A FREE BOUNDARY PROBLEM FOR TWO-DIMENSIONAL GAS DYNAMICS EQUATIONS

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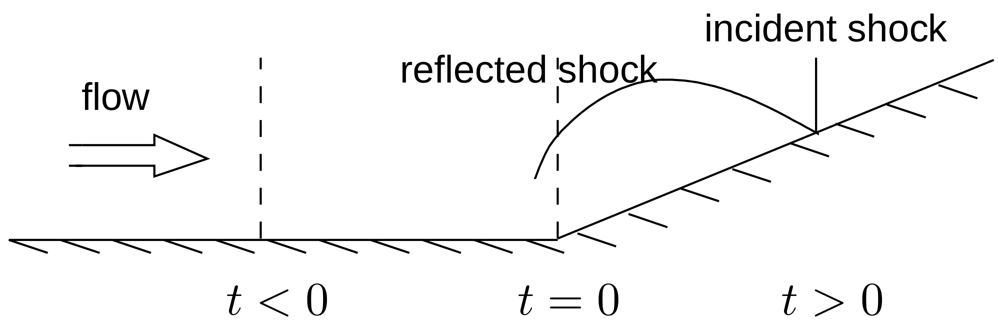
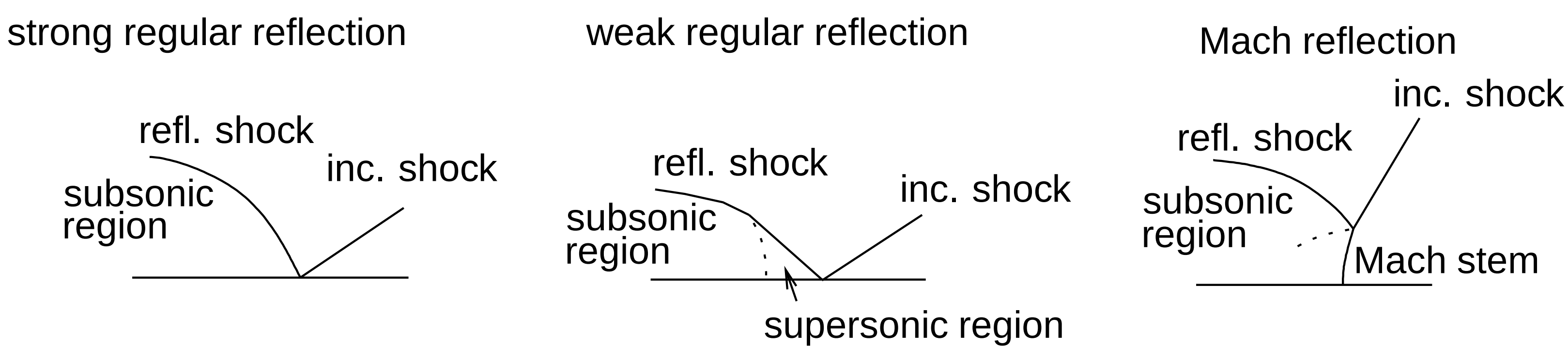
INTRODUCTION

- Reflection of a shock by a wedge
  - two-dimensional Riemann problems for systems of conservation laws
    - ∗ steady / unsteady transonic small disturbance equations
    - ∗ nonlinear wave system
    - ∗ gas dynamics equations (isentropic, adiabatic)
  - shock reflection
    - ∗ regular reflection (strong, weak)
    - ∗ Mach reflection
    - ∗ Guderley reflection
  - self-similar coordinates $\Rightarrow \begin{cases} \text{mixed type (hyperbolic, elliptic) system} \\ \text{free boundary value problem} \\ \text{nonlinear problem} \end{cases}$
  - theory of second order elliptic equations with mixed boundary conditions (Dirichlet and oblique) and fixed point theory
- Result
  - Riemann problem for the isentropic gas dynamics equations
  - strong regular reflection
  - existence of a local solution in weighted Holder spaces

SHOCK REFLECTION

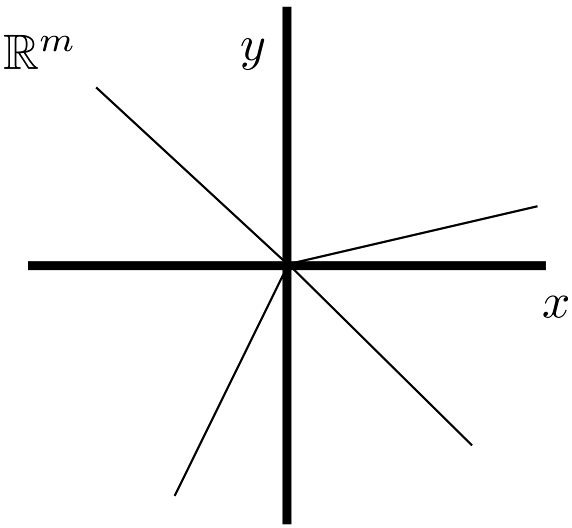


- motivating problem: airflow past object
- types of reflection



TWO-DIMENSIONAL RIEMANN PROBLEMS

$U_t + F(U)_x + G(U)_y = 0,\ U : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}^m$



similarity variables $\xi = x/t$ and $\eta = y/t$

reduced system:

$$\partial_\xi (F - \xi U) + \partial_\eta (G - \eta U) = -2U$$

initial data becomes boundary data at infinity

- the reduced system changes type:
  hyperbolic in the far field, mixed type near the origin
- resolve one-dimensional discontinuities in the far field
- formulate a free boundary problem for the reflected shock and the subsonic state
  - $*$ theory of second order elliptic equations with mixed boundary conditions (Gilbarg, Trudinger, Lieberman)
  - $*$ fixed point theorems

- adiabatic gas dynamics equations $E = \frac{1}{\gamma-1}\frac{p}{\rho} + \frac{u^2+v^2}{2}$

$$\begin{array}{ccccccc} \rho_t & + & (\rho u)_x & + & (\rho v)_y & = & 0 \\ (\rho u)_t & + & (\rho u^2 + p)_x & + & (\rho uv)_y & = & 0 \\ (\rho v)_t & + & (\rho uv)_x & + & (\rho v^2 + p)_y & = & 0 \\ (\rho E)_t & + & (u(\rho E + p))_x & + & (v(\rho E + p))_y & = & 0 \end{array}$$

- isentropic gas dynamics equations $p = p(\rho)$

$$\begin{array}{ccccccc} \rho_t & + & (\rho u)_x & + & (\rho v)_y & = & 0 \\ (\rho u)_t & + & (\rho u^2 + p)_x & + & (\rho uv)_y & = & 0 \\ (\rho v)_t & + & (\rho uv)_x & + & (\rho v^2 + p)_y & = & 0 \end{array}$$

- nonlinear wave system $p = p(\rho)$

$$\begin{array}{ccccccc} \rho_t & + & (\rho u)_x & + & (\rho v)_y & = & 0 \\ (\rho u)_t & + & p_x & & & = & 0 \\ (\rho v)_t & & & + & p_y & = & 0 \end{array}$$

- unsteady transonic small disturbance equation

$$u_t + uu_x + v_y = 0, \quad -v_x + u_y = 0$$

- Euler equations for potential flow, pressure gradient system

ISENTROPIC GAS DYNAMICS EQUATIONS

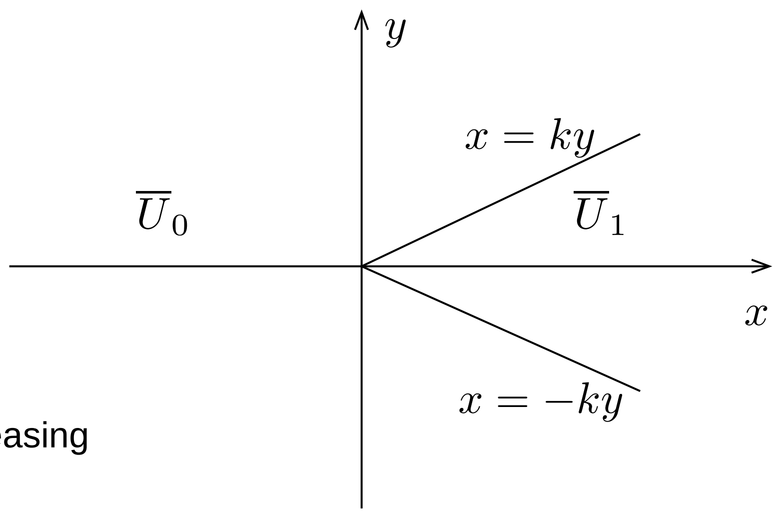
$$\rho_t + (\rho u)_x + (\rho v)_y = 0$$

$$(\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y = 0$$

$$(\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y = 0$$

$p = p(\rho),\ c^2(\rho) := p'(\rho) > 0$ and increasing

$\overline{U} = (\rho, u, v)$



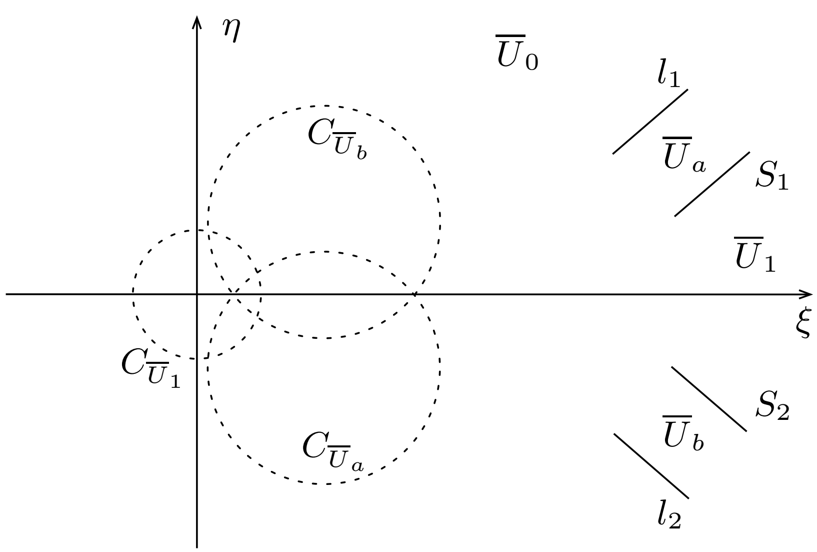
Initial data: $\overline{U}_0 = (\rho_0, u_0, 0), \overline{U}_1 = (\rho_1, 0, 0)$

- fix $\rho_0 > \rho_1 > 0$
- fix $k \in (k_C(\rho_0, \rho_1), \infty)$
- set $u_0 = \sqrt{1+k^2}\sqrt{\frac{(\rho_0-\rho_1)(p(\rho_0)-p(\rho_1))}{\rho_0\rho_1}}$

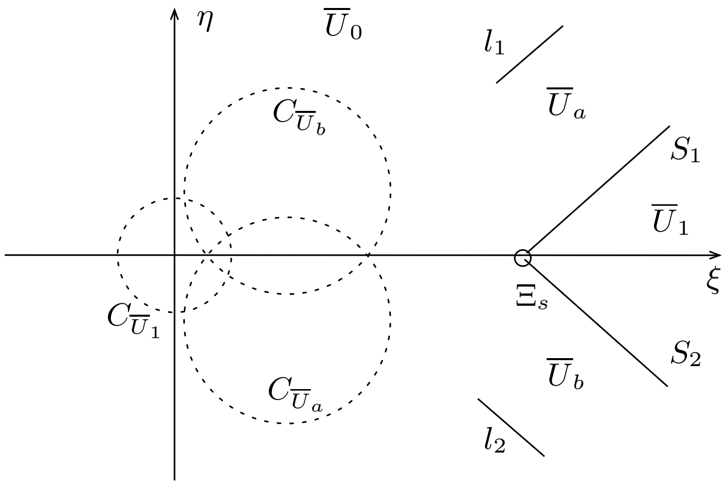
- Formulate the problem in self-similar coordinates: $\xi = x/t\,,\ \eta = y/t$

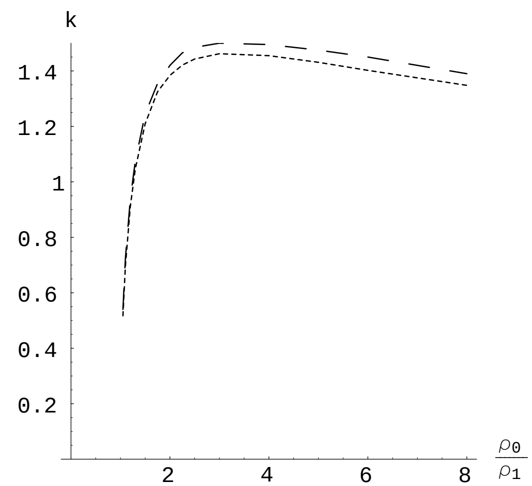
$$(\rho U)_\xi + (\rho V)_\eta + 2\rho = 0$$
$$(U,V)\cdot\nabla U + U + p_\xi/\rho = 0$$
$$(U,V)\cdot\nabla V + V + p_\eta/\rho = 0$$

- here, $U := u - \xi$ and $V := v - \eta$
- when linearized about a constant state $\overline{U}_* = (\rho_*, u_*, v_*)$, the system is hyperbolic outside the circle $C_{\overline{U}_*}$: $(u_* - \xi)^2 + (v_* - \eta)^2 = c^2(\rho_*)$



- Solution in the hyperbolic part of the domain





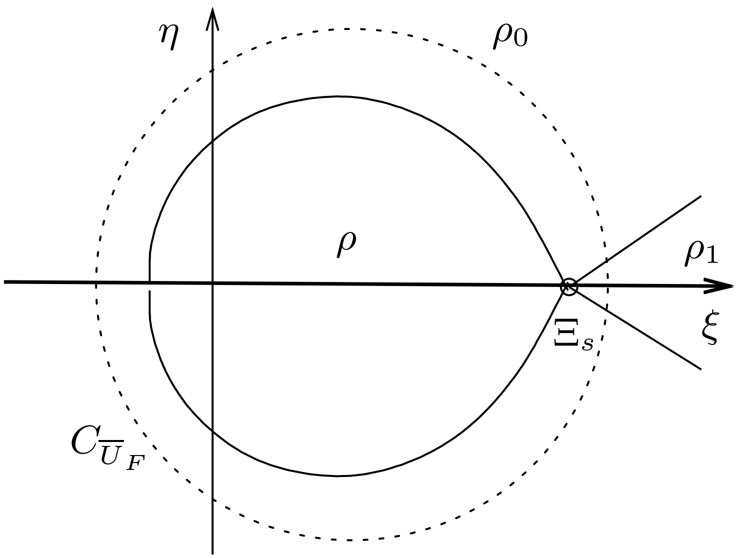
If $k \in (k_C(\rho_0, \rho_1), \infty)$, then:

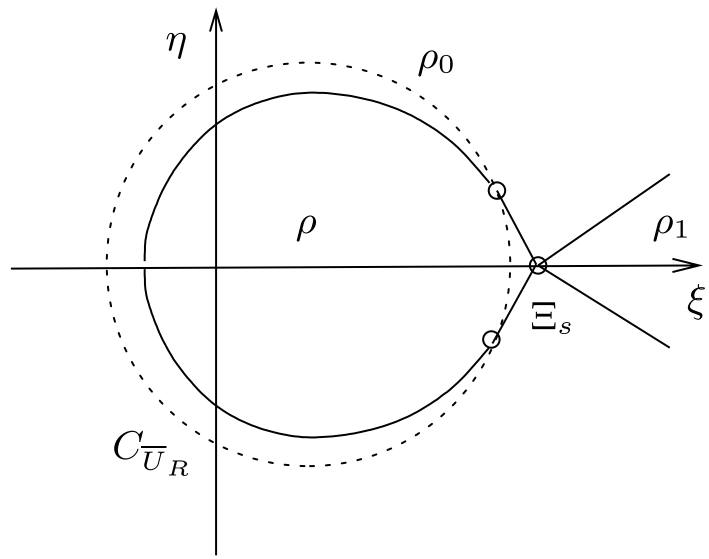
1) Ξ_s is outside the sonic circles $C_{\overline{U}_1}, C_{\overline{U}_a}, C_{\overline{U}_b}$

2) quasi-1-d Riemann problem at Ξ_s has a solution (shock – shock)

- two solutions: $\overline{U}_F = (\rho_F, u_F, 0)$ and $\overline{U}_R = (\rho_R, u_R, 0)$
  - Ξ_s is subsonic w.r.t. $\overline{U}_F$ when $k \in (k_C, \infty)$
  - Ξ_s is subsonic w.r.t. $\overline{U}_R$ when $k \in (k_C, k_*)$
  - Ξ_s is supersonic w.r.t. $\overline{U}_R$ when $k \in (k_*, \infty)$

- Two types of regular reflection

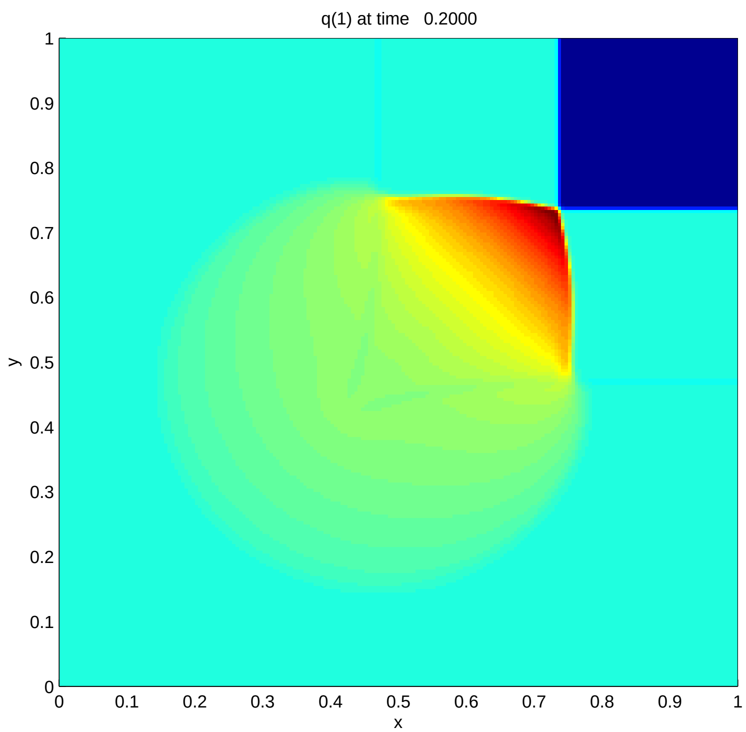


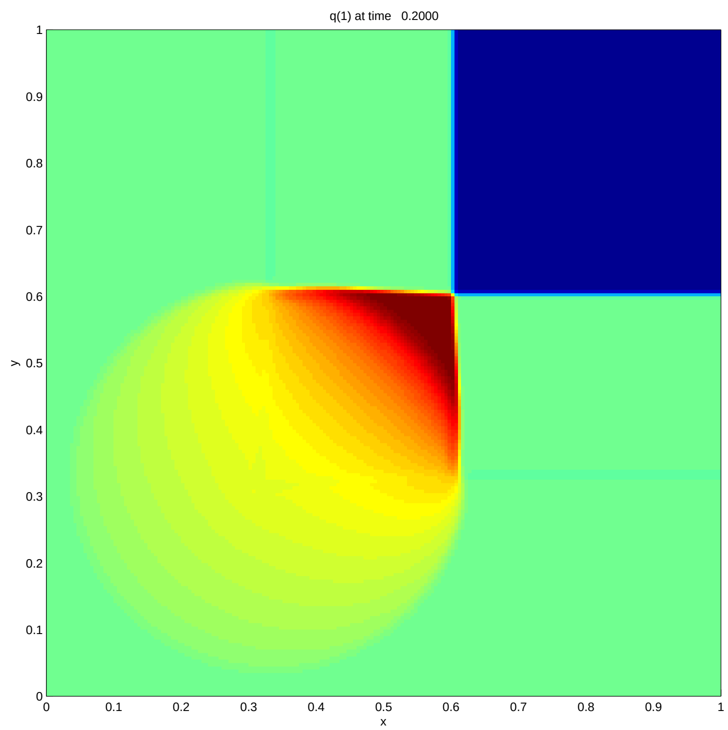


Case 1: Ξ_s is subsonic w.r.t. the solution at $\Xi_s \Rightarrow$ strong regular reflection

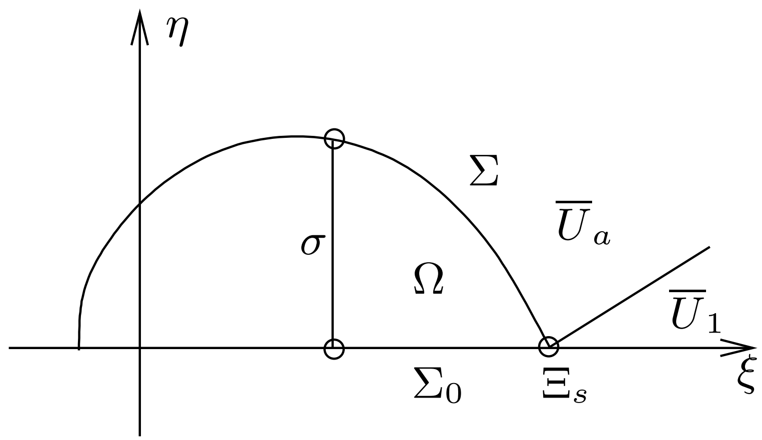
Case 2: Ξ_s is supersonic w.r.t. the solution at $\Xi_s \Rightarrow$ weak regular reflection

- Numerical simulations for the adiabatic gas dynamics equations
  - using CLAWPACK software
  - transonic (strong) and supersonic (weak) regular reflection





- Free boundary problem describing transonic regular reflection



reduced system in Ω

jump conditions along Σ

symmetry conditions along Σ_0

Dirichlet condition for ρ on σ

Dirichlet conditions at Ξ_s

Theorem: There exists a local solution $\rho \in H^{(-\gamma)}_{1+\alpha}$, $U, V \in H^{(-\gamma)}_{1+\epsilon}$ and $\Sigma \in H_{1+\alpha}$ of the above free boundary problem. $(0 < \epsilon < \alpha < \gamma < 1)$

Main idea.

reduced system in $\Omega \Leftrightarrow \begin{cases} \text{2nd order equation for } \rho \text{ in } \Omega \\ \text{two 1st order "transport" equations for } U \text{ and } V \text{ in } \Omega \end{cases}$

jump conditions on $\Sigma \Leftrightarrow \begin{cases} \text{oblique derivative boundary condition for } \rho \text{ along } \Sigma \\ \text{Dirichlet conditions for } U \text{ and } V \text{ along } \Sigma \\ \text{shock evolution equation } d\eta/d\xi = \Psi(\rho, \xi, \eta) \end{cases}$

- Holder spaces

  Let $S \subseteq \mathbb{R}^2$, $u : S \to \mathbb{R}$. Define

$$|u|_{0;S} := \sup |u(x)| \qquad \text{supermum norm}$$

$$[u]_{\alpha;S} := \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \qquad \alpha\text{-Holder seminorm}$$

$$|u|_{\alpha;S} := |u|_{0;S} + [u]_{\alpha;S} \qquad \alpha\text{-Holder norm}$$

$$|u|_{k+\alpha;S} := \sum_{j=0}^{k} |D^j u|_{0;S} + [D^k u]_{\alpha;S} \qquad (k+\alpha)\text{-Holder norm}$$

Let $T \subseteq \partial S$ and $\delta > 0$. Let

$$S_{\delta;T} := \{x \in S : \text{dist}(x, T) > \delta\},$$

and for $a > 0$ and b such that $a - b \geq 0$, define

$$u^{(-b)}_{a;\overline{S}\setminus T} := \sup_{\delta > 0} \delta^{a-b} |u|_{a;S_{\delta;T}} \qquad \text{weighted interior Holder norm}$$

Theorem: There exists a local solution $\rho \in H_{1+\alpha}^{(-\gamma)}$, $U, V \in H_{1+\epsilon}^{(-\gamma)}$ and $\Sigma \in H_{1+\alpha}$ of the free boundary problem

$$\left.\begin{array}{l} a_{ij}(\rho,U,V)D^{ij}\rho + b_i(\rho,U,V)D^i\rho + c_{ij}(\rho,U,V)D^i\rho D^j\rho = 0 \\ (U,V)\cdot\nabla U + U + p_\xi/\rho = 0 \\ (U,V)\cdot\nabla V + V + p_\eta/\rho = 0 \end{array}\right\} \text{ in } \Omega$$

$$\left.\begin{array}{l} \beta(\rho,U,V)\cdot\nabla\rho = F(\rho,U,V) \\ U = G(\rho,\xi,\eta) \\ V = H(\rho,\xi,\eta) \\ \frac{d\eta}{d\xi} = \Psi(\rho,\xi,\eta) \end{array}\right\} \text{ on } \Sigma : \eta = \eta(\xi)$$

$$\rho_\eta = U_\eta = V = 0 \text{ on } \Sigma_0$$

$$\rho = f \text{ on } \sigma$$

$$\overline{U}(\Xi_s) = \overline{U}_F, \qquad \eta(\xi_s) = 0$$

Remark. We introduce cut-off functions to ensure that:

· the 2nd order equation for ρ is uniformly elliptic in Ω

· the condition for ρ on Σ is uniformly oblique

· the shock evolution equation is well-defined

Step 1. given $\Sigma_m \in \mathcal{K} \subset H_{1+\alpha}$, find ρ_m, U_m, V_m solving the fixed BP in Ω_m

$$\left.\begin{array}{c} a_{ij}(\rho,U,V)D^{ij}\rho + b_i(\rho,U,V)D^i\rho + c_{ij}(\rho,U,V)D^i\rho D^j\rho = 0 \\ (U,V)\cdot\nabla U + U + p_\xi/\rho = 0, \qquad (U,V)\cdot\nabla V + V + p_\eta/\rho = 0 \end{array}\right\} \text{ in } \Omega$$

$$\beta(\rho,U,V)\cdot\nabla\rho = F(\rho,U,V) \text{ on } \Sigma, \quad \rho_\eta = 0 \text{ on } \Sigma_0, \quad \rho|_\sigma = f, \quad \rho|_{\Xi_s} = \rho_s$$

$$U|_\Sigma = G(\rho), \qquad U|_{\Xi_s} = u_F - \xi_s, \qquad U_\eta = 0 \text{ on } \Sigma_0$$

$$V|_\Sigma = H(\rho), \qquad V|_{\Xi_s} = 0, \qquad V = 0 \text{ on } \Sigma_0$$

- fix $\omega, W, Z \in H_{1+\epsilon}^{(-\gamma)}$
- linearize the second order problem for density using ω, W, Z in the coefficients
- show that there exists a solution $\rho \in H_{1+\alpha}^{(-\gamma)}$ of this linear problem
- show that the map $\omega \mapsto \rho$ has a fixed point $\rho[W, Z]$
- linearize the first order problem for pseudo-velocities using $W, Z, \rho[W, Z]$
- show that there exists a solution $U, V \in H_{1+\epsilon}^{(-\gamma)}$ of this linear problem
- show that the map $(W, Z) \mapsto (U, V)$ is a contraction

Step 2. find Σ_{m+1} using the shock evolution equation $d\eta/d\xi = \Psi(\rho_m, \xi, \eta)$

- show that the map $\Sigma_m \mapsto \Sigma_{m+1}$ has a fixed point Σ

RELATED WORK

- steady transonic small disturbance equation
  - shock perturbation: Čanić-Keyfitz-Lieberman
- unsteady transonic small disturbance equation
  - strong regular reflection: Čanić-Keyfitz-Kim
  - weak regular reflection: Čanić-Keyfitz-Kim
  - Guderley reflection: Tesdall-Hunter
- nonlinear wave system
  - Mach reflection: Čanić-Keyfitz-Kim, Sever
  - strong regular reflection: Jegdić-Keyfitz-Čanić
  - weak regular reflection: Jegdić
  - Guderley reflection: Tesdall-Sanders-Keyfitz
- pressure-gradient system
  - weak regular reflection: Y.Zheng-D.Wang
- Euler equations for potential flow
  - weak regular reflection: G.Q.Chen-M.Feldman
- adiabatic/isentropic gas dynamics equations
  - T.Chang-G.Q.Chen, S.X.Chen et al., T.Zhang-Y.Zheng, T.P.Liu-V.Elling, A.Tesdall-R.Sanders

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