.

Proposition : There is a function $T(k|p,q)$ such that:

$$\int_0^{+\infty} a(k|012)\,\delta(k0)\,dk = T(0|12)$$
$$\int_0^{+\infty} b(k|012)\,\delta(k1)\,dk = T(1|20)$$
$$\int_0^{+\infty} c(k|012)\,\delta(k2)\,dk = T(2|01)$$

Proof

Recall that

$$T_o(k|012) = T_c(012)\left[\delta(k1)H(2k) - \delta(k2)H(k1)\right]$$

Let's build towards $T(k|012)$ by adding these together in the following pattern:

$$T_o(k|012) + T_o(k|021) = a_o(k|012)\delta(k0) + b_o(k|012)\delta(k1) + c_o(k|012)\delta(k2)$$
$$T_o(k|120) + T_o(k|102) = a_o(k|120)\delta(k1) + b_o(k|120)\delta(k2) + c_o(k|120)\delta(k0)$$
$$T_o(k|201) + T_o(k|210) = a_o(k|201)\delta(k2) + b_o(k|201)\delta(k0) + c_o(k|201)\delta(k1)$$

The first equation defines a_o, b_o, c_o. The next two equations follow from the first via cyclic rotation of the k_0, k_1, k_2 variables. From the first equation, and our expression for T_o further above, we may read off a_o, b_o, c_o:

$$a_o(k|012) = 0$$
$$b_o(k|012) = T_c(012)H(2k) - T_c(021)H(k2)$$
$$c_o(k|012) = T_c(021)H(1k) - T_c(012)H(k1)$$

and by adding the three equations together we may establish the relation between a, b, c and a_o, b_o, c_o by collecting with respect to $\delta(k0)$, $\delta(k1)$, $\delta(k2)$:

$$a(k|012) = a_o(k|012) + b_o(k|201) + c_o(k|120)$$
$$b(k|012) = a_o(k|120) + b_o(k|012) + c_o(k|201)$$
$$c(k|012) = a_o(k|201) + b_o(k|120) + c_o(k|012)$$

From these relations it follows that

$$a(k|012) = b(k|201) = c(k|120)$$

If we <u>define</u> $T(0|12)$ via

$$T(0|12) = \int_0^{+\infty} a(k|012)\,\delta(k0)\,dk$$

then the other two relations follow by employing the above relation between a, b, c. For example:

$$\begin{aligned} T(1|20) &= \int_0^{+\infty} a(k|120)\,\delta(k1)\,dk = && \text{(definition)} \\ &= \int_0^{+\infty} b(k|012)\,\delta(k1)\,dk && \text{(identity bt. } a, b, c\text{)} \end{aligned}$$

and

$$\begin{aligned} T(2|01) &= \int_0^{+\infty} a(k|201)\,\delta(k2)\,dk = && \text{(definition)} \\ &= \int_0^{+\infty} b(k|120)\,\delta(k2)\,dk = && \text{(identity)} \\ &= \int_0^{+\infty} c(k|012)\,\delta(k2)\,dk. && \text{(identity)} \end{aligned}$$

□

Without using a closed form for $T(k|p,q)$ we can prove kraichnan's identity of triad interactions as follows:

<u>Proposition</u>:

$$\boxed{\begin{aligned} &T(k|p,q) + T(p|q,k) + T(q|k,p) = 0 \\ &T(k|p,q) = T(k|q,p) \end{aligned}}$$

<u>Proof</u>

a) Recall the detailed conservation theorem from p. 33:

$$\int_0^{+\infty} T_0(k|012)\,dk = 0$$

From the definition of $T_0(k|012)$ (see p.35) it follows that:

$$\int_0^{+\infty} T(k|012)\,dk = 0$$

However note that this integral can also be written as:

$$I = \int_0^{+\infty} T(k|012)dk = \int_0^{+\infty}\left[a(k|012)\delta(k0) + b(k|012)\delta(k1) + c(k|012)\delta(k2)\right]dk$$

$$= \int_0^{+\infty} a(k|012)\delta(k0)dk + \int_0^{+\infty} b(k|012)\delta(k1)dk + \int_0^{+\infty} c(k|012)\delta(k2)dk =$$

$$= T(0|12) + T(1|20) + T(2|01)$$

therefore: $T(0|12) + T(1|20) + T(2|01) = 0$

b) Recall that:

$$T(k|012) = a(k|012)\delta(k0) + b(k|012)\delta(k1) + c(k|012)\delta(k2)$$
$$T(k|021) = a(k|021)\delta(k0) + b(k|021)\delta(k2) + c(k|021)\delta(k1)$$

By the symmetric definition of $T(k|012)$ we have:

$$T(k|012) = T(k|021) \Rightarrow a(k|012) = a(k|021)$$

consequently,

$$T(0|12) = \int_0^{+\infty} a(k|012)\delta(k0)dk = \int_0^{+\infty} a(k|021)\delta(k0)dk =$$

$$= T(0|21) \qquad \square$$

Now, we will derive an analytic relation between $T(k|p,q)$ and the convolution integral $T_c(k,p,q)$.

<u>Theorem</u> : $\boxed{\begin{aligned} T(0|12) = [T_c(102) + T_c(201)] - [T_c(201) + T_c(210)]\, H(01) \\ - [T_c(102) + T_c(120)]\, H(02) \end{aligned}}$

<u>Proof</u>

Review p. 36-38 for the notation used in this proof.

$$T(0|12) = \int_0^{+\infty} a(k|012)\,\delta(k0)\,dk =$$
$$= \int_0^{+\infty} [a_0(k|012) + b_0(k|201) + c_0(k|120)]\,\delta(k0)\,dk$$

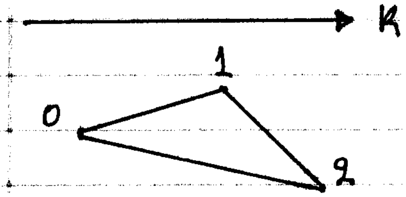
Recall that : (see p. 37)

$$a_0(k|012) = 0$$
$$b_0(k|201) = T_c(201)H(1k) - T_c(210)H(k1)$$
$$c_0(k|120) = T_c(102)H(2k) - T_c(120)H(k2)$$

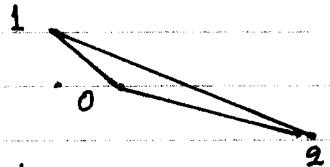
consequently, after we do the integral, we get:

$$\begin{aligned} T(0|12) &= [T_c(201)H(10) - T_c(210)H(01)] + [T_c(102)H(20) - T_c(120)H(02)] = \\ &= T_c(201)[1 - H(01)] - T_c(210)H(01) + \\ &\quad + T_c(102)[1 - H(02)] - T_c(120)H(02) = \\ &= [T_c(102) + T_c(201)] - [T_c(201) + T_c(210)]H(01) \\ &\qquad - [T_c(102) + T_c(120)]H(02) \qquad \square \end{aligned}$$

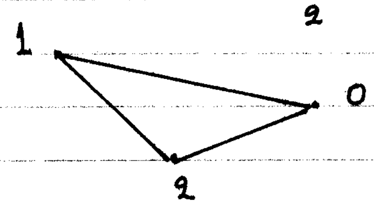
This expression can be simplified for the special triads k_1, k_2, k_3 shown in the following figure :



$$\boxed{\begin{aligned} &H(01) = H(02) = 0 \qquad (0<1<2) \\ &T(0|12) = T_c(102) + T_c(201) \end{aligned}}$$



$$\boxed{\begin{aligned} &H(01) = 1\ ,\ H(02) = 0 \qquad (1<0<2) \\ &T(0|12) = T_c(102) - T_c(210) \end{aligned}}$$



$$\boxed{\begin{aligned} &H(01) = H(02) = 1 \qquad (1<2<0) \\ &T(0|12) = -T_c(210) - T_c(120) \end{aligned}}$$

Note that thanks to the symmetry $T(k|p,q) = T(k|q,p)$ it is not necessary to consider the case $2<1$.

Let us summarize for a moment our analysis so far. We begin with the abstract form of the flux term $\langle ab^< c^> \rangle$ and show that it implies triadic interactions and the detailed conservation theorem. Then we introduce the idea that a, b, c-modes are indistinguishable and that leads to the introduction of Kraichnan's transfer function $T(k|p,q)$. This function tells us how much energy is transfered to shell k by (k,p,q) interactions. Then we prove the fundamental identities:

$$T(k|p,q) + T(p|q,k) + T(q|k,p) = 0 \qquad (1)$$

$$T(k|p,q) = T(k|q,p) \qquad (2)$$

and express $T(k|p,q)$ in terms of the integral $T_c(k,p,q)$. The actual meaning of $T(k|p,q)$ can be difficult to grasp. Essentially, equation (1) says that <u>any energy transfered to k, by this interaction, is energy lost by p and q.</u> To see that, just transfer the 2nd and 3rd term in equation (1) to the RHS. Our next step is to focus on the physical meaning of $T(k|p,q)$ by relating it with the transfer and flux spectra.

The flux and transfer spectra

We have seen that the governing equations for cumulative spectra of conserved quantities take the following general form:

$$\boxed{\frac{\partial \mathcal{A}(k)}{\partial t} + \langle a b^{<}(k) c^{>}(k) \rangle = \mathcal{J}(k) - \mathcal{X}(k)}$$

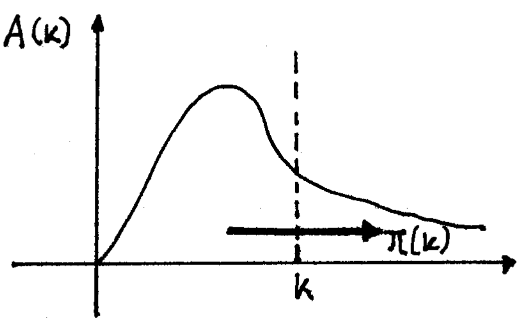
where

$\mathcal{A}(k) =$ cumulative spectrum of conserved quantity
$\mathcal{J}(k) =$ injection cumulative spectrum
$\mathcal{X}(k) =$ dissipation cumulative spectrum

Now consider the physical meaning of the term

$$\boxed{\Pi(k) = \langle a b^{<}(k) c^{>}(k) \rangle}$$

We already know that this term does not create or destroy "energy"; it only moves it around. More specifically, $\Pi(k)$ is the rate with which "energy" moves from the $(0,k)$ interval to $(k,+\infty)$:



spectrum vs. wavenumber

To see this, note that $\mathcal{A}(k)$ is the total "energy" in the $[0,k)$ interval and the flux term contributes $-\Pi(k)$ to its rate of change. Therefore a positive $\Pi(k)$

moves "energy" away from that interval, towards larger wavenumbers.

Also note that the mathematical structure of the flux term imposes boundary conditions:

$$\boxed{\Pi(k) = 0} \quad \boxed{\lim_{k\to+\infty} \Pi(k) = \overset{0}{\cancel{+\infty}}}$$

To see this note that when $k=0 \Rightarrow b^{<}(k) = 0$. In the limit $k\to+\infty$, $\lim_{k\to+\infty} c^{>}(k) = 0$. The physical interpretation is that "energy" may not flow through the boundary of the $[0,+\infty)$ interval; the flux term does not create it nor destroys it.

If we differentiate both sides of the governing equation for the cumulative spectrum wrt k, we obtain the governing equation for the spectrum itself:

$$\boxed{\frac{\partial A(k)}{\partial t} = T(k) + J(k) - D(k)}$$

where

$$\boxed{T(k) = -\frac{\partial \Pi(k)}{\partial k}} \quad \boxed{J(k) = \frac{\partial \mathcal{J}(k)}{\partial k}} \quad \boxed{D(k) = \frac{\partial \mathcal{D}(k)}{\partial k}}$$

The physical meaning of $T(k)$ is the rate with which "energy" is brought to k through the flux term from other wavenumbers. An immediate consequence of the boundary conditions of $\Pi(k)$ is that

$$\boxed{\int_0^{+\infty} T(k)\, dk = 0}$$

It follows that when the flow reaches steady state:

$$\boxed{\text{steady state} \Rightarrow \int_0^{+\infty} J(k)\,dk = \int_0^{+\infty} D(k)\,dk}$$

Note that under random stirring, or in a hydrodynamically unstable system, only ensemble averages of the above may reach a steady state.

Using the boundary condition, we may also show that $\Pi(k)$ and $T(k)$ are related by:

$$\boxed{\Pi(k) = \int_k^{+\infty} T(k_0)\,dk_0}$$

Now, we would like to establish a relation between $T(k)$ and $T(k|p,q)$, and place the physical meaning of the latter on solid ground. On physical basis, we expect that integrating over all relevant p and q should yield $T(k)$. It is important however not to overcount the interactions, and to straighten out the signs.

Theorem : $\boxed{T(k) = -\frac{1}{2}\int_{[0,+\infty)} dp\,dq\; T(k|p,q)}$

Proof

$$T(k) = -\frac{\partial \Pi(k)}{\partial k} = -\frac{\partial}{\partial k}\langle a b^{<}(k) c^{>}(k)\rangle =$$

$$= -\int d\vec{k}_0\, d\vec{k}_1\, d\vec{k}_2\; \hat{\tau}(k,\vec{k}_0,\vec{k}_1,\vec{k}_2)\,\hat{a}(k_0)\,\hat{b}(k_1)\,\hat{c}(k_2) \overset{(p.31)}{\underset{(p.34)}{=}}$$

$$= -\int_{[0,+\infty)^3} dk_0\,dk_1\,dk_2\; T_0(k|k_0,k_1,k_2)$$

(p. 35)

$$= -\frac{1}{6}\int_{[0,+\infty)^3} dk_0 dk_1 dk_2 \; T(k|k_0,k_1,k_2) =$$

$$= -\frac{1}{6}\int_{[0,+\infty)^3} dk_0 dk_1 dk_2 \; a(k|k_0,k_1,k_2)\,\delta(k-k_0)$$

$$- \frac{1}{6}\int_{[0,+\infty)^3} dk_0 dk_1 dk_2 \; b(k|k_0,k_1,k_2)\,\delta(k-k_1)$$

$$- \frac{1}{6}\int_{[0,+\infty)^3} dk_0 dk_1 dk_2 \; c(k|k_0,k_1,k_2)\,\delta(k-k_2) =$$

$$= -\frac{1}{6}\int_{[0,+\infty)^2} dk_1 dk_2 \; a(k|k,k_1,k_2) - \frac{1}{6}\int_{[0,+\infty)^2} dk_0 dk_2 \; b(k|k_0,k,k_2)$$

$$- \frac{1}{6}\int_{[0,+\infty)^2} dk_0 dk_1 \; c(k|k_0,k_1,k) =$$

(proposition p. 36).

$$= -\frac{1}{6}\int_{[0,+\infty)^2} dk_1 dk_2 \; T(k|k_1,k_2) - \frac{1}{6}\int_{[0,+\infty)^2} dk_0 dk_2 \; T(k|k_0,k_2)$$

$$- \frac{1}{6}\int_{[0,+\infty)^2} dk_0 dk_1 \; T(k|k_0,k_1) =$$

$$= -\frac{1}{2}\int_{[0,+\infty)^2} dp\,dq \; T(k|p,q) \qquad \square$$

The following observations follow immediately:

a) $T(k|p,q)$ is the "energy" that moves <u>from k to p,q</u> due to <u>all</u> triad interactions between (k,p,q). Because we add up all permutations between a-modes, b-modes and c-modes, the (k,p,q) interactions are the same as (k,q,p). Therefore $T(k|p,q) + T(k|q,p)$ would be an overcount. Because our integral does overcount, a factor of $1/2$ appears.

b) We can eliminate the 1/2 factor by restricting the domain of integration to $p \leq q$. However we can restrict it further. The numbers k, p, q must be sides of a triangle in order for the interaction to take place. A necessary and sufficient condition for this is the triangle inequality:

$$\boxed{\begin{array}{l} |p-q| \leq k \leq p+q \\ k \geq 0 \;,\; p \geq 0 \;,\; q \geq 0 \end{array}}$$

Note that:

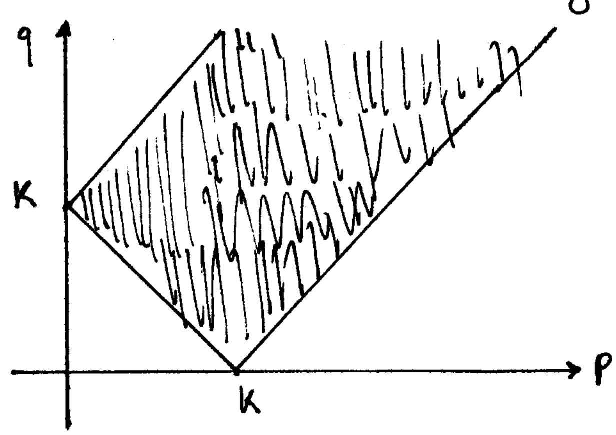
$$\begin{aligned} |p-q| \leq k &\Leftrightarrow -k \leq p-q \leq k \Leftrightarrow \\ &\Leftrightarrow -k \leq p-q \quad \text{and} \quad p-q \leq k \Leftrightarrow \\ &\Leftrightarrow \underline{q \leq p+k} \quad \text{and} \quad \underline{q \geq p-k} \end{aligned}$$

and: $\quad k \leq p+q \Leftrightarrow \underline{q \geq -p+k}$

The solution of the system of inequalities

$$q \leq p+k \quad \text{and} \quad q \geq p-k \quad \text{and} \quad q \geq -p+k$$

is shown in the following graph:



If we denote this region as

$$\boxed{S(k) = \{(p,q) \in [0,+\infty)^2 \mid |p-q| \leq k \leq p+q\}}$$

then the integral for the transfer function simplifies as follows:

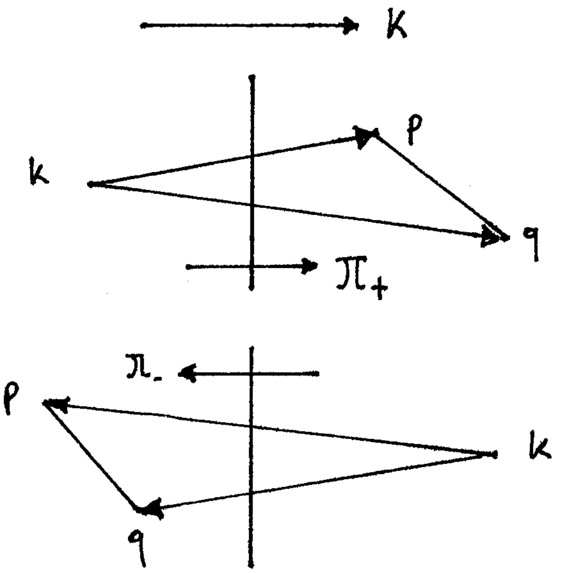
$$\boxed{T(k) = -\frac{1}{2}\int_{S(k)} dp\,dq\; T(k|p,q)}$$

From this result, Kraichnan obtained using a clever physical argument an expression for the flux $\Pi(k)$.

Let Δ be the set of all triad interactions (k,p,q):

$$\boxed{\Delta = \bigcup_{k \in [0,+\infty)} \{k\} \times S(k)}$$

and let $k_0 > 0$ be a given wavenumber. Now consider the following two groups of interactions:



Let $\Delta^+(k_0)$ be all the interactions with two vertices on the right of k_0 and one on the left of k_0, and $\Delta^-(k_0)$ be all the interactions with one vertex on the right of k_0 and two vertices on the left of k_0. Also let $\Pi_+(k_0)$ and $\Pi_-(k_0)$ be the flux accross k_0 contributed by each group of interactions. Mathematically:

$$\boxed{\begin{aligned} \Delta^{+}(k_0) &= \{(k,p,q)\in\Delta \mid k<k_0 \wedge p>k_0 \wedge q>k_0\} \\ \Delta^{-}(k_0) &= \{(k,p,q)\in\Delta \mid k>k_0 \wedge p<k_0 \wedge q<k_0\} \end{aligned}}$$

and

$$\boxed{\begin{aligned} \Pi_+(k_0) &= \frac{1}{2}\int_{\Delta_+(k_0)} dk\,dp\,dq\ T(k|p,q) \\ \Pi_-(k_0) &= \frac{1}{2}\int_{\Delta_-(k_0)} dk\,dp\,dq\ T(k|p,q) \end{aligned}}$$

Kraichnan's argument was that no other interactions contribute to flux accross k_0, therefore $\Pi(k) = \Pi_+(k) - \Pi_-(k)$. Although this is taken as obvious by intuitionist mathophobes, it turns out that the mathematical proof is tricky, so we will write it down in full:

<u>Theorem</u>: $\boxed{\Pi(k) = \Pi_+(k) - \Pi_-(k)}$

<u>Proof</u>

Employing what has already been shown does not take us very far:

$$\Pi(k_0) = \int_{k_0}^{+\infty} T(k)\,dk = -\int_0^{k_0} T(k)\,dk + \int_0^{+\infty} T(k)\,dk =$$

$$= -\int_0^{k_0} T(k)\,dk = -\int_0^{k_0} dk\left[-\frac{1}{2}\int_{\delta(k)} dp\,dq\ T(k|p,q)\right]$$

$$= \int_0^{k_0} dk \int_{\delta(k)} dp\,dq\ T(k|p,q)$$

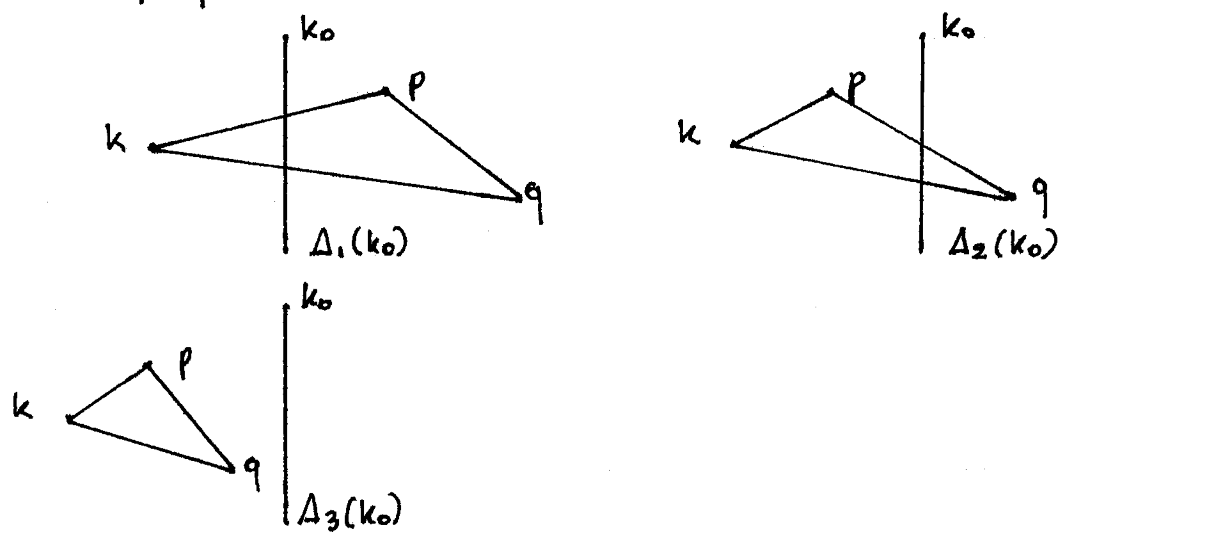
If we define $\Delta(k_0) = \{(k,p,q) \in \Delta \mid k < k_0 \wedge p < q\}$, then we may rewrite the flux integral as:

$$\Pi(k_0) = \int_{\Delta(k_0)} dk\,dp\,dq\; T(k|p,q)$$

Note that the restriction $p<q$ eliminated overcounting. Now partition the domain of integration to the following three subspaces:

$$\Delta(k_0) = \Delta_1(k_0) \cup \Delta_2(k_0) \cup \Delta_3(k_0)$$

Each subspace corresponds to a certain relative position of k,p,q with respect to k_0. Given the constraints $k<k_0$ and $p<q$, there are three possibilities left:



So, mathematically:

$$\begin{aligned}
\Delta_1(k_0) &= \{(k,p,q) \in \Delta(k_0) \mid p > k_0 \wedge q > k_0\} \\
\Delta_2(k_0) &= \{(k,p,q) \in \Delta(k_0) \mid p < k_0 \wedge q > k_0\} \\
\Delta_3(k_0) &= \{(k,p,q) \in \Delta(k_0) \mid p < k_0 \wedge q < k_0\}
\end{aligned}$$

Let Π_1, Π_2, and Π_3 be the fluxes contributed by each subspace. In general:

$$\Pi_a(k_0) = \int_{\Delta_a(k_0)} dk\,dp\,dq \; T(k|p,q)$$

It is obvious that:

$$\Pi(k_0) = \Pi_1(k_0) + \Pi_2(k_0) + \Pi_3(k_0)$$

We will now show that $\Pi_1(k_0) = \Pi_+(k_0)$, $\Pi_2(k_0) = -\Pi_-(k_0)$, and $\Pi_3(k_0) = 0$.

a) Note that:

$$\begin{aligned}
\Delta_+(k_0) &= \{(k,p,q) \in \Delta \mid k < k_0 \wedge p > k_0 \wedge q > k_0\} \\
&= \{(k,p,q), (k,q,p) \in \Delta \mid p < q \wedge k < k_0 \wedge p > k_0 \wedge q > k_0\} = \\
&= \{(k,p,q), (k,q,p) \in \Delta \mid (k,p,q) \in \Delta_1(k_0)\} \\
&= \{(k,p,q), (k,q,p) \mid (k,p,q) \in \Delta_1(k_0)\}
\end{aligned}$$

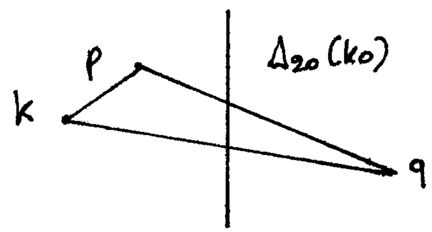
therefore:

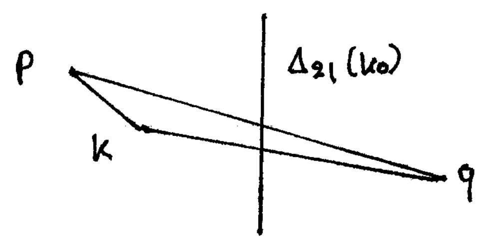
$$\Pi_1(k_0) = \int_{\Delta_1(k_0)} dk\,dp\,dq \; T(k|p,q) = \frac{1}{2}\int_{\Delta_+(k_0)} dk\,dp\,dq \; T(k|p,q) = \Pi_+(k_0)$$

since $\Delta_+(k_0)$ is essentially an overcount of $\Delta_1(k_0)$. Their only difference is the restriction $p<q$.

b) In the $\Delta_2(k_0)$ interactions there are no restrictions on the relative position of k, p, so we partition $\Delta_2(k_0)$ to the following subspaces:

$$\Delta_{20}(k_0) = \{(k,p,q) \in \Delta_2(k_0) \mid k < p\}$$
$$\Delta_{21}(k_0) = \{(k,p,q) \in \Delta_2(k_0) \mid k > p\}$$





Note that:

$$\begin{aligned} \Delta_{20}(k_0) &= \{(k,p,q) \in \Delta_2(k_0) \mid k < p\} = \\ &= \{(p,k,q) \in \Delta_2(k_0) \mid k > p\} = \\ &= \{(p,k,q) \mid (k,p,q) \in \Delta_{21}(k_0)\} \qquad (1) \end{aligned}$$

and

$$\begin{aligned} \Delta_-(k_0) &= \{(k,p,q) \in \Delta \mid k > k_0 \wedge p < k_0 \wedge q < k_0\} = \\ \text{(I)} \quad &= \{(k,p,q), (k,q,p) \underline{\in \Delta} \mid p < q \wedge k > k_0 \wedge \underline{p < k_0} \wedge q < k_0\} = \\ &= \{(k,p,q), (k,q,p) \mid (p,q,k) \in \Delta(k_0) \wedge p < q \wedge \underline{k > k_0} \wedge \underline{q < k_0}\} \\ &= \{(k,p,q), (k,q,p) \mid (p,q,k) \in \Delta_2(k_0) \wedge p < q\} \\ &= \{(k,p,q), (k,q,p) \mid (p,q,k) \in \Delta_{20}(k_0)\} \qquad (2) \end{aligned}$$

In the above derivations we employ with care the definitions of all the Δ-sets introduced so far.
In step (I) we employ $p < k_0$ and $q < k$. The latter is not shown, but it follows from $q < k_0$ and $k > k_0$.
The underlined parts are absorbed by the definitions of the sharper Δ-sets in the next step. So $\Delta_-(k_0)$ is a peculiar overcount of $\Delta_{20}(k_0)$.
From all this, we may compute π_2:

$$\Pi_2(k_0) = \int_{\Delta_2(k_0)} dk\,dp\,dq\; T(k|p,q) =$$

$$= \int_{\Delta_{20}(k_0)} dk\,dp\,dq\; T(k|p,q) + \int_{\Delta_{21}(k_0)} dk\,dp\,dq\; T(k|p,q) = \quad (1)$$

$$= \int_{\Delta_{20}(k_0)} dk\,dp\,dq\; T(k|p,q) + \int_{\Delta_{20}(k_0)} dp\,dk\,dq\; T(k|p,q) =$$

$$= \int_{\Delta_{20}(k_0)} dk\,dp\,dq\; T(k|p,q) + \int_{\Delta_{20}(k_0)} dk\,dp\,dq\; T(p|k,q)$$

$$= -\int_{\Delta_{20}(k_0)} T(q|k,p)\, dk\,dp\,dq \qquad \text{(detailed conservation thm)}$$

$$= -\frac{1}{2}\int_{\Delta_{20}(k_0)} dp\,dq\,dk \left[T(k|p,q) + T(k|q,p) \right] = \quad (2)$$

$$= -\frac{1}{2}\int_{\Delta_{-}(k_0)} dk\,dp\,dq\; T(k|p,q) = -\Pi_{-}(k_0)$$

c) The difference between $\Delta_3(k_0)$ and $\Delta_2(k_0)$ interactions is that we remove the restriction $q > k_0$ and keep only $p<q$. Now we have three possible relative positions of k,p,q and henceforth three further subspaces:

$$\Delta_{30}(k_0) = \{ (k,p,q) \in \Delta_3(k_0) \mid k<p<q \}$$
$$\Delta_{31}(k_0) = \{ (k,p,q) \in \Delta_3(k_0) \mid p<k<q \}$$
$$\Delta_{32}(k_0) = \{ (k,p,q) \in \Delta_3(k_0) \mid p<q<k \}$$

Note that the constraint $p<q$ remains in effect.
To evaluate $\Pi_3(k_0)$ we need the following set relations:

$$\Delta_{31}(k_0) = \{ (k,p,q) \in \Delta_3(k_0) \mid p<k<q \}$$
$$= \{ (p,k,q) \in \Delta_3(k_0) \mid k<p<q \} =$$
$$= \{ (p,k,q) \mid (k,p,q) \in \Delta_{30}(k_0) \}$$

and

$$\Delta_{32}(k_0) = \{ (k,p,q) \in \Delta_3(k_0) \mid p<q<k \} =$$
$$= \{ (q,k,p) \in \Delta_3(k_0) \mid k<p<q \} =$$
$$= \{ (q,k,p) \mid (k,p,q) \in \Delta_{30}(k_0) \}$$

It follows that

$$\Pi_{31}(k_0) = \int_{\Delta_{31}(k_0)} dk\,dp\,dq\; T(k|p,q) = \int_{\Delta_{30}(k_0)} dp\,dk\,dq\; T(k|p,q) =$$
$$= \int_{\Delta_{30}(k_0)} dk\,dp\,dq\; T(p|k,q)$$

and

$$\Pi_{32}(k_0) = \int_{\Delta_{32}(k_0)} dk\,dp\,dq\; T(k|p,q) = \int_{\Delta_{30}(k_0)} dq\,dk\,dp\; T(k|p,q) =$$
$$= \int_{\Delta_{30}(k_0)} dk\,dp\,dq\; T(q|k,p).$$

and, of course,

$$\Pi_{30} = \int_{\Delta_{30}(k_0)} dk\,dp\,dq\; T(k|p,q)$$

and using the detailed conservation theorem:

$$\Pi_3(k_0) = \Pi_{30}(k_0) + \Pi_{31}(k_0) + \Pi_{32}(k_0) =$$
$$= \int_{\Delta_{30}(k_0)} dk\,dp\,dq \left[T(k|p,q) + T(p|k,q) + T(q|k,p) \right] =$$
$$= 0$$

Combining all the results so far:

$$\Pi(k_0) = \Pi_1(k_0) + \Pi_2(k_0) + \Pi_3(k_0) = \\ = \Pi_+(k_0) - \Pi_-(k_0) + 0 = \Pi_+(k_0) - \Pi_-(k_0). \qquad \square$$

In full, this result states:

$$\boxed{\Pi(k_0) = \frac{1}{2}\int_{\Delta_+(k_0)} dk\,dp\,dq\; T(k|p,q) - \frac{1}{2}\int_{\Delta_-(k_0)} dk\,dp\,dq\; T(k|p,q)}$$

Sometimes we are interested in the fluxeds Π_+ and Π_-. In particular $\Pi_+(k_0)$ can be used to introduce the concept of <u>eddy viscosity.</u>

Recall our definition of the convolution integral:

$$\boxed{\begin{aligned} T_c(k,p,q) = \int d\Omega(A)\, d\Omega(B)\, d\Omega(C)\, \delta(kA\vec{e} + pB\vec{e} + qC\vec{e}) \\ \times\, \hat{a}(kA\vec{e})\, \hat{b}(pB\vec{e})\, \hat{c}(qC\vec{e}) \end{aligned}}$$

from p. 34. We may employ the relation between $T_c(k,p,q)$ and $T(k|p,q)$ to compute $\Pi_+(k_0)$ and $\Pi_-(k_0)$. In particular we require the following results from p. 41:

$$\boxed{\begin{aligned} k<p<q &\Rightarrow T(k|p,q) = T_c(p,k,q) + T_c(q,k,p) \\ p<q<k &\Rightarrow T(k|p,q) = -T_c(p,q,k) - T_c(q,p,k). \end{aligned}}$$

written in full, instead of, in abbreviated notation.

<u>Theorem</u> :

$$\boxed{\begin{aligned} \Pi_+(k_0) &= \int_{\Delta_+(k_0)} dk\,dp\,dq\; T_c(p,k,q) \\ \Pi_-(k_0) &= -\int_{\Delta_-(k_0)} dk\,dp\,dq\; T_c(p,q,k) \end{aligned}}$$

Proof

Recall that $\Delta_+(k_0), \Delta_-(k_0)$ do not include the restriction $p<q$, so we introduce subspaces that do include it:

$$\Delta_+^\circ(k_0) = \{(k,p,q) \in \Delta_+(k_0) \mid p<q\}$$
$$\Delta_-^\circ(k_0) = \{(k,p,q) \in \Delta_-(k_0) \mid p<q\}$$

Note that

$$\begin{aligned}(k,p,q) \in \Delta_+^\circ(k_0) &\Rightarrow (k,p,q) \in \Delta_+(k_0) \wedge p<q \Rightarrow \\ &\Rightarrow k<k_0 \wedge p>k_0 \wedge q>k_0 \wedge p<q \Rightarrow \\ &\Rightarrow k<p<q \Rightarrow \\ &\Rightarrow T(k|p,q) = T_c(p,k,q) + T_c(q,k,p)\end{aligned}$$

and similarly

$$(k,p,q) \in \Delta_-^\circ(k_0) \Rightarrow p<q<k \Rightarrow T(k|p,q) = -T_c(p,q,k) - T_c(q,p,k)$$

It follows that

$$\begin{aligned}\Pi_+(k_0) &= \int_{\Delta_+(k_0)} dk\,dp\,dq\, \frac{T(k|p,q)}{2} = \int_{\Delta_+^\circ(k_0)} dk\,dp\,dq\; T(k|p,q) = \\ &= \int_{\Delta_+^\circ(k_0)} dk\,dp\,dq\, \left[T_c(p,k,q) + T_c(q,k,p) \right] \\ &= \int_{\Delta_+(k_0)} dk\,dp\,dq\; T_c(p,k,q)\end{aligned}$$

Similarly: $\Pi_-(k_0) = \int_{\Delta_-(k_0)} dk\,dp\,dq\, \left[-T_c(p,q,k) \right]$ □

Energy transfer equation

We now return from this abstract detour to the job at hand. Recall the energy flux equation:

$$\frac{\partial \mathcal{E}(k)}{\partial t} + \Pi(k) = \mathcal{F}(k) - 2\nu\,\Omega(k)$$

where:

$$\Pi(k) = \langle u_\alpha^< u_\beta \partial_\beta u_\alpha^> \rangle$$

$$2\Omega(k) = -\langle u_\alpha^< \nabla^2 u_\alpha^< \rangle = \langle \omega_\alpha^< \omega_\alpha^< \rangle = \langle s_{\alpha\beta}^< s_{\alpha\beta}^< \rangle$$

$$\mathcal{F}(k) = \langle f_\alpha^< u_\alpha^< \rangle$$

are the flux, dissipation rate, and energy injection, and

$$\mathcal{E}(k) = \langle u_\alpha^< u_\alpha^< \rangle / 2$$

is the cumulative energy spectrum.

We have already introduced the transfer rate $T(k)$ and the energy spectrum $E(k)$:

$$T(k) = -\frac{\partial \Pi(k)}{\partial k} \qquad E(k) = \frac{\partial \mathcal{E}(k)}{\partial k}$$

Similarly, we introduce the energy injection spectrum:

$$F(k) = \frac{\partial \mathcal{F}(k)}{\partial k}$$

As for $\Omega(k)$, we use the following result:

<u>Proposition</u> : $\boxed{\dfrac{\partial \Omega}{\partial k} = -k^2 E(k)}$

<u>Proof</u>

Let $v_a = \nabla^2 u_a$, and let $\hat{u}_a = \mathcal{F} u_a$, $\hat{v}_a = \mathcal{F} v_a$ be the Fourier transforms of u_a and v_a. Recall from p. 28 that $E(k)$ is given by:

$$E(k) = \int d\vec{k}_0 \, \delta(k - \|\vec{k}_0\|) \left(|\hat{u}_a|^2 / 2 \right)$$

Since: $\hat{v}_a(\vec{k}) = \mathcal{F} v_a = \mathcal{F} \nabla^2 u_a = -\|\vec{k}\|^2 \mathcal{F} u_a = -\|\vec{k}\|^2 \hat{u}_a(\vec{k})$
it follows that

$$\frac{\partial \Omega}{\partial k} = -\frac{\partial}{\partial k} \left\langle \frac{u_a^< \nabla^2 u_a^<}{2} \right\rangle = -\frac{\partial}{\partial k} \left\langle \frac{u_a^< v_a^<}{2} \right\rangle =$$

$$= -\int d\vec{k}_0 \, \delta(k - \|\vec{k}_0\|) \frac{\hat{u}_a^*(\vec{k}_0) \hat{v}_a(\vec{k}_0) + \hat{u}_a(\vec{k}_0) \hat{v}_a^*(\vec{k}_0)}{4}$$

$$= -\int d\vec{k}_0 \, \delta(k - \|\vec{k}_0\|) (-\|\vec{k}_0\|^2) \frac{\hat{u}_a^*(\vec{k}_0) \hat{u}_a(\vec{k}_0) + \hat{u}_a(\vec{k}_0) \hat{u}_a^*(\vec{k}_0)}{4}$$

$$= +k^2 \int d\vec{k}_0 \, \delta(k - \|\vec{k}_0\|) \frac{|\hat{u}_a(\vec{k}_0)|^2}{2} = +k^2 E(k) \quad \square$$

Putting it all together we obtain the energy transfer equation:

$$\boxed{\frac{\partial E(k)}{\partial t} = T(k) + F(k) - 2\nu k^2 E(k)}$$

We will discuss $T(k)$ further later.

Helicity transfer equation

A transfer equation can also be derived for helicity conservation. Recall (p.23) that

$$\frac{\partial \mathcal{H}(k)}{\partial t} + \mathcal{P}(k) = -2\nu\, W(k) + \eta(k)$$

where:

$$\mathcal{P}(k) = \langle \varepsilon_{\alpha\beta\gamma}\, u_\alpha\, w_\beta^{>}\, w_\gamma^{<} \rangle$$
$$W(k) = -\langle u_\alpha^{<}\, \nabla^2 w_\alpha^{<} \rangle = \frac{1}{2}\langle w_\alpha^{<} z_\alpha^{<} \rangle$$
$$\mathcal{F}(k) = \langle f_\alpha^{<} w_\alpha^{<} \rangle$$

are the flux, dissipation rate, and ~~energy~~ helicity injection, and

$$\mathcal{H}(k) = \langle u_\alpha^{<} w_\alpha^{<} \rangle / 2$$

is the cumulative helicity spectrum. Following our discussion of the energy transfer equation, we introduce the helicity spectrum, the transfer rate $T_H(k)$, and the helicity injection rate $F_H(k)$:

$$H(k) = \frac{\partial \mathcal{H}(k)}{\partial k} \qquad T_H(k) = -\frac{\partial \mathcal{P}}{\partial k} \qquad F_H(k) = \frac{\partial \eta}{\partial k}$$

With a similar line of argument as before, we have

$$\frac{\partial W}{\partial k} = -\frac{\partial}{\partial k}\langle u_\alpha^{<} \nabla^2 w_\alpha^{<} \rangle = -(-k^2)\frac{\partial}{\partial k}\langle u_\alpha^{<} w_\alpha^{<} \rangle = k^2 H(k)$$

and so we obtain the helicity transfer equation:

$$\boxed{\frac{\partial H(k)}{\partial t} = T_H(k) + F_H(k) - 2\nu k^2 H(k)}$$

Proof

Recall that $\Delta_+(k_0), \Delta_-(k_0)$ do not include the restriction $p<q$, so we introduce subspaces that do include it:

$$\Delta_+^o(k_0) = \{(k,p,q) \in \Delta_+(k_0) \mid p<q\}$$
$$\Delta_-^o(k_0) = \{(k,p,q) \in \Delta_-(k_0) \mid p<q\}$$

Note that

$$\begin{aligned}(k,p,q) \in \Delta_+^o(k_0) &\Rightarrow (k,p,q) \in \Delta_+(k_0) \wedge p<q \Rightarrow \\ &\Rightarrow k<k_0 \wedge p>k_0 \wedge q>k_0 \wedge p<q \Rightarrow \\ &\Rightarrow k<p<q \Rightarrow \\ &\Rightarrow T(k|p,q) = T_c(p,k,q) + T_c(q,k,p)\end{aligned}$$

and similarly

$$(k,p,q) \in \Delta_-^o(k_0) \Rightarrow p<q<k \Rightarrow T(k|p,q) = -T_c(p,q,k) - T_c(q,p,k)$$

It follows that

$$\begin{aligned}\Pi_+(k_0) &= \int_{\Delta_+(k_0)} dk\,dp\,dq\, \frac{T(k|p,q)}{2} = \int_{\Delta_+^o(k_0)} dk\,dp\,dq\, T(k|p,q) = \\ &= \int_{\Delta_+^o(k_0)} dk\,dp\,dq\, \left[T_c(p,k,q) + T_c(q,k,p)\right] \\ &= \int_{\Delta_+(k_0)} dk\,dp\,dq\, T_c(p,k,q)\end{aligned}$$

Similarly : $\Pi_-(k_0) = \int_{\Delta_-(k_0)} dk\,dp\,dq\, \left[-T_c(p,q,k)\right]$ $\square$

Energy transfer equation

We now return from this abstract detour to the job at hand. Recall the energy flux equation:

$$\frac{\partial \mathcal{E}(k)}{\partial t} + \Pi(k) = \mathcal{F}(k) - 2\nu\,\Omega(k)$$

where:

$$\Pi(k) = \langle u_a^< u_\beta \partial_\beta u_a^> \rangle$$
$$2\,\Omega(k) = -\langle u_a^< \nabla^2 u_a^< \rangle = \langle \omega_a^< \omega_a^< \rangle = \langle s_{ab}^< s_{ab}^< \rangle$$
$$\mathcal{F}(k) = \langle f_a^< u_a^< \rangle$$

are the flux, dissipation rate, and energy injection, and

$$\mathcal{E}(k) = \langle u_a^< u_a^< \rangle / 2$$

is the cumulative energy spectrum.

We have already introduced the transfer rate $T(k)$ and the energy spectrum $E(k)$:

$$T(k) = -\frac{\partial \Pi(k)}{\partial k} \qquad E(k) = \frac{\partial \mathcal{E}(k)}{\partial k}$$

Similarly, we introduce the energy injection spectrum:

$$F(k) = \frac{\partial \mathcal{F}(k)}{\partial k}$$

As for $\Omega(k)$, we use the following result:

Proposition : $\boxed{\dfrac{\partial \Omega}{\partial k} = -k^2 E(k)}$

Proof

Let $v_a = \nabla^2 u_a$, and let $\hat{u}_a = \mathcal{F} u_a$, $\hat{v}_a = \mathcal{F} v_a$ be the Fourier transforms of u_a and v_a. Recall from p. 28 that $E(k)$ is given by:

$$E(k) = \int d\vec{k}_0 \; \delta(k - \|\vec{k}_0\|) \left(|\hat{u}_a|^2 / 2 \right)$$

Since: $\hat{v}_a(\vec{k}) = \mathcal{F} v_a = \mathcal{F} \nabla^2 u_a = -\|\vec{k}\|^2 \mathcal{F} u_a = -\|\vec{k}\|^2 \hat{u}_a(\vec{k})$ it follows that

$$\frac{\partial \varrho}{\partial k} = -\frac{\partial}{\partial k} \left\langle \frac{u_a^< \nabla^2 u_a^<}{2} \right\rangle = -\frac{\partial}{\partial k} \left\langle \frac{u_a^< v_a^<}{2} \right\rangle =$$

$$= -\int d\vec{k}_0 \; \delta(k - \|\vec{k}_0\|) \; \frac{\hat{u}_a^*(\vec{k}_0) \hat{v}_a(\vec{k}_0) + \hat{u}_a(\vec{k}_0) \hat{v}_a^*(\vec{k}_0)}{4}$$

$$= -\int d\vec{k}_0 \; \delta(k - \|\vec{k}_0\|) (-\|\vec{k}_0\|^2) \; \frac{\hat{u}_a^*(\vec{k}_0) \hat{u}_a(\vec{k}_0) + \hat{u}_a(\vec{k}_0) \hat{u}_a^*(\vec{k}_0)}{4}$$

$$= +k^2 \int d\vec{k}_0 \; \delta(k - \|\vec{k}_0\|) \; \frac{|\hat{u}_a(\vec{k}_0)|^2}{2} = +k^2 E(k) \qquad \square$$

Putting it all together we obtain the energy transfer equation:

$$\boxed{\frac{\partial E(k)}{\partial t} = T(k) + F(k) - 2\nu k^2 E(k)}$$

We will discuss $T(k)$ further later.

The case of hyperviscosity

It is common in numerical simulations to dissipate energy using a high-order hyperviscosity term. The governing equation is now:

$$\frac{\partial u_a}{\partial t} + u_b \partial_b u_a = -\partial_a \varphi + \nu \nabla^{2\gamma} u_a + f_a$$

$$\partial_a u_a = 0$$

where $\gamma = 1, 2, 3, \ldots$ is an integer. Since the nonlinear term is left unchanged, most of our discussion of the conservation laws follows over. The flux equations for energy and helicity remain

$$\frac{\partial \mathcal{E}(k)}{\partial t} + \Pi(k) = \mathcal{F}(k) - 2\nu \underline{\Omega}(k)$$

$$\frac{\partial \mathcal{H}(k)}{\partial t} + \mathcal{P}(k) = \eta(k) - 2\nu W(k)$$

as before (p. 55, 57), however $\underline{\Omega}(k)$ and $W(k)$ are now given by

$$\boxed{\begin{aligned} \underline{\Omega}(k) &= -\langle u_a^< \nabla^{2\gamma} u_a^< \rangle \\ W(k) &= -\langle u_a^< \nabla^{2\gamma} \omega_a^< \rangle \end{aligned}}$$

The transfer equations now become:

$$\boxed{\frac{\partial E(k)}{\partial t} = T(k) + F(k) - 2\nu k^{2\gamma} E(k)}$$

$$\boxed{\frac{\partial H(k)}{\partial t} = T_H(k) + F_H(k) - 2\nu k^{2\gamma} H(k)}$$

Conservation laws in physical space

So far we have studied conservation laws, and nonlinear transfer in particular, in the most native space: spectral space. We did this by studying the governing equations of the energy and helicity cumulative spectra:

$$\mathcal{E}(k) = \frac{1}{2}\langle u_a^< u_a^< \rangle$$

$$\mathcal{H}(k) = \frac{1}{2}\langle u_a^< \omega_a^< \rangle$$

In the limit $k \to +\infty$ they reduce to the coveted conservation laws. For finite k we obtain information about the transfer of energy/helicity due to triadic interactions.

In the physical space approach, we instead use correlation functions:

$$B_{ab}(\vec{\ell}) = \langle u_a(0) u_b(\vec{\ell}) \rangle$$
$$\Gamma_{ab}(\vec{\ell}) = \langle u_a(0) \omega_b(\vec{\ell}) \rangle$$

We employ a rather eccentric notation: The brackets, of course, denote a spatial integral. The arguments in parenthesis, when they appear inside such brackets, represent <u>shifting</u>, not evaluation. For example:

$$B_{ab}(\vec{\ell}) = \int d\vec{r}\ \ u_a(\vec{r}) u_b(\vec{r}+\vec{\ell})$$

This is sensible since the brackets take <u>functions</u> of $\vec{r}$. Note that this is not an ensemple average, and no homogeneity or isotropy assumptions are made.

In the limit $\vec{l} \to 0$ we recover, again, the total energy and helicity. We will show that the spectral and physical space descriptions are equivalent, and develop the physical space formulation in detail.

The spherical Fourier transform

We now briefly review some mathematics. The derivations are relegated to an appendix.

Let ${}^{m\times n}\mathbb{R}$ be the set of all $m\times n$ real matrices. This is a specialization of the standard notation ${}^{A}B$ for the set of all functions $f: A \to B$. We introduce the group of orthogonal $d\times d$ matrices $O(d)$:

$$\boxed{O(d) = \{ A \in {}^{d\times d}\mathbb{R} \mid A^T A = A A^T = I \}}$$

The columns, or equivalently rows, of an $A \in O(d)$ can be interpreted as the unit vectors of an orthonormal coordinate system in d dimensions. We may then interpret $O(d)$ as the set of all such coordinate systems. We may also interpret it as a set of transformations that map one coordinate system into another. These transformations include rotations and reflections. We may isolate the rotations by noting that

$$A \in O(d) \Rightarrow \det A = 1 \vee \det A = -1$$

and adding the restriction $\det A = 1$. We obtain:

$$\boxed{SO(d) = \{ A \in O(d) \mid \det A = 1 \}}$$

which can be interpreted as the set of all rotations or the set of all right-handed coordinate systems in d dimensions.

We use this to introduce generalized polar coordinates, and define integration over $SO(d)$. Note that $SO(d)$ is isomorphic to an d-sphere. To see this, take an $\vec{v} \in \mathbb{R}^d$ with $\|v\| = r$ the desired radius. Then

$$SO(d)\vec{v} = \{ A\vec{v} \mid A \in SO(d) \} = \\ = \{ \vec{v} \mid \|\vec{v}\| = r \}$$

A d-sphere can in turn be represented using d-1 angles $\vec{\varphi} \in [0,\pi]^{d-2} \times [0, 2\pi)$. as follows:

$$\boxed{\begin{aligned} x_1 &= r\cos\varphi_1 \\ x_a &= r\cos\varphi_a \prod_{b=1}^{a-1} \sin\varphi_a \\ x_d &= r\prod_{a=1}^{d-1} \sin\varphi_a \end{aligned}}$$

It can be shown that the isomorphism between $SO(n)$ and the angles φ_a can be written as a product of matrix exponentials

$$A = \prod_{a=1}^{d-1} \exp[i\varphi_a E_a]$$

where E_a are generators that correspond to successive rotations around each axis. We use this isomorphism to introduce a measure on $SO(d)$ which is to be interpreted as the "area" of a differential patch of the d-sphere.

It can be shown that the volume measure on $\mathbb{R}^d$ can be rewritten in polar coordinates as follows:

$$dV = \prod_{a=1}^{d} dx_a = r^{d-1}\, dr \prod_{a=1}^{d-1} (\sin\varphi_a)^{d-1-a}\, d\varphi_a$$

We define the product of angle differentials as the <u>solid-angle measure</u> on $SO(d)$

$$\boxed{d\Omega(A) = \prod_{a=1}^{d-1} (\sin\varphi_a)^{d-1-a}\, d\varphi_a}$$

$$\boxed{\vec{\varphi} \in [0,\pi]^{d-2} \times [0, 2\pi)}$$

Let $\vec{e}$ be some arbitrary unit vector. We introduce the Fourier-Bessel special function as:

$$\boxed{\Phi_d(x) = \frac{\int_{SO(d)} \exp[ix\vec{e}\cdot(A\vec{e})]\, d\Omega(A)}{\int_{SO(d)} d\Omega(A)}}$$

This is a spherical average of the exponential over the surface of a d-sphere. We denote the integral at the denominator as γ_d and it represents the surface area of a d-sphere of radius 1.

After some algebra, γ_d evaluates as

$$\boxed{\gamma_d = \int_{SO(d)} d\Omega(A) = \frac{2\pi^{d/2}}{\Gamma(d/2)}}$$

The function $\Phi_d(x)$ can be reduced to a trigonometric integral which can be written in closed form in terms of Bessel functions $J_\nu(x)$:

$$\boxed{\Phi_d(x) = \frac{\gamma_{d-1}}{\gamma_d}\int_0^{\pi} \cos(x\cos\varphi)\sin^{d-2}\varphi\, d\varphi = \frac{(2\pi)^{d/2}}{\gamma_d}\,\frac{J_{(d-2)/2}(x)}{x^{(d-2)/2}}}$$

These expressions simplify when $d=2$ or $d=3$:

$$\gamma_2 = 2\pi \qquad \Phi_2(x) = J_0(x)$$
$$\gamma_3 = 4\pi \qquad \Phi_3(x) = \frac{\sin x}{x}$$

The Fourier-Bessel functions emerge when one attempts to evaluate the Fourier transform of an isotropic function $f(\vec{r}) = \varphi(\|\vec{r}\|)$:

$$\hat{f}(\vec{k}) = \frac{1}{(2\pi)^d}\int d\vec{r}\; f(\vec{r})\exp[i\vec{k}\cdot\vec{r}]$$

It can be shown that $\hat{f}(\vec{k})$ is given by:

$$\boxed{\hat{f}(\vec{k}) = \gamma_d \|\vec{k}\|^{d-1}\,\hat{\varphi}(\|\vec{k}\|)}$$

where $\hat{\varphi}$ and φ are related by the spherical Fourier transform pair (also called "the Fourier-Bessel transform"):

$$\boxed{\begin{aligned} \varphi(\rho) &= \int_0^{+\infty} \hat{\varphi}(k)\,\Phi_d(k\rho)\,dk \\ \hat{\varphi}(k) &= \frac{\gamma_d^2}{(2\pi)^d}\int_0^{+\infty} (k\rho)^{d-1}\varphi(\rho)\,\Phi_d(k\rho)\,d\rho \end{aligned}}$$

For $d=2$ and $d=3$ the Fourier-Bessel transform simplifies as follows:

$$\xleftarrow{\quad d=2 \quad}$$

$$\varphi(\rho) = \int_0^{+\infty} \hat{\varphi}(k)\, J_0(k\rho)\, dk \quad \Bigg| \quad \hat{\varphi}(k) = \int_0^{+\infty} k\rho\, \varphi(\rho)\, J_0(k\rho)\, d\rho$$

$$\varphi(\rho) = \int_0^{+\infty} \hat{\varphi}(k)\, \frac{\sin(k\rho)}{k\rho}\, dk \quad \Bigg| \quad \hat{\varphi}(k) = \frac{2}{\pi}\int_0^{+\infty} \varphi(\rho)\, k\rho \sin(k\rho)\, d\rho$$

$$\xleftarrow{\quad d=3 \quad}$$

The relation that serves as a point of departure for all the above is:

$$\boxed{\int_{\mathbb{R}^d} d\vec{r}\ \varphi(\|\vec{r}\|) \exp(i\vec{k}\cdot\vec{r})\, d\vec{r} = \int_0^{+\infty} \gamma_d\, \rho^{d-1}\, \varphi(\rho)\, \Phi_d(\|\vec{k}\|\rho)\, d\rho}$$

Finally, for completeness we mention a group of simple properties of Φ_d that can be shown by the definition:

$$\Phi_d(0) = 1$$
$$\Phi_d(-x) = \Phi_d(x) \qquad \lim_{x\to 0^+} \frac{1-\Phi_d(x)}{x^2} = \frac{1}{2d}$$
$$|\Phi_d(x)| \leqslant 1$$

The limit property in particular implies that the Taylor expansion of Φ_d around $x=0$ is given by:

$$\boxed{\Phi_d(x) = 1 - \frac{x^2}{2d} + O(x^4)}$$

Equivalence with spectral representation

The equivalence of the physical space representation with the spectral representation is well-known but not fully appreciated outside the context of "homogeneous - isotropic turbulence".

It is our intention to compare

$$E(k) = \frac{1}{2}\frac{d}{dk}\langle a^{<}(k)\, b^{<}(k)\rangle \quad \text{with} \quad B(\vec{\ell}) = \langle a(0)\, b(\vec{\ell})\rangle$$

where a, b are two arbitrary fields. Recall that the brackets simply denote a spatial integral. We act as follows:

1) Note that E is invariant under the transposition of $a \leftrightarrow b$ whereas B is not. We introduce therefore

$$\Gamma(\vec{\ell}) = \frac{\langle a(\vec{\ell})\, b(0)\rangle + \langle a(0)\, b(\vec{\ell})\rangle}{2}$$

which exhibits the same invariance.

2) We spherically average $\Gamma(\vec{\ell})$ since we want to match it with $E(k)$ which is spherically averaged by definition

$$\Gamma(\rho) = \frac{1}{\gamma_n}\int_{SO(n)} d\Omega(A)\, \Gamma(\rho A\vec{e})$$

3) Expand $\Gamma(\rho)$ in terms of Fourier- Bessel modes:

$$\Gamma(\rho) = \int_0^{+\infty} \gamma(k)\, \Phi_n(k\rho)\, dk$$

We will show that $E(k) = \gamma(k)/2$. To do that, let $\hat{a} = \mathcal{F}a$ and $\hat{b} = \mathcal{F}b$ and define:

$$\tilde{F}(\vec{p},\vec{q}) = \frac{\hat{a}^*(\vec{p})\,\hat{b}(\vec{q}) + \hat{a}(\vec{p})\,\hat{b}^*(\vec{q})}{2}$$

We have already shown (p.28) that

$$E(k) = \int d\vec{k}_0\; \delta(k - \|\vec{k}_0\|)\, \tilde{F}(\vec{k}_0,\vec{k}_0)\; (2\pi)^n/2$$

We now perform the corresponding evaluation for $\gamma(k)$.

Proposition : $\boxed{\Gamma(\vec{\ell}) = (2\pi)^n \int_{\mathbb{R}^n} d\vec{k}\; \tilde{F}(k,k)\, \exp(-i\vec{k}\cdot\vec{r})}$

Proof

First note that

$$\langle a(0)\, b(\vec{\ell})\rangle = \int d\vec{r}\; a(\vec{r})\, b(\vec{r}+\vec{\ell}) =$$

$$= \int d\vec{r} \int d\vec{p}\, d\vec{q}\;\; \hat{a}(\vec{p})\, \hat{b}^*(\vec{q})\, \exp[\,i\vec{p}\cdot\vec{r} - i\vec{q}\cdot(\vec{r}+\vec{\ell})\,] =$$

$$= \int d\vec{p}\, d\vec{q}\;\; \hat{a}(\vec{p})\, \hat{b}^*(\vec{q})\, \exp(-i\vec{q}\cdot\vec{\ell}) \left\{ \int_{\mathbb{R}^n} d\vec{r}\; \exp[\,i(\vec{p}-\vec{q})\cdot\vec{r}\,] \right\} =$$

$$= \int d\vec{p}\, d\vec{q}\;\; \hat{a}(\vec{p})\, \hat{b}^*(\vec{q})\, \exp(-i\vec{q}\cdot\vec{\ell})\; (2\pi)^n\, \delta(\vec{p}-\vec{q})$$

$$= (2\pi)^n \int d\vec{k}\;\; \hat{a}(\vec{k})\, \hat{b}^*(\vec{k})\, \exp(-i\vec{k}\cdot\vec{r})$$

Similarly:

$$\langle a(\vec{\ell})\, b(0)\rangle = (2\pi)^n \int d\vec{k}\;\; \hat{a}^*(\vec{k})\, \hat{b}(\vec{k})\, \exp(-i\vec{k}\cdot\vec{r})$$

Areraging these two identities together we obtain:

$$\Gamma(\vec{\ell}) = (2\pi)^n \int_{\mathbb{R}^n} d\vec{k}\ \tilde{F}(\vec{k},\vec{k}) \exp(-i\vec{k}\cdot\vec{r}) \qquad \square$$

Proposition :

$$\boxed{\gamma(k) = (2\pi)^n k^{n-1} \int_{SO(n)} \tilde{F}(kA\vec{e}, kA\vec{e})\, d\Omega(A)}$$

Proof

$$\Gamma(\rho) = \frac{1}{\gamma_n} \int_{SO(n)} d\Omega(A)\ \Gamma(\rho A\vec{e}) =$$

$$= \frac{1}{\gamma_n} \int_{SO(n)} d\Omega(A) \left\{ (2\pi)^n \int d\vec{k}\ \tilde{F}(\vec{k},\vec{k}) \exp(-i\vec{k}\cdot(\rho A\vec{e})) \right\} =$$

$$= \frac{(2\pi)^n}{\gamma_n} \int_{\mathbb{R}^n} d\vec{k}\ \tilde{F}(\vec{k},\vec{k}) \left\{ \int_{SO(n)} d\Omega(A) \exp[-i\vec{k}\cdot(\rho A\vec{e})] \right\} =$$

$$= \frac{(2\pi)^n}{\gamma_n} \int_{\mathbb{R}^n} d\vec{k}\ \tilde{F}(\vec{k},\vec{k})\, \gamma_n\, \Phi_n(\|\vec{k}\|\rho)$$

$$= (2\pi)^n \int_0^{+\infty} dk \int_{SO(n)} d\Omega(A)\ k^{n-1} \tilde{F}(kA\vec{e}, kA\vec{e})\, \Phi_n(k\rho)$$

$$= (2\pi)^n \int_0^{+\infty} dk \left\{ k^{n-1} \int_{SO(n)} d\Omega(A)\ \tilde{F}(kA\vec{e}, kA\vec{e}) \right\} \Phi_n(k\rho) \Rightarrow$$

$$\Rightarrow \quad \gamma(k) = (2\pi)^n k^{n-1} \int_{SO(n)} \tilde{F}(kA\vec{e}, kA\vec{e})\, d\Omega(A) \qquad \square$$

Comparing this relation with the one for $E(k)$ (p. 66) we find:

$$\boxed{E(k) = \gamma(k)/2}$$

This result is known as the Wienier-khinchin theorem. Note that this is NOT a statistical result. It is a purely mathematical result of Fourier theory. In fact the derivation of the "same" result for homogeneous and isotropic turbulence requires a different proof, a statistical proof! This detail is not well-appreciated.

Wienier-khinchin relations

An immediate consequence of our result is the Wienier-khinchin relations:

(1) Conservation of energy

The energy spectrum $E(k)$ can be related with the trace of the covariance tensor of u_a. Both are defined as:

$$E(k) = \frac{1}{2} \frac{d}{dk} \langle \cancel{d} u_a^{<} u_a^{<} \rangle \qquad B$$