Dynamics and Stabilities of Generalized Forchheimer Flows in Porous Media

Luan Hoang[†], Akif Ibragimov

Department of Mathematics and Statistics, Texas Tech University
[†] *http://www.math.umn.edu/∼lhoang/*
luan.hoang@ttu.edu
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Outline

Introduction

Fluid flows in porous media with velocity u and pressure p:

- Darcy's Law:

$$\alpha u = -\nabla p,$$

- the "two term" law

$$\alpha u + \beta |u|\, u = -\nabla p,$$

- the "power" law

$$c^n |u|^{n-1}\, u + au = -\nabla p,$$

- the "three term" law

$$\mathcal{A} u + \mathcal{B}\, |u|\, u + \mathcal{C} |u|^2\, u = -\nabla p.$$

Here α, β, c, $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are empirical positive constants.

General Forchheimer equations

Generalizing the above equations as follows

$$g(|u|)u = -\nabla p.$$

Let $G(s) = sg(s)$. Then $G(|u|) = |\nabla p| \Rightarrow |u| = G^{-1}(|\nabla p|)$. Hence

$$u = -\frac{\nabla p}{g(G^{-1}(|\nabla p|))} = -K(|\nabla p|)\nabla p,$$

where

$$K(\xi) = K_g(\xi) = \frac{1}{g(s)} = \frac{1}{g(G^{-1}(\xi))}, \quad sg(s) = \xi.$$

We derived a non-linear equation of Darcy type from Forchheimer equations.

Equations of Fluids

Let ρ be the density. Continuity equation

$$\frac{d\rho}{dt} + \nabla \cdot (\rho u) = 0.$$

For slightly compressible fluid:

$$\frac{d\rho}{dp} = \frac{1}{\kappa}\rho,$$

where $\kappa \gg 1$. Substituting this into the continuity equation yields

$$\frac{d\rho}{dp}\frac{dp}{dt} + \rho\nabla \cdot u + \frac{d\rho}{dp} u \cdot \nabla p = 0,$$

$$\frac{dp}{dt} + \kappa\nabla \cdot u + u \cdot \nabla p = 0.$$

Since $\kappa \gg 1$, we neglect the second term in continuity equation

$$\frac{dp}{dt} = -\kappa\nabla \cdot u\,.$$

Combining the equation of pressure and the Forchheimer equation, one gets after scaling:

$$\frac{dp}{dt} = \nabla \cdot \big(K(|\nabla p|)\,\nabla p\big)\,.$$

Consider the equation on a bounded domain U in $\mathbb{R}^n$, $n \geq 2$, with boundary Γ.

- Dirichlet condition: previous works with E.Aulisa, L.Bloshanskaya and A.Ibragimov (2009), with A.Ibragimov (2011).
- Here we focus on the flux condition on Γ:

$$u \cdot N = -K(|\nabla p|)\frac{\partial p}{\partial N} = \psi(x,t) \text{ is known },$$

  where N is the outward normal vector on Γ.

Class $FP(N, \vec{\alpha})$

We introduce a class of "Forchheimer polynomials"

Definition

A function $g(s)$ is said to be of class $FP(N, \vec{\alpha})$ if

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_N s^{\alpha_N} = \sum_{j=0}^{N} a_j s^{\alpha_j},$$

where $N > 0$, $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_N$, and $a_0, a_N > 0$, $a_1, \ldots, a_{N-1} \geq 0$.

Notation: $\alpha_N = \deg(g)$, $\vec{a} = (a_0, a_1, \ldots, a_N)$,

$$a = \frac{\alpha_N}{\alpha_N + 1} \in (0, 1), \quad b = \frac{a}{2 - a} = \frac{\alpha_N}{\alpha_N + 2} \in (0, 1).$$

Degeneracy - The Degreee Condition

Lemma

Let $g(s)$ be in class FP($N, \vec{\alpha}$). One has for any $\xi \geq 0$ that

$$\frac{C_1}{(1+\xi)^a} \leq K(\xi) \leq \frac{C_2}{(1+\xi)^a},$$

$$C_3(\xi^{2-a} - 1) \leq K(\xi, \vec{a})\xi^2 \leq C_2\xi^{2-a}.$$

Degree Condition (DC)

$$\deg(g) \leq \frac{4}{n-2} \iff 2 \leq (2-a)^* = \frac{n(2-a)}{n-(2-a)}.$$

Under the (DC), the Sobolev space $W^{1,2-a}(U)$ is continuously embedded into $L^2(U)$.

- No Degree Condition: minimal information

We assume (DC) throughout. By Poincaré-Sobolev inequality + non-linear differential inequality $\Rightarrow$ better results.

Functions H

Define:

$$H(\xi) = \int_0^{\xi^2} K(\sqrt{s})ds, \quad \xi \geq 0.$$

Relations between H and K:

$$K(|\nabla p|)|\nabla p|^2 \leq H(|\nabla p|) \leq 2K(|\nabla p|)|\nabla p|^2 \sim |\nabla p|^{2-a}.$$

Why?

$$\int_U K(\nabla p|)\nabla p \cdot \nabla p_t dx = \frac{1}{2}\frac{d}{dt}\int_U H(|\nabla p|)dx.$$

Poincaré–Sobolev inequality

Lemma

Let u belong to $W^{1,2-a}(U)$ and satisfy either $\int_U u dx = 0$ or $u\big|_\Gamma = 0$. Then

$$\int_U u^2 dx \le \varphi_{c_0,\gamma_0}\left(\int_U H(|\nabla u|)dx\right),$$

where $\gamma_0 = 2/(2-a)$, $c_0 > 0$, and

$$\varphi_{c,\gamma}(z) = cz + c^\gamma z^\gamma, \text{ for } c > 0, \ \gamma > 0, \ z \ge 0.$$

Consequently,

$$\int_U H(|\nabla u|)dx \ge \varphi^{-1}_{c_0,\gamma_0}\left(\int_U u^2 dx\right).$$

We also obtain suitable Trace Theorem.

Definitions and Notations

Let $\bar{p}(x,t) = p(x,t) - \frac{1}{|U|}\int_U p(x,t)dx$, then $\int_U \bar{p}(x,t)dx = 0$ and

$$\bar{p}(x,t) \quad = \quad p(x,t) \; - \; \frac{1}{|U|}\int_U p(x,0)dx \; + \; \frac{1}{|U|}\int_0^t \int_\Gamma \psi(x,\tau)d\sigma d\tau.$$

For a function $u(x,t)$ defined on $U \times [0,\infty)$, denote

$$J[u](t) = \int_U u^2(x,t)dx \text{ and } J_H[u](t) = \int_U H(|\nabla u(x,t)|)dx.$$

We will obtain various bounds for $J[\bar{p}],\ J[\bar{p}_t],$ and $J_H[p] = J_H[\bar{p}]$.

Main differential inequalities

Define $f(t) = \|\psi(t)\|_{L^\infty}^2 + \|\psi(t)\|_{L^\infty}^{\frac{2-a}{1-a}}$, $\tilde{f}(t) = \|\psi_t(t)\|_{L^\infty}^2 + \|\psi_t(t)\|_{L^\infty}^{\frac{2-a}{1-a}}$.

Lemma

We have for all $t > 0$ that

$$\frac{d}{dt} J[\bar{p}](t) \le -J_H[p](t) + Cf(t),$$

$$\frac{d}{dt} J_H[p](t) \le -2J[\bar{p}_t](t) + \varepsilon \int_U K(|\nabla p|)|\nabla \bar{p}_t|^2 dx + \varepsilon J_H[p](t) + C_\varepsilon f(t),$$

$$\frac{d}{dt} J_H[p](t) \le -2J[\bar{p}_t](t) + \varepsilon J_H[p](t) + 2\frac{d}{dt}\left(\int_\Gamma \psi \bar{p} d\sigma\right) + C_\varepsilon \tilde{f}(t),$$

$$\frac{d}{dt} J[\bar{p}_t](t) \le -(1-a)\int_U K(|\nabla p|)|\nabla \bar{p}_t|^2 dx + \varepsilon J_H[p](t) + C_\varepsilon \tilde{f}(t),$$

where $\varepsilon > 0$ is arbitrary and $C_\varepsilon > 0$ depends on ε.

Non-linear differential inequality

$$\frac{d}{dt}J[\bar{p}](t) \le -\varphi^{-1}_{C_1,\frac{2}{2-a}}(J[\bar{p}](t)) + C_2 f(t),$$

where $C_1, C_2 > 0$, and

$$\varphi_{c,\gamma}(z) = cz + c^\gamma z^\gamma, \text{ for } c > 0,\ \gamma > 0,\ z \ge 0.$$

In general, consider

$$y'(t) \le -\phi^{-1}(y(t)) + f(t),$$

- $f(t)$ is a non-negative continuous function on $[0,\infty)$,
- $\phi(z)$ is a continuous, strictly increasing function from $[0,\infty)$ onto $[0,\infty)$. Note that $\phi(0) = 0$ and $\phi(\infty) = \infty$.

Let $y(t)$ be a non-negative solution that belongs to $C([0,\infty))$ and $C^1((0,\infty))$.

Lemma

For all $t \geq 0$

$$y(t) \leq \max\{y(0), \phi(F(t))\} \leq y(0) + \phi(F(t)),$$

where $F(t)$ *is a continuous, increasing majorant of* $f(t)$ *on* $[0, \infty)$*.*
If $A \stackrel{\text{def}}{=\!=} \limsup_{t\to\infty} f(t) < \infty$ *then*

$$\limsup_{t\to\infty} y(t) \leq \phi(A).$$

Proposition

Let $\phi = cz + c^\gamma z^\gamma$, for $z \geq 0$, where $c > 0$, $\gamma \in (1, 2]$.
(i) Then

$$y(t) \leq y(0) + C_\gamma F(t) \text{ for all } t \geq 0,$$

where $C_\gamma = c + \max\{2, c^\gamma\}$.
(ii) Let $A = \limsup_{t\to\infty} f(t)$ and $\beta = \limsup_{t\to\infty}[f'(t)]^-$. Then

$$\limsup_{t\to\infty} y(t) \leq C_\gamma A,$$

and there is $T > 0$ such that for $t \geq T$,

$$y(t) \leq \max\{3, c, c^2\}\big(1 + \beta + f(t)\big).$$

Note on the proof. Combination of above non-linear estimate and the following linear estimate: $\phi^{-1}(y) \geq c^{-1}(y - 2)$, therefore $y(t)$ satisfies

$$y' \leq -c^{-1}y + 2c^{-1} + f(t), \quad t > 0.$$

Estimates of solutions - I

Set $A = \limsup_{t\to\infty} f(t)$ and $\beta = \limsup_{t\to\infty} [f'(t)]^-$.

Theorem

$$J[\bar{p}](t) \le J[\bar{p}](0) + CM_f(t) \textit{ for all } t \ge 0.$$

If $A < \infty$ *then*

$$\limsup_{t\to\infty} J[\bar{p}](t) \le CA.$$

If $\beta < \infty$ *then there is* $T > 0$ *such that*

$$J[\bar{p}](t) \le C\big(1 + \beta + f(t)\big) \textit{ for all } t > T.$$

Estimates of solutions - II

Theorem

$$J_H[p](t) \le CJ[\bar{p}](0) + CM_f(t) + C\int_{t-1}^{t} \tilde{f}(\tau)d\tau \text{ for all } t \ge 1.$$

If $A < \infty$ then

$$\limsup_{t\to\infty} J_H[p](t) \le C\Big(A + \limsup_{t\to\infty} \int_{t-1}^{t} \tilde{f}(\tau)d\tau\Big).$$

If $\beta < \infty$ then there is $T > 0$ such that

$$J_H[p](t) \le C\Big(1 + \beta + f(t-1) + f(t) + \int_{t-1}^{t} f(\tau) + \tilde{f}(\tau)d\tau\Big) \text{ for all } t > T.$$

Estimates of solutions - III

Note: May happen $\limsup_{t\to 0+} \int_U \bar{p}_t^2(x,t)dx = \infty$.

Proposition

$$J[\bar{p}_t](t) \le CJ[\bar{p}](0) + CM_f(t) + C\int_{t-1}^{t} \tilde{f}(\tau)d\tau, \quad t \ge 1.$$

If $A < \infty$ then

$$\limsup_{t\to\infty} J[\bar{p}_t](t) \le C\Big(A + \limsup_{t\to\infty} \int_{t-1}^{t} \tilde{f}(\tau)d\tau\Big).$$

If $\beta < \infty$, then there is $T > 0$ such that for all $t > T$,

$$J[\bar{p}_t](t) \le C\Big(1 + \beta + f(t-1) + f(t) + \int_{t-1}^{t} f(\tau) + \tilde{f}(\tau)d\tau\Big).$$

Dependence on the boundary data

Let $p_1(x,t)$ and $p_2(x,t)$ be two solutions having fluxes ψ_1 and ψ_2, and initial data $p_1(x,0)$ and $p_2(x,0)$, respectively.
Let $\Psi = \psi_1 - \psi_2$, $P = p_1 - p_2$, and $\bar{P} = P - |U|^{-1}\int_U P dx$. Hence

$$\bar{P}(x,t) = P(x,t) - \frac{1}{|U|}\int_U P(x,0)dx + \frac{1}{|U|}\int_0^t \int_\Gamma \Psi(x,\tau) d\sigma d\tau.$$

We have

$$\frac{1}{2}\frac{d}{dt}\int_U \overline{P}^2 dx = -\int_U (K(|\nabla p_1|)\nabla p_1 - K(|\nabla p_2|)\nabla p_2)\cdot(\nabla p_1 - \nabla p_2)dx - \int_\Gamma \Psi \overline{P} d\sigma.$$

The degenerate monotonicity

Proposition

(i) For any $y, y' \in \mathbb{R}^n$, one has

$$\big(K(|y|)y - K(|y'|)y'\big) \cdot \big(y - y'\big) \geq aK(|y| \vee |y'|)|y - y'|^2.$$

(ii) For any functions p_1 and p_2 one has

$$\int_U \big(K(|\nabla p_1|, \vec{a})\nabla p_1 - K(|\nabla p_2|)\nabla p_2\big) \cdot \big(\nabla p_1 - \nabla p_2\big)dx$$

$$\geq C_5 \left(\int_U |\nabla p_1 - \nabla p_2|^{2-a} dx\right)^{\frac{2}{2-a}} \big(1 + \|\nabla p_1\|_{L^{2-a}(U)} \vee \|\nabla p_2\|_{L^{2-a}(U)}\big)^{-a},$$

where $C_5 = C_5(N, \deg(g), \vec{a})$.

Differential inequality for the difference

$$\frac{d}{dt}\int_U \bar{P}^2 dx \le -C\Lambda(t)^{-b}\int_U \bar{P}^2 dx + C\|\Psi\|_{L^\infty}^2 \Lambda(t)^b.$$

where $\Lambda(t) = 1 + J_H[\bar{p}_1](t) + J_H[\bar{p}_2](t)$.
Notation. We define for $i = 1, 2$,

$$f_i(t) = \|\psi_i(t)\|_{L^\infty}^2 + \|\psi_i(t)\|_{L^\infty}^{\frac{2-a}{1-a}}, \ \tilde{f}_i(t) = \|\psi_{it}(t)\|_{L^\infty}^2 + \|\psi_{it}(t)\|_{L^\infty}^{\frac{2-a}{1-a}}.$$

For $i = 1, 2$, we assume $f_i(t), \tilde{f}_i(t) \in C([0,\infty))$ and when needed $f_i(t) \in C^1((0,\infty))$; let

$$A_i = \limsup_{t\to\infty} f_i(t) \text{ and } \beta_i = \limsup_{t\to\infty} [f_i'(t)]^-.$$

Set $\bar{A} = A_1 + A_2$, $\bar{\beta} = \beta_1 + \beta_2$.
Let $F(t) = f_1(t) + f_2(t)$, $M_F(t) = M_{f_1}(t) + M_{f_2}(t)$, where $M_{f_i}(t)$, $i = 1, 2$, is a continuous increasing majorant of $f_i(t)$ on $[0, \infty)$.
Let $\tilde{F}(t) = \tilde{f}_1(t) + \tilde{f}_2(t)$.

Continuous dependence for pressure

Let $W(t) = W_1(t) = 1 + M_F(t) + \int_{t-1}^{t} \tilde{F}(\tau)d\tau$ in general case, or $W(t) = W_2(t) = 1 + F(t) + F(t-1) + \int_{t-1}^{t} \tilde{F}(\tau)d\tau$ in case $\bar{\beta} < \infty$,

Theorem

(i) If $\bar{A} < \infty$ and $\int_1^\infty \left(1 + \int_{\tau-1}^{\tau} \tilde{F}(s)ds\right)^{-b} d\tau = \infty$ then

$$\limsup_{t\to\infty} J[\bar{P}](t) \le C \limsup_{t\to\infty} \left\{ \|\Psi(t)\|_{L^\infty}^2 \left(1 + \bar{A} + \int_{t-1}^{t} \tilde{F}(\tau)d\tau\right)^{2b} \right\}.$$

(ii) If $\bar{A} = \infty$ and $\int_1^\infty W^{-b}(t)dt = \infty$ then

$$\limsup_{t\to\infty} J[\bar{P}](t) \le C \limsup_{t\to\infty} \left\{ \|\Psi(t)\|_{L^\infty}^2 W^{2b}(t) \right\}.$$

Continuous dependence for pressure gradient

Theorem

(i) If $\bar{A} < \infty$ *and* $\int_{t-1}^{t} \tilde{F}(\tau)d\tau$ *is uniformly bounded on* $[1,\infty)$*, then*

$$\limsup_{t\to\infty} \left(\int_U |\nabla P(x,t)|^{2-a} dx \right)^{\frac{2}{2-a}} \le C M_4^{2b+1/2} \limsup_{t\to\infty} \|\Psi(t)\|_{L^\infty} + C M_4^{2b} \limsup_{t\to\infty} \|\Psi(t)\|_{L^\infty}^2,$$

where $M_4 = 1 + \bar{A} + \limsup_{t\to\infty} \int_{t-1}^{t} \tilde{F}(\tau)d\tau$*.*
(ii) If $\bar{A} = \infty$ *and* $\lim_{t\to\infty} W'(t)W^{b-1}(t) = 0$ *and* $\int_1^\infty W^{-b}(\tau)d\tau = \infty$*, then*

$$\limsup_{t\to\infty} \left(\int_U |\nabla P(x,t)|^{2-a} dx \right)^{\frac{2}{2-a}} \le C \limsup_{t\to\infty} \left(\|\Psi(t)\|_{L^\infty} W^{2b+1/2}(t) \right) + C \limsup_{t\to\infty} \left(\|\Psi(t)\|_{L^\infty}^2 W^{2b}(t) \right).$$

Dependence on the Forchheimer polynomials

Let $N > 0$ and the exponent vector $\vec{\alpha} = (0, \alpha_1, \ldots, \alpha_N)$. Denote

$$R(N) = \{\vec{a} = (a_0, a_1, \ldots, a_N) : a_0, a_N > 0, a_1, \ldots, a_{N-1} \geq 0\}.$$

Let D be a compact set in $R(N)$. For $\vec{a} \in R(N)$, we define

$$\chi(\vec{a}) = \max\left\{a_0, a_1, \ldots, a_N, \frac{1}{a_0}, \frac{1}{a_N}\right\} \in [1, \infty).$$

and $\hat{\chi}(D) = \max\{\chi(\vec{a}) : \vec{a} \in D\}$.
Then for $\vec{a} \in D$, all constants appearing in estimates in the previous sections can be made independent of $\vec{a}$, and depend on U, $\hat{\chi}(D)$, and $\vec{\alpha}$ only. We denote them by C_* here aferwards.

Equation for the difference of solutions

Let $g_1(s) = g(s, \vec{a}^{(1)})$ and $g_2(s) = g(s, \vec{a}^{(2)})$ be two functions of class $\mathrm{FP}(N, \vec{\alpha})$, where $\vec{a}^{(1)}$ and $\vec{a}^{(2)}$ belong to D. Let $p_k = p_k(x, t; \vec{a}^{(k)})$ for $k = 1, 2$ be two solutions with the same given boundary flux $\psi(x, t)$. Let $P = p_1 - p_2$, $\bar{P} = \bar{p}_1 - \bar{p}_2$.

$$\frac{1}{2}\frac{d}{dt}\int_U \bar{P}^2 dx = -\int_U (K(|\nabla \bar{p}_1|, \vec{a}^{(1)})\nabla \bar{p}_1 - K(|\nabla \bar{p}_2|, \vec{a}^{(2)})\nabla \bar{p}_1)\cdot(\nabla \bar{p}_1 - \nabla \bar{p}_2)dx$$

Perturbed Monotonicity

Let $\vec{a}$ and $\vec{a}'$ be two arbitrary vectors. We define max and min vectors $\vec{a} \vee \vec{a}'$ and, resp., $\vec{a} \wedge \vec{a}'$ by

$$(\vec{a} \vee \vec{a}')_j = \max\{a_j, a_j'\} \quad \text{and} \quad (\vec{a} \wedge \vec{a}')_j = \min\{a_j, a_j'\}.$$

Define $\chi(\vec{a}, \vec{a}') = \max\{\chi(\vec{a}), \chi(\vec{a}')\}$.

Lemma

Let $\vec{a}, \vec{a}' \in R(N) = \{(a_0, a_1, \ldots, a_N) : a_0, a_N > 0, a_1, \ldots, a_{N-1} \geq 0\}$. Then for any y, y' in $\mathbb{R}^n$, one has

$$(K(|y|, \vec{a})y - K(|y'|, \vec{a}')y') \cdot (y - y') \geq (1 - a)K(|y| \vee |y'|, \vec{a} \vee \vec{a}')|y - y'|^2 - N\chi(\vec{a}, \vec{a}')|\vec{a} - \vec{a}'|K(|y| \vee |y'|, \vec{a} \wedge \vec{a}')(|y| \vee |y'|)|y - y'|.$$

Main relations

$$\frac{1}{2}\frac{d}{dt}\int_U \bar{P}^2 dx \le -C_*\Big(\int_U |\nabla \bar{P}(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}}\Lambda^{-b}(t) + C_*|\vec{a}^{(1)} - \vec{a}^{(2)}|\Lambda(t).$$

By Poincaré's inequality:

$$\frac{1}{2}\frac{d}{dt}\int_U \bar{P}^2 dx \le -C_*\Lambda^{-b}(t)\int_U \bar{P}^2 dx + C_*|\vec{a}^{(1)} - \vec{a}^{(2)}|\Lambda(t).$$

$$\Big(\int_U |\nabla P(x,t)|^{2-a}dx\Big)^{\frac{2}{2-a}} \le C_*\Lambda(t)^b\big(J[\bar{P}_t](t)\big)^{1/2}\big(J[\bar{P}](t)\big)^{1/2} + C_*\Lambda(t)^{b+1}|\vec{a}^{(1)} - \vec{a}^{(2)}|.$$

Limsup estimates

Theorem

If $f(t)$ $(t \geq 0)$ and $\int_{t-1}^{t} \tilde{f}(\tau)d\tau$ $(t \geq 1)$ are uniformly bounded, then

$$\limsup_{t\to\infty} J[\bar{P}](t) \leq C_* M_8^{b+1} |\vec{a}^{(1)} - \vec{a}^{(2)}|,$$

$$\limsup_{t\to\infty} \Big(\int_U |\nabla \bar{P}(x,t)|^{2-a} dx \Big)^{\frac{2}{2-a}} \leq C_* M_8^{3b/2+1} |\vec{a}^{(1)} - \vec{a}^{(2)}|^{1/2} + C_* M_8^{b+1} |\vec{a}^{(1)} - \vec{a}^{(2)}|,$$

where $M_8 = 1 + A + \limsup_{t\to\infty} \int_{t-1}^{t} \tilde{f}(\tau)d\tau$.

THANK YOU FOR YOUR ATTENTION!