The Amazing World of Trigonometry.

One of the things that make mathematics fascinating is the general nature of the ideas. When you ask your math teacher how to solve a specific problem he will probably say to you: "Look, this is how we handle the general case" and then he will mumble over some crazy variables. "But I want to do this simple problem!" will shout the poor student lost in the jungle of some bizarre symbols and the teacher will calmly retort: "Then set your variables to the given situation!". This is how it usually ~~we~~ goes and personally I find this attitude quite enjoying (though my classmates do not share my opinion). Wherever there are general things, I ~~am~~ ~~on~~ such as the binomial theorem, series, solutions of third-degree polynomial equations, I am on, I take them and I have fun with them. Whenever my teacher mumbles over specific things, I try to work out the general case. I had one shot of this in trigonometry. At first, I was just curious. Then, I made some more analysis on the thing and I discovered some amazing facts. The problem is to figure out the cos and sin of $(2v+1)a$. What is it?

Well, we can use the $(a+b)$ formulae, which say that

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$

and figure out some special formulas for the $2a$:

$$\cos 2a = 2\cos^2 a - 1.$$

$$\sin 2a = 2 \sin a \cos a$$

Then, we can take the $(2v+1)a$ this way:

$$\cos 3a = \cos(a+2a) = \cos a \cos 2a - \sin a \sin 2a = \ldots$$

and let it rip. It will turn out that

$$\cos 3a = 4\cos^3 a - 3\cos a.$$

Doing the same thing with the sin, we'll get

$$\sin 3a = -4\sin^3 a + 3\sin a$$

which looks pretty interesting.

Then we can try the $5a$. We will write $5a$ as

3a+2a and use the a+b formula once again. The calculation will be a little messy but in the end, we will get two beatiful formulae for the 5a:

$$\cos 5a = 16\cos^5 a - 20\cos^3 a + 5\cos a$$
$$\sin 5a = 16\sin^5 a - 20\sin^5 a + 5\sin a$$

7a will be much harder but if you work it out it ~~w~~ ~~ig~~ is going to turn out to be: ($7a = 5a+2a$)

$$\cos 7a = 64\cos^7 a - 112\cos^5 a + 56\cos^3 a - 7\cos a$$
$$\sin 7a = -64\sin^7 a + 112\sin^5 a - 56\sin^3 a + 7\sin a$$

This is wonderful! But... this is how far our teacher got. He gave us an appreciation of these wonderful jewels and then he stopped! I could not rest, for I knew there was a pattern in those formulae. In the first place, the ones ~~o~~for the cos were almost the same ~~for~~ as those ~~o~~ for the sin. Were they to keep going like this? Then, the coefficients at the edges were a geometric and an arithmetic sequance respectively. Why? Third notice: the signs flip-flop, starting with a plus in all cos-formulae and a minus in a few sin-formulae! What is this? What is the meaning, the cause of all these? In other words, what is the general principle? Well; let's find it.

▼ Newton's binomial and De Moivre theorema

When dealing with trigonometry, we can easily get stuck by asking crazy questions. When this happens, we ~~will~~ have to resort to tricky geometric situations. For example, to prove $\sin^2 x + \cos^2 x = 1$ with no background, will make us resort to the pythagorean theorem. To calculate the ultimate formula for $\cos(a+b)$, we will have to draw a circle and equate the two different expressions of two equal things; the lengths of two equal chords. So we see that to prove a rough theorem requires geometric figures and ~~as~~ because too!

of this, a lot of wits. However, De Moivre found a different route.

We all know what e is. It is merely the limit of

$$a_v = \left(1+\frac{1}{v}\right)^v$$ ~~e~~

and it turns out to be a transcendental number, approximately equal to 2,7. The $f(x) = e^x$ is called as exponential fuction and it is usually evaluated by the series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^v}{v!} + \cdots$$

De Moivre asked himself: «What is the e^{ix}, that is what will happen if I set x equal to ix?». Well, half the terms will turn out real, and the rest imaginary. If we collect the real ones away from the imaginary, our expression for e^{ix} will be of the form

$$e^{ix} = c(x) + i \cdot s(x)$$

where $c(x), s(x)$ some fuctions of x. What are they? De Moivre discovered the astonishing fact that they are the cos and the sin fuctions of trigonometry and he wrote:

$$e^{ix} = \cos x + i \sin x \qquad (1)$$

I don't know how he did it but at least I know how it <u>must</u> be done. It is a lot of work and one day I am going to do it.

If (1) is true, then the conjugate expression

$$e^{-ix} = \cos x - i \sin x \qquad (2)$$

has to be true as well. By adding and subtracting we can get

$$\boxed{\begin{aligned} 2\cos x &= e^{ix} + e^{-ix} \\ 2i \sin x &= e^{ix} - e^{-ix} \end{aligned}} \begin{matrix} (a) \\ (b) \end{matrix} \quad (3)$$

which is music to my ears! We no longer have to refer to geometric sketches; we can think of sin and cos as algebraic fuctions defined by (3) and figure out everything we want by pure algebra, including the $(2v+1)a$ formula!

Let's do it! Applying (3a) we have

$$2\cos[(2v+1)a] = (e^{ia})^{2v+1} + (e^{-ia})^{2v+1}$$

which we can rewrite as

$$2\cos[(2v+1)a] = (\cos a + i\sin a)^{2v+1} + (\cos a - i\sin a)^{2v+1}$$

Now, all we have to do is to unwrap the parentheses and set the thing to a decent form. This is a job for the binomial theorem which tells us how to raise $(a+b)$ to an integer power. It is

$$\boxed{(a\pm b)^v = \sum_{k=0}^{v} (\pm 1)^k \binom{v}{k} a^{v-k} b^k} \quad (4)$$

where $\binom{v}{k}$ is the number of combinations of ~~a~~ v things taken k at a time. This number is given by the formula

$$\boxed{\binom{v}{k} = \frac{v!}{k!(v-k)!}} \quad , \; v \geqslant k \geqslant 0 \qquad (5)$$

and in some books it is written out as $_kC_v$. As we are going to get involved into some cumbersome expressions, we shall prefer the $\binom{v}{k}$ notation. There are a couple of identities that may prove handy so we'll ~~just~~ mention them here, right away. The first one says that

$$\boxed{\binom{a}{b} = \binom{a}{a-b}} \quad (6)$$

and you can easily check it out for yourself. The second one is more involved. It was discovered by Pascal and it is used to ~~prov~~ build the famous "Pascal triangle". It states that

$$\boxed{\binom{a-1}{b-1} + \binom{a-1}{b} = \binom{a}{b}} \quad (7)$$

Again, it can been proved from the definition like this:

$$\binom{a-1}{b-1} + \binom{a-1}{b} = \frac{(a-1)!}{(b-1)!(a-b)!} + \frac{(a-1)!}{b!(a-b-1)!} = \frac{(a-1)!}{(b-1)!(a-b-1)!}\left[\frac{1}{a-b} + \frac{1}{b}\right] =$$

$$= \frac{(a-1)!}{(b-1)!(a-b-1)!}\,\frac{a}{b(a-b)} = \frac{a!}{b!(a-b)!} = \binom{a}{b}$$

What makes this relationship so important is the fact that we can use it to get a recursive definition for the $\binom{a}{b}$-thing. We simply set $\binom{1}{0} = \binom{a}{0} = \binom{a}{a} = 1$

and use Pascal's formula to get the rest. We can do this process by building up the coefficients like this:

$$\begin{array}{l|llllll l l}
\multicolumn{9}{l}{\xrightarrow{\hspace{3em}} k} \\
 & 1 & 1 & & & & & & \\
 & 1 & 2 & 1 & & & & & (a\pm b)^2 = a^2 \pm 2ab + b^2 \\
v & 1 & \boxed{3 \;\; 3} & 1 & & & & & (a\pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3 \\
\downarrow & 1 & 4 & \boxed{6} & 4 & 1 & & & (a\pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4 \\
 & 1 & 5 & 10 & 10 & 5 & 1 & & (a\pm b)^5 = a^5 \pm 5a^4b + 10a^3b^2 \pm 10a^2b^3 + 5ab^4 + b^5
\end{array}$$

and so on. You can see the pattern; at the edges it is all aces. The rest of the junk comes by adding together the number above ~~b~~ and the number before that, which is exactly what the Pascal identity says. The whole building is known as Pascal's triangle and it gives a recursive way of calculating the binomial coefficients.

▼ The $\cos[(2v+1)a]$ formula

Now, back to our job. We must unwrap the

$$2\cos[(2v+1)a] = (\cos a + i\sin a)^{2v+1} + (\cos a - i\sin a)^{2v+1}$$

The first term will give

$$(\cos a + i\sin a)^{2v+1} = \sum_{k=0}^{2v+1} \binom{2v+1}{k} \cos^{2v+1-k} a \, (i\sin a)^k =$$

$$= \sum_{k=0}^{2v+1} i^k \binom{2v+1}{k} \cos^{2v+1-k} a \, \sin^k a$$

and similarly, the second one will be

$$(\cos a - i\sin a)^{2v+1} = \sum_{k=0}^{2v+1} (-1)^k \binom{2v+1}{k} \cos^{2v+1-k} a \, (i\sin a)^k =$$

$$= \sum_{k=0}^{2v+1} (-1)^k i^k \binom{2v+1}{k} \cos^{2v+1-k} a \cdot \sin^k a$$

If we put it back together, we'll get

$$2\cos[(2v+1)a]=\sum_{k=0}^{2v+1}\left[i^{k}\binom{2v+1}{k}\cos^{2v+1-k}a\cdot\sin^{k}a+(-1)^{k}i^{k}\binom{2v+1}{k}\cos^{2v+1-k}a\cdot\sin^{k}a\right]$$

which is longer than it should be. Observe that as the counter k runs over 1, 3, 5, ..., 2v+1, everything gets wiped out so we can forget about them and let k run only through 0, 2, 4, ..., 2v. Now, we don't have a symbol to designate the fact that we run k at step 2 so we just switch to another variable, say λ, that will run at the natural step 1. How can we do that? Well, we simply say that

$$\lambda=\frac{k}{2}$$

Then, as we run k over 0, 2, ..., 2v, the λ variable will run over 0, 1, 2, 3, ..., v which is good. And one more thing: we replace all k's with 2λ's because $k=2\lambda$. When we do this trick, we get

$$\cancel{2}\cos[(2v+1)a]=\cancel{2}\sum_{\lambda=0}^{v} i^{2\lambda}\binom{2v+1}{2\lambda}\cos^{2v+1-2\lambda}a\cdot\sin^{2\lambda}a$$

which looks better. As you can see, we have got rid of all the imaginary terms. You wouldn't expect a cosine to turn out ~~come~~ imaginary, would you? Certainly not. So, if there are some imaginary terms, they had better go and that's just what they did! Where do we go from here? We can go ~~eve~~ wherever we want to and what we want is to put cos[(2v+1)a] into a form P(cosa) where P will be some polynomial. You recall that

$$\cos 3a=4\cos^{3}a-3\cos a$$
$$\cos 5a=16\cos^{5}a-20\cos^{3}a+5\cos a$$
$$\cos 7a=64\cos^{7}a-112\cos^{5}a+56\cos^{3}a-7\cos a$$

We want this sort of expression and to get it, we must exterminate the $\sin^{2\lambda}a$. This can be done with the oldest trick in the book. $\sin^2 a$ is

$$\sin^2 a = 1-\cos^2 a$$

so,

$$\sin^{2\lambda} a = (1-\cos^2 a)^\lambda = \sum_{\mu=0}^{\lambda} (-1)^\mu \binom{\lambda}{\mu} 1^{\lambda-\mu}\cos^{2\mu} a = \sum_{\mu=0}^{\lambda}(-1)^\mu\binom{\lambda}{\mu}\cos^{2\mu}a$$

As for $i^{2\lambda}$, it looks awkward so we write it as $(-1)^\lambda$ and we get the following expression for $\cos[(2v+1)a]$

$$\cos[(2v+1)a] = \sum_{\lambda=0}^{v}\left[(-1)^\lambda\binom{2v+1}{2\lambda}\cos^{2(v-\lambda)+1}a\cdot\left(\sum_{\mu=0}^{\lambda}(-1)^\mu\binom{\lambda}{\mu}\cos^{2\mu}a\right)\right]$$

which is rather messy. To improve the shape a little bit, we use the generalized distributive property which says that

$$\sum_{\lambda=0}^{v}\left(a_\lambda\cdot\sum_{\mu=0}^{\lambda}b_\mu\right) = \sum_{\lambda=0}^{v}\left[\sum_{\mu=0}^{\lambda}(a_\lambda b_\mu)\right] = \sum_{\lambda=0}^{v}\sum_{\mu=0}^{\lambda}(a_\lambda b_\mu)$$

The desired result will be

$$\cos[(2v+1)a] = \sum_{\lambda=0}^{v}\sum_{\mu=0}^{\lambda}\left[(-1)^\lambda\binom{2v+1}{2\lambda}\cos^{2(v-\lambda)+1}a\cdot(-1)^\mu\binom{\lambda}{\mu}\cos^{2\mu}a\right] =$$

$$= \sum_{\lambda=0}^{v}\sum_{\mu=0}^{\lambda}\left[(-1)^{\lambda+\mu}\binom{\lambda}{\mu}\binom{2v+1}{2\lambda}\cos^{2(\mu+v-\lambda)+1}a\right] \qquad (8)$$

Now it looks better! We may already understand why the $\cos a$ is found only in odd powers since they will be of the form $2(\mu+v-\lambda)+1$. We may also see what the min power will be if we let μ have the min value and λ the max. $\mu=0$ and $\lambda=v$ and this will yield first power. So it is! More over, we can see that the max power is $2v+1$, if we let ~~get~~ λ get minimum (which is $\lambda=0$) and μ maximum (which is $\mu=\lambda=0$, what a max!). So, we are on the right track. There is only one more problem we have to take care of. We must put the like terms together, that is we must sort out of this garbled mix the $\cos a$-terms, the $\cos^3 a$-ones and so on and find the total coefficients. We can set the question like this: "What is the $2b+1$-degree term?". Well, we have a lot of terms in there and for

some specific values of λ and μ we will get the terms we want. So we must find some info about λ and μ.
One useful relation is

$$2b+1 = 2(\mu+v-\lambda)+1$$

which gives $b = \mu+v-\lambda \Leftrightarrow \boxed{\mu = b+\lambda-v}$

What does it mean? It says that as we run the λ, <u>only one value for μ</u> will yield a $2b+1$-degree term, the one given by the equation at the box. So we merely run the λ and use for μ the $b+\lambda-v$. Now, there comes another problem. Which values of λ should we take? The ones that give a reasonable μ of course! We can see from the summation signs that

- $0 \leqslant \mu \leqslant \lambda$

By setting $\mu = b+\lambda-v$ we have

$$0 \leqslant b+\lambda-v \leqslant \lambda \Leftrightarrow -\lambda \leqslant b-v \leqslant 0 \Leftrightarrow \underline{\lambda \geqslant v-b}.$$

If we add to this the fact that $\lambda \leqslant v$, we can see that we must run the λ from $v-b$ to v, that is:

$$\boxed{v-b \leqslant \lambda \leqslant v}$$

To sum it up, in order to get the $2b+1$ term, we take the thing inside the $[\cdots]$ marks and let λ run from $v-b$ to v using $b+\lambda-v$ instead of μ! (This was the toughest part to grasp). When we do that, we will get:

$$\left\{\sum_{\lambda=v-b}^{v}\left[(-1)^{\lambda+(b+\lambda-v)}\binom{\lambda}{b+\lambda-v}\binom{2v+1}{2\lambda}\right]\right\}\cos^{2b+1} a$$

Notice that $(-1)^{\lambda+(b+\lambda-v)} = (-1)^{b-v}\cdot(\overset{+1}{\cancel{-1}})^{2\lambda} = (-1)^{b}(\overset{(-1)^{v}}{\cancel{-1}})^{-v} = (-1)^{v+b}$.
If we use for shorcut the notation

$$A_b^v \cos^{2b+1} a$$

we can see that A_b^v should be

$$\boxed{A_b^v = \sum_{\lambda=v-b}^{v}\left[(-1)^{b+v}\binom{\lambda}{b+\lambda-v}\binom{2v+1}{2\lambda}\right]} \quad (9)$$

We have already explained that $2b+1$ should vary from $1,3,5,\ldots,2v+1$ which means that $0 \leqslant b \leqslant v$. We can therefore conjure up the wonderful theorema that

THEOREMA 1:

$$\boxed{\begin{array}{l} \cos[(2v+1)a] = \sum\limits_{k=0}^{v} A_k^v \cos^{2k+1} a \\ \text{where, } A_k^v = \sum\limits_{\lambda=v-k}^{v} \left[(-1)^{k+v} \binom{\lambda}{k+\lambda-v} \binom{2v+1}{2\lambda} \right] \end{array}}$$

This IS the general formula but does it really work? Let's try it on the $5a$. We are expecting

$$\cos 5a = 16\cos^5 a - 20\cos^3 a + 5\cos a$$

which means that $A_2^2 = 16$, $A_1^2 = -20$, $A_0^2 = 5$. It's easy to check these out. Let's do the A_2^2. We set $v=2$ and $k=2$. Then, from the inequality $v-b \leqslant \lambda \leqslant v$ we see that λ must take the values $0,1,2$. This will give:

$$A_2^2 = \overset{\lambda=0}{(-1)^4 \binom{0}{0}\binom{5}{0}} + \overset{\lambda=1}{(-1)^4 \binom{1}{1}\binom{5}{2}} + \overset{\lambda=2}{(-1)^4 \binom{2}{2}\binom{5}{3}} =$$

$$= \frac{0!}{0!\,0!}\,\frac{5!}{0!\,5!} + \frac{1!}{1!\,0!}\,\frac{5!}{2!\,3!} + \frac{2!}{2!\,0!}\cdot\frac{5!}{4!\,1!} = 1+10+5 = 16$$

which is exactly what we want it to be! What about A_1^2? This time we set $v=2$, $k=1$ and let λ take the the values 1 and 2. What we get is

$$A_1^2 = (-1)^3 \binom{1}{0}\binom{5}{2} + (-1)^3 \binom{2}{1}\binom{5}{4} = -\frac{1!}{0!\,1!}\,\frac{5!}{2!\,3!} - \frac{2!}{1!\,1!}\,\frac{5!}{4!\,1!} =$$

$$= -1\cdot 10 - 2\cdot 5 = -20 \quad !!$$

One more too go. We set $v=2$ and $k=0$. The inequality yields $2 \leqslant \lambda \leqslant 2$ which means that we are to let λ have only one value, that is 2. We will get just one term for A_0^2:

$$A_0^2 = (-1)^2 \binom{2}{0}\binom{5}{4} = \frac{2!}{0!\,2!}\,\frac{5!}{4!\,1!} = 5$$

so you see that it really works. The path is clear for us to work out some party ~~fast~~ formulae: We can confirm the 5a and 7a and find the 9a one. It will be

$$\cos 9a = 256\cos^9 a - 576\cos^7 a + 432\cos^5 a - 120\cos^3 a + 9\cos a$$

Next comes the 11a:

$$\cos 11a = 1029\cos^{11} a - 2816\cos^9 a + 2816\cos^7 a - 1232\cos^5 a + 220\cos^3 a - 11\cos a$$

Why not the lucky 13a?

$$\cos 13a = 4096\cos^{13} a - 13312\cos^{11} a + 16640\cos^9 a \mp 9984\cos^7 a + 2912\cos^5 a - 364\cos^3 a + 13\cos a.$$

and with a good computer, you can get the 15a, 17a, 19a and so on. As a matter of fact, a formula for 23a is actually just junk, however it is fun trying to work it out.

You must have already noticed two particularly annoying things. In the first place, the cosa term has a 2v+1 coefficient. We can really guess that the cos45a will give a 45cosa term! Then, we also notice that the max-degree terms have coefficients which are all powers of 4 ~~is~~ and are, in fact, just 4^v! Originally, I just wanted to work out the general formulae. But then I noticed these crazy properties. I finally managed to work them out and we'll do that later. Firstly we must get the sin[(2v+1)a] formula.

▼ The sin[(2v+1)a] formula

It is merely the same story, only this time we shall use the relation

$$2i\sin x = e^{ix} - e^{-ix}$$

instead. Setting $x = [2v+1)a$ we get

$$\cos[(2v+1)a] = (e^{ia})^{2v+1} - (e^{-ia})^{2v+1} = (\cos a + i\sin a)^{2v+1} - (\cos a - i\sin a)^{2v+1} =$$

$$= \sum_{k=0}^{2v+1}\left[i^k \binom{2v+1}{k} \cos^{2v+1-k} a \cdot \sin^k a - (-1)^k i^k \binom{2v+1}{k} \cos^{2v+1-k} a \cdot \sin^k a \right]$$

Once again, half the terms cancel, ~~this~~ that is the real terms. The terms that remain are the ones with

$$k = 1, 3, 5, \ldots, 2v+1.$$

To keep track of them, we switch to a variable λ, this time defined by

$$\lambda = \frac{\kappa - 1}{2}$$

As κ runs through $1, 3, \dots, 2v+1$, λ will naturally run from 0 up to v. Solving for κ we get what we must use for κ:

$$\kappa = 2\lambda + 1$$

and we rewrite the thing as

$$\cancel{2i} \cdot \sin[(2v+1)a] = 2 \sum_{\lambda=0}^{v} i^{2\lambda+1} \binom{2v+1}{2\lambda+1} \cos^{2v+1-2\lambda-1} a \cdot \sin^{2\lambda+1} a \quad (\bullet\, i^{2\lambda+1} = i(-1)^{\lambda})$$

$$= \cancel{2i} \sum_{\lambda=0}^{v} (-1)^{\lambda} \binom{2v+1}{2\lambda+1} \cos^{2(v-\lambda)} a \cdot \sin^{2\lambda+1} a$$

Then we play the old trick that $\cos^2 a = 1 - \sin^2 a$, which in this case will give

$$\sin[(2v+1)a] = \sum_{\lambda=0}^{v} (-1)^{\lambda} \binom{2v+1}{2\lambda+1} (1 - \sin^2 a)^{v-\lambda} \cdot \sin^{2\lambda+1} a =$$

$$= \sum_{\lambda=0}^{v} \left[(-1)^{\lambda} \binom{2v+1}{2\lambda+1} \sin^{2\lambda+1} a \cdot \left(\sum_{\mu=0}^{v-\lambda} \binom{v-\lambda}{\mu} (-1)^{\mu} \sin^{2\mu} a \right) \right]$$

$$= \sum_{\lambda=0}^{v} \sum_{\mu=0}^{v-\lambda} \left[(-1)^{\mu+\lambda} \binom{2v+1}{2\lambda+1} \binom{v-\lambda}{\mu} \sin^{2(\mu+\lambda)+1} a \right]$$

So here we are! Now to sort out the terms. We are looking for the $2b+1$ degree term which means that

$$2b+1 = 2(\mu+\lambda)+1 \Leftrightarrow$$
$$\Leftrightarrow b = \mu + \lambda \Leftrightarrow \boxed{\mu = b - \lambda}$$

for each value of λ, we must use $b-\lambda$ for μ to get a $2b+1$ term. Then we must be careful enough so that $b-\lambda$ be meaningful. According to the summation sign, μ's range is

$0 \leqslant \mu \leqslant v-\lambda \Leftrightarrow 0 \leqslant b-\lambda \leqslant v-\lambda \Leftrightarrow \lambda \leqslant b \leqslant v$

so we see that λ must be $\lambda \leqslant b$. On the other hand, λ has its own range: $0 \leqslant \lambda \leqslant v$. Putting it all together, we see that only the λ between 0 and b will work, that is:

$$\boxed{0 \leqslant \lambda \leqslant b}$$

So we just ~~take~~ get the λ going and use $b-\lambda$ for μ.
We get

$$B_b^v \sin^{2b+1} a = \left[\sum_{\lambda=0}^{b} (-1)^b \binom{2v+1}{2\lambda+1} \binom{v-\lambda}{b-\lambda}\right] \sin^{2b+1} a$$

What values is b allowed to have? Firstly, by $0 \leqslant \lambda \leqslant b$ we conclude that b is positive or zero. Then, because $b = \mu + \lambda$, we must use the max values for μ and λ to get the max b. The max λ is $\lambda = v$ and the max μ is $\mu = v - \lambda = v - v = 0$, therefore ~~$\mu$~~ b shouldn't excede $2v+1$. Thus, we get our second theorema:

<u>THEOREMA 2:</u>

$$\boxed{\begin{gathered} \sin[(2v+1)a] = \sum_{k=0}^{v} B_k^v \sin^{2k+1} a \\ \text{where } B_k^v = \sum_{\lambda=0}^{k} \left[(-1)^k \binom{2v+1}{2\lambda+1} \binom{v-\lambda}{k-\lambda}\right] \end{gathered}}$$

Now comes the fun part. We said, a moment ago, that

$$\cos 5a = 16\cos^5 a - 20\cos^3 a + 5\cos a$$

What is the corresponding formula for the sinus function? It is, according to our theorema, just

$$\sin 5a = 16\sin^5 a - 20\sin^3 a + 5\sin a$$

and lo and behold! It is exactly the same with the cos formula! Is it a coincidence? Then, let's try the 7a ones. For the cos it is

$$\cos 7a = 64\cos^7 a - 112\cos^5 a + 56\cos^3 a - 7\cos a$$

and for the sin

$$\sin 7a = -64\cos^7 a + 112\cos^5 a - 56\cos^3 a + 7\cos a$$

which again is the same thing, this time with the signs reversed! Hm... something spooky is going on in here...

Let's do the damn thing generally! What we want is to compare one to one the coefficients A_k^v with the B_k^v. If they are to be almost the same, there has to be some general relation that will tell us ~~what~~ exactly what is happening. To begin with, we have the following two results:

$$A^{v}_{k} = \sum_{\lambda=v-k}^{v}\left[(-1)^{k+v}\binom{\lambda}{(k-v)+\lambda}\binom{2v+1}{2\lambda}\right]$$

$$B^{v}_{k} = \sum_{\lambda=0}^{k}\left[(-1)^{k}\binom{v-\lambda}{k-\lambda}\binom{2v+1}{2\lambda+1}\right]$$

They do not seem alike but maybe there is a reason for that. In the A^v_k expression, the sum runs from $v-k$ to v while in the B^v_k it runs from 0 to k. To give the <u>same</u> results, they <u>must</u> look different! Then, we'll just make the counters run over the same range and see what happens. We will take A^v_k and switch to another counter, say λ', that will go from 0 to k. When $\lambda=v-k$, λ' must be 0 so the definition of λ' should be

$$\lambda' = \lambda-(v-k)$$

Indeed, when $\lambda=v$, $\lambda'=v-(v-k)=k$, so λ' runs over the same range as does the counter at B^v_k. So, we replace λ with $\lambda=\lambda'+(v-k)$ and see what happens:

$$A^{v}_{k} = \sum_{\lambda'=0}^{k}\left[(-1)^{k+v}\binom{\lambda'+(v-k)}{(k-v)+\lambda'+(v-k)}\binom{2v+1}{2[\lambda'+(v-k)]}\right]$$

$$= \sum_{\lambda'=0}^{k}\left[(-1)^{k+v}\binom{\lambda'+v-k}{\lambda'}\binom{2v+1}{2(\lambda'+v-k)}\right]$$

Nothing happened! It is still different from B^v_k and very cumbersome. Yet it stills gives the same results! What is wrong? We have used the same range but still there's something different. What is it? It can be only one thing: the order of the terms. If A^v_k is to give three terms, they will look like a+b+c. But B^v_k will give, maybe, c+b+a that is the same terms but in different order. If we match them one to one, they don't match because we are not using the proper order. So all we have to do is to make λ' run backwards! How can we do that is easy to answer. We merely switch to $\lambda'' = \kappa - \lambda'$. As we run λ' backwards, λ'' goes forward ~~from~~ over the beloved range $0-\kappa$. Now, this had better ~~work!~~ work!, We start again from

$$A_{\kappa}^{v} = \sum_{\lambda=v-\kappa}^{v} \left[(-1)^{\kappa+v} \binom{\lambda}{(\kappa-v)+\lambda} \binom{2v+1}{2\lambda} \right]$$

Since $\lambda' = \lambda - (v-\kappa)$, we can ~~express~~ relate λ'' with λ and solve for λ:

$$\lambda'' = \cancel{\lambda+}\, \kappa - [\lambda - (v-\kappa)] =$$

$$= \kappa - \lambda + v - \kappa = v - \lambda \Rightarrow \boxed{\lambda = v - \lambda''}$$

We just change the range and replace λ with $v-\lambda''$ (this is getting a routine!)

$$A_{\kappa}^{v} = \sum_{\lambda''=0}^{\kappa} \left[(-1)^{\kappa+v} \binom{v-\lambda''}{(\kappa-v)+(v-\lambda'')} \binom{2v+1}{2(v-\lambda'')} \right] =$$

$$= \sum_{\lambda''=0}^{\kappa} \left[(-1)^{\kappa+v} \binom{v-\lambda''}{\kappa-\lambda''} \binom{2v+1}{2(v-\lambda'')} \right]$$

This looks almost the same with B_{κ}^{v}. Only $\binom{2v+1}{2(v-\lambda'')}$ needs to be fixed. We can do that with the

$$\binom{a}{b} = \binom{a}{a-b}$$

identity. In our case, it will yield

$$\binom{2v+1}{2(v-\lambda'')} = \binom{2v+1}{2v+1-2(v-\lambda'')} = \binom{2v+1}{2\lambda''+1}$$

and if we pull out the $(-1)^{v}$ factor, the rest will be just B_{v}^{κ}: (we can, by the way, forget about the primes and use λ)

$$A_{\kappa}^{v} = (-1)^{v} \sum_{\lambda=0}^{\kappa} \left[(-1)^{\kappa} \binom{v-\lambda}{\kappa-\lambda} \binom{2v+1}{2\lambda+1} \right] = (-1)^{v} B_{\kappa}^{v}$$

Isn't that wonderful! We found, just out of the blue, that the A_{κ}^{v} and B_{κ}^{v} are exactly the same in those cases where v is even and with reversed signs when v is odd. That's why the 5a formulae were the same and moreover that's the agency that makes the signs get reversed in the 7a formulae. As we progress to the 9a, again they will be identical. The 11a will have the signs reversed and 13a right up again! Isn't that fascinating! It is, and it's one of those things that makes mathematics beautiful. We can use it to put our two theoremas into one supergrand marvelous theorema:

THEOREMA

$$\boxed{\begin{aligned} \sin[(2v+1)a] &= \sum_{K=0}^{v}\left[(-1)^v A_K^v \sin^{2K+1} a\right] \\ \cos[(2v+1)a] &= \sum_{K=0}^{v}\left[A_K^v \cos^{2K+1} a\right] \\ A_K^v &= \sum_{\lambda=0}^{K}\left[(-1)^{v+K}\binom{v-\lambda}{K-\lambda}\binom{2v+1}{2\lambda+1}\right] \end{aligned}}$$

This is the centre of the universe for trigonometry, these particularly elegant things. Why, we can even invent a glorious symbol, say $\mathbb{T}_v(x)$, that will stand for the "trigonometry polynomials" defined as

$$\boxed{\mathbb{T}_v(x) = \sum_{K=0}^{v}\left(A_K^v x^{2K+1}\right)}$$

The first few of them would be

$$\mathbb{T}_1(x) = 4x^3 - 3x$$
$$\mathbb{T}_2(x) = 16x^5 - 20x^3 + 5x$$
$$\mathbb{T}_3(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$

and we would have the fun of writting our results us

$$\boxed{\begin{aligned} \sin[(2v+1)a] &= (-1)^v\,\mathbb{T}_v(\sin a) \\ \cos[(2v+1)a] &= \mathbb{T}_v(\cos a) \end{aligned}}$$

Indeed, the best part of mathematics is inventing glorious symbols! Now, what about those polynomials? Do they have any crazy properties as polynomials? I have only found one, so far. To impress you, I'll just state it out: One of the roots is zero! I know it's trivial, but it is better than nothing! (By the way, another trite property is that $\mathbb{T}_v(-x) = -\mathbb{T}_v(x)$). However, I haven't found out yet something interesting about those polynomials. There are though, a few interesting things about the coefficients A^v_k and that's where we shall now turn to.

▼ Properties of the A^v_k coefficients

Mathematicians love to figure out elegant properties so we'll follow the tradition and do just that. Our first property will be this

$$\boxed{A^v_0 = (-1)^v(2v+1)}$$

To prove it, we take the formula

$$A^v_k = \sum_{\lambda=0}^{k}\left[(-1)^{v+k}\binom{v-\lambda}{k-\lambda}\binom{2v+1}{2\lambda+1}\right]$$

and set $k=0$. Then, λ will take only one value, $\lambda=0$ and we'll get

$$A^v_0 = (-1)^v\binom{v}{0}\binom{2v+1}{1} = (-1)^v\,\frac{v!}{0!\,v!}\;\frac{(2v+1)!}{1!\,(2v)!} = (-1)^v(2v+1)$$

That was almost too easy. The next one, will be harder. What we want to do is to prove that

$$\boxed{A^v_v = 4^v}$$

Now, we cannot use the formula directly. Yet, it will help up to a point to set $k=v$ and see what ~~will~~ happens:

$$A^v_v = \sum_{\lambda=0}^{v}\left[\overset{+1}{(\cancel{-1})^{v-v}}\binom{v-\lambda}{v-\lambda}\binom{2v+1}{2\lambda+1}\right]$$. But $\binom{v-\lambda}{v-\lambda} = \dfrac{(v-\lambda)!}{0!\,(v-\lambda)!} = 1$, so

we see that

$$A^v_v = \sum_{\lambda=0}^{v}\binom{2v+1}{2\lambda+1}$$

Where do we go from here? Well, all we have to do is to prove that

$$A_{v+1}^{v+1} = 4A_v^v \text{ and } A_1^1 = 4.$$

This is enough work to conclude that $A_v^v = 4^v$. It is just the induction method. You can check out for yourself that $A_1^1 = 4$. from the $\cos 3a = 4\cos^3 a - 3\cos a$. But what about proving that $A_{v+1}^{v+1} = 4A_v^v$? This is some quite involved work but I discovered that it can be done this way:

We start with the expression for A_{v+1}^{v+1} which is

$$A_{v+1}^{v+1} = \sum_{\lambda=0}^{v+1} \binom{2v+3}{2\lambda+1}$$

We have to relate it with

$$A_v^v = \sum_{\lambda=0}^{v} \binom{2v+1}{2\lambda+1}$$

and to do it we must make the $2v+3$ become $2v+1$, that is we must, sort of reduce the expression. We can do that with Pascal's identity which says: (I told you it might come in handy!)

$$\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$$

In our case, it tells us that

$$\binom{2v+3}{2\lambda+1} = \binom{2v+2}{2\lambda} + \binom{2v+2}{2\lambda+1}$$

and by applying it ~~to~~ to each of the $(2v+2)$-terms it gives

$$\binom{2v+3}{2\lambda+1} = \left[\binom{2v+1}{2\lambda-1} + \binom{2v+1}{2\lambda}\right] + \left[\binom{2v+1}{2\lambda} + \binom{2v+1}{2\lambda+1}\right] =$$

$$= \binom{2v+1}{2\lambda-1} + 2\binom{2v+1}{2\lambda} + \binom{2v+1}{2\lambda+1} \qquad (10)$$

That's good, but we must be very careful because there are some restrictions. You recall that

$$\binom{a}{b} = \frac{a!}{b!\,(a-b)!}$$

The domain for the expression is $a \geqslant 0, b \geqslant a, a-b \geqslant 0$ which is the same as $a \geqslant b \geqslant 0$. If we apply this fact to the ~~right~~ expression at (10), we'll get three inequalities:

$$\begin{cases} 2v+1 \geqslant 2\lambda-1 \geqslant 0 \\ 2v+1 \geqslant 2\lambda \geqslant 0 \\ 2v+1 \geqslant 2\lambda-1 \geqslant 0 \end{cases}$$

which, you can prove, are equivalent to $\boxed{v \geqslant \lambda \geqslant 1}$. On the other hand, λ's true range is $0 \leqslant \lambda \leqslant v+1$ ~~so~~ which is wider than it should be so we are in ~~true~~ trouble. The only way out of this is to take special care of the terms with $\lambda=0$ and $\lambda=v+1$. So, we'll just say that:

$$A_{v+1}^{v+1} = \sum_{\lambda=0}^{v+1} \binom{2v+3}{2\lambda+1} = \binom{2v+3}{1} + \sum_{\lambda=1}^{v} \left[\binom{2v+1}{2\lambda-1} + 2\binom{2v+1}{2\lambda} + \binom{2v+1}{2\lambda+1} \right] + \binom{2v+3}{2\lambda+3}$$

and reduce the edge-terms by some other trick. Like how? Notice that $\binom{a}{1} = \frac{a!}{1!(a-1)!} = a$. It's a joke! We just write the first term like this:

$$\binom{2v+3}{1} = 2v+3 = 2+[2v+1] = 2+\binom{2v+1}{1}$$

The last term can be taken care of by noting that

$$\binom{a}{a} = \frac{a!}{0!\,a!} = 1 = \binom{b}{b}$$

Though we could merely say that is 1, it will be useful to rewrite it as $\binom{2v+1}{2v+1}$.

Putting all the junk back together, we get

$$A_{v+1}^{v+1} = \underline{\binom{2v+1}{1}} + 2 + \sum_{\lambda=1}^{v} \left[\underline{\underline{\binom{2v+1}{2\lambda-1}}} + 2\binom{2v+1}{2\lambda} + \underline{\binom{2v+1}{2\lambda+1}} \right] + \underline{\underline{\binom{2v+1}{2v+1}}}$$

Which is total chaos! But not for long. Look at the single underlined terms. They are

$$\binom{2v+1}{1} + \sum_{\lambda=1}^{v} \binom{2v+1}{2\lambda+1} = \sum_{\lambda=0}^{v} \binom{2v+1}{2\lambda+1} = A_v^v$$

which looks good. If we can take out three more A_v^v, we will have proved that $A_{v+1}^{v+1}=4A_v^v$. We can have another one, this time by the double underlined terms

$$\sum_{\lambda=1}^{v}\left[\binom{2v+1}{2\lambda-1}\right]+\binom{2v+1}{2v+1}=\sum_{\lambda=0}^{v-1}\left[\binom{2v+1}{2\lambda+1}\right]+\binom{2v+1}{2v+1}=\sum_{\lambda=0}^{v}\binom{2v+1}{2\lambda+1}=A_v^v$$

We are on the right track, so far. ~~To~~ All this gives:

$$A_{v+1}^{v+1}=A_v^v+A_v^v+\sum_{\lambda=1}^{v}\left[2\binom{2v+1}{2\lambda}\right]+2$$

To get our job done, we merely have to prove that

$$1+\sum_{\lambda=1}^{v}\left[\binom{2v+1}{2\lambda}\right]=\sum_{\lambda=0}^{v}\binom{2v+1}{2\lambda+1}=A_v^v$$

It is rather tough but we can do it this way. We start with the right-hand part of the identity and extract the 1:

$$\sum_{\lambda=0}^{v}\binom{2v+1}{2\lambda+1}=\left[\sum_{\lambda=0}^{v-1}\binom{2v+1}{2\lambda+1}\right]+\binom{2v+1}{2v+1}=\left[\sum_{\lambda=0}^{v-1}\binom{2v+1}{2(v-\lambda)}\right]+1$$

Our job is almost done. All we have to do is to switch the range and make it run over 1, 2, ..., v. We merely switch to

$$\lambda'=\lambda+1\Rightarrow\lambda=\lambda'-1$$

and
$$\sum_{\lambda=0}^{v}\binom{2v+1}{2\lambda+1}=\left[\sum_{\lambda'=1}^{v}\binom{2v+1}{2(v-\lambda'+1)}\right]+1 \qquad (11)$$

Damn it! It didn't work! Yet, having played over the same game before, we know what to do next: Reverse the order of the summation. We switch to an λ'' which running from 1 to v makes λ' run from v to 1. It should look like

$$\lambda''=v-\lambda'+a.$$

What is a? Let's think! If we set $\lambda''=v\Rightarrow\lambda'=1$ so

$$v=v-1+a\Rightarrow a=1$$

and therefore $\lambda''=v-\lambda'+1$

Looking at (11) we see λ'' already laid up and by replacing $v-\lambda'+1$ with λ'' we readily get back to the identity we wanted to prove. So, we see that

$$A_{v+1}^{v+1} = A_v^v + A_v^v + A_v^v + A_v^v = 4A_v^v$$

This result, along with $A_1^1 = 4$ says that indeed $A_v^v = 4^v$! So, who cares! No-one I know. Yet, we had noticed something peculiar before and now we have explained it. You recall ~~so~~ that the coefficients at the edges were a geometric and an ~~arithem~~ arithmetic sequance. We can now guarantee that they ~~we~~ will keep going like this.