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One Solution of the Duffing Equation by Using the Finite Element Method

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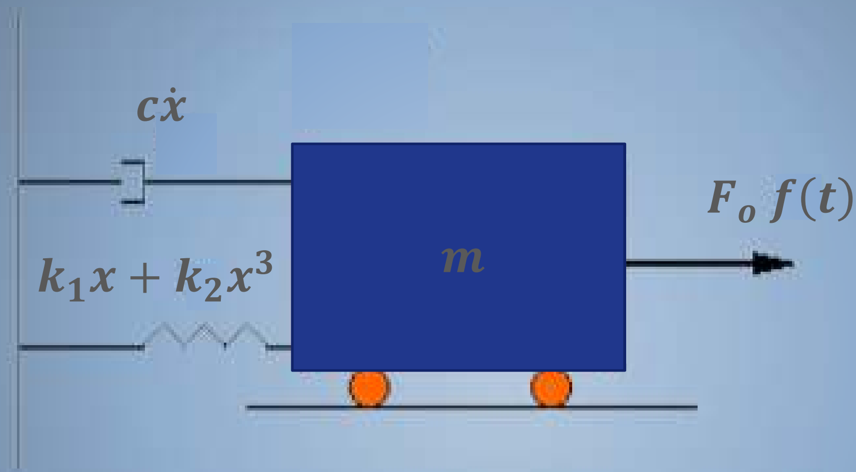
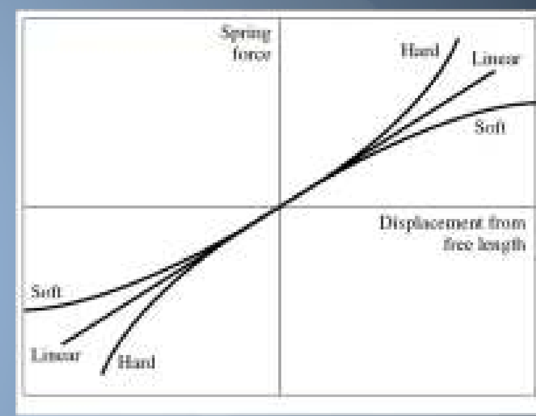
Duffing Equation

Mechanical Vibrations





$$\ddot{x} + \delta\,\dot{x} + \gamma\,x + \beta\,x^3 = a_o f(t)$$

$$\delta = \frac{c}{m}\ ,\ \gamma = \frac{k_1}{m},\ \beta = \frac{k_2}{m}\ ,\ a_o = \frac{F_o}{m}$$

m mass of the body
c damping coefficient
k_1 spring coefficient
$k_2 > 0$ hard spring
$k_2 < 0$ soft spring

Duffing Equation

We propose an approximate solution

$$\tilde{x} = \sum_{j=1}^{n} a_j \; \varphi_j(t)$$

a_j unknown parameters

φ_j known functions

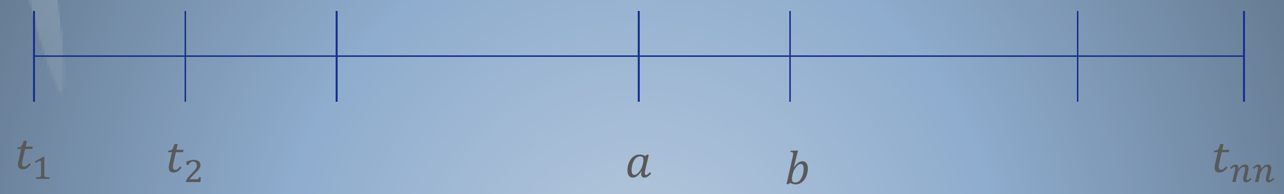
n number of terms

$$\boldsymbol{R} = \ddot{\tilde{\boldsymbol{x}}} + \boldsymbol{\delta}\,\dot{\tilde{\boldsymbol{x}}} + \boldsymbol{\gamma}\,\tilde{\boldsymbol{x}} + \boldsymbol{\beta}\,\tilde{\boldsymbol{x}}^{3} - \boldsymbol{a_o f(t)}$$

$\boldsymbol{R}$ Residual function

Duffing Equation

We divide the domain of the time in finite elements



For each element we can propose the following:

The average weighted of the residual function will be equal to zero

$$\int_a^b R(t)\, \varphi_i(t)\, dt = 0$$

$$i = 1, 2, \ldots, n$$

Duffing Equation

We obtain an integral equation

$$\int_a^b \ddot{\tilde{x}}\ \varphi_i(t)\ dt + \int_a^b \delta\ \dot{\tilde{x}}\ \varphi_i(t)\ dt + \int_a^b \gamma\ \tilde{x}\ \varphi_i(t)\ dt\ +$$

$$\int_a^b \beta\ \tilde{x}^3\ \varphi_i(t)\ dt - \int_a^b a_o\ f(t)\ \varphi_i(t)\ dt = 0$$

Now we perform a parts integration with the first term of the integral equation

$$-\int_a^b \dot{\tilde{x}}\ \dot{\varphi}_i(t)\ dt + \int_a^b \delta\ \dot{\tilde{x}}\ \varphi_i(t)\ dt + \int_a^b \gamma\ \tilde{x}\ \varphi_i(t)\ dt\ +$$

$$\int_a^b \beta\ \tilde{x}^3\ \varphi_i(t)\ dt = \int_a^b a_o\ f(t)\ \varphi_i(t)\ dt - \dot{\tilde{x}}\varphi_i\Big|_a^b$$

Duffing Equation

The approximate solution

$$\tilde{x} = \sum_{j=1}^{n} a_j \, \varphi_j \qquad \dot{\tilde{x}} = \sum_{j=1}^{n} a_j \, \dot{\varphi}_j$$

We obtain the following elemental equation

$$\sum_{j=1}^{n} R_{ij} \, a_j = F_i$$

Duffing Equation

The terms

$$R_{ij} = -M_{ij} + C_{ij} + K_{ij} + D_{ij}$$

$$M_{ij} = \int_a^b \dot{\varphi}_i \;\, \dot{\varphi}_j \, dt \qquad C_{ij} = \int_a^b \delta \, \varphi_i \;\, \dot{\varphi}_j \, dt$$

$$K_{ij} = \int_a^b \gamma \, \varphi_i \;\, \varphi_j \, dt \qquad D_{ij} = \int_a^b \beta \left(\sum_{l=1}^{n} a_l \, \varphi_l \right)^2 \varphi_i \;\, \varphi_j \, dt$$

Non linear term

$$F_i = \int_a^b a_o \, f(t) \, \varphi_i \, dt - \dot{\tilde{x}} \, \varphi_i \Big|_a^b$$

Duffing Equation

First Approximation

$$n = 2 \qquad \tilde{x} = a_1\varphi_1 + a_2\varphi_2$$

$$\varphi_1 = \frac{b-t}{L} \qquad \varphi_2 = \frac{t-a}{L} \qquad L = b - a$$

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Elemental Matrix

Duffing Equation

The terms of the elemental matrix

$$R_{11} = \frac{1}{3}\gamma\, L - \frac{1}{2}\delta - \frac{1}{L} + \frac{1}{30}\beta\, L\, (6\, a_1^2 + 3\, a_1\, a_2 + a_2^2)$$

$$R_{12} = \frac{1}{6}\gamma\, L + \frac{1}{2}\delta + \frac{1}{L} + \frac{1}{60}\beta\, L\, (3\, a_1^2 + 4\, a_1\, a_2 + 3\, a_2^2)$$

$$R_{21} = \frac{1}{6}\gamma\, L - \frac{1}{2}\delta + \frac{1}{L} + \frac{1}{60}\beta\, L\, (3\, a_1^2 + 4\, a_1\, a_2 + 3\, a_2^2)$$

$$R_{22} = \frac{1}{3}\gamma\, L + \frac{1}{2}\delta - \frac{1}{L} + \frac{1}{30}\beta\, L\, (a_1^2 + 3\, a_1\, a_2 + 6\, a_2^2)$$

$$F_1 = \left(\frac{a_o}{L}\right)\int_a^b f(t)\,(b - t)\,dt + v_a \qquad F_2 = \left(\frac{a_o}{L}\right)\int_a^b f(t)\,(t - a)\,dt - v_b$$

Duffing Equation

The assembly of the elements

$$\begin{bmatrix} R_{11}{}^{(1)} & R_{12}{}^{(1)} & 0 \\ R_{21}{}^{(1)} & R_{22}{}^{(1)} + R_{11}{}^{(2)} & R_{12}{}^{(2)} \\ 0 & R_{21}{}^{(2)} & R_{22}{}^{(2)} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} g_1 + v_1 \\ g_2 \\ g_3 - v_3 \end{Bmatrix}$$

Initial Conditions

$$a_1 = x_o \qquad v_1 = v_o$$

Duffing Equation

Applying the initial conditions

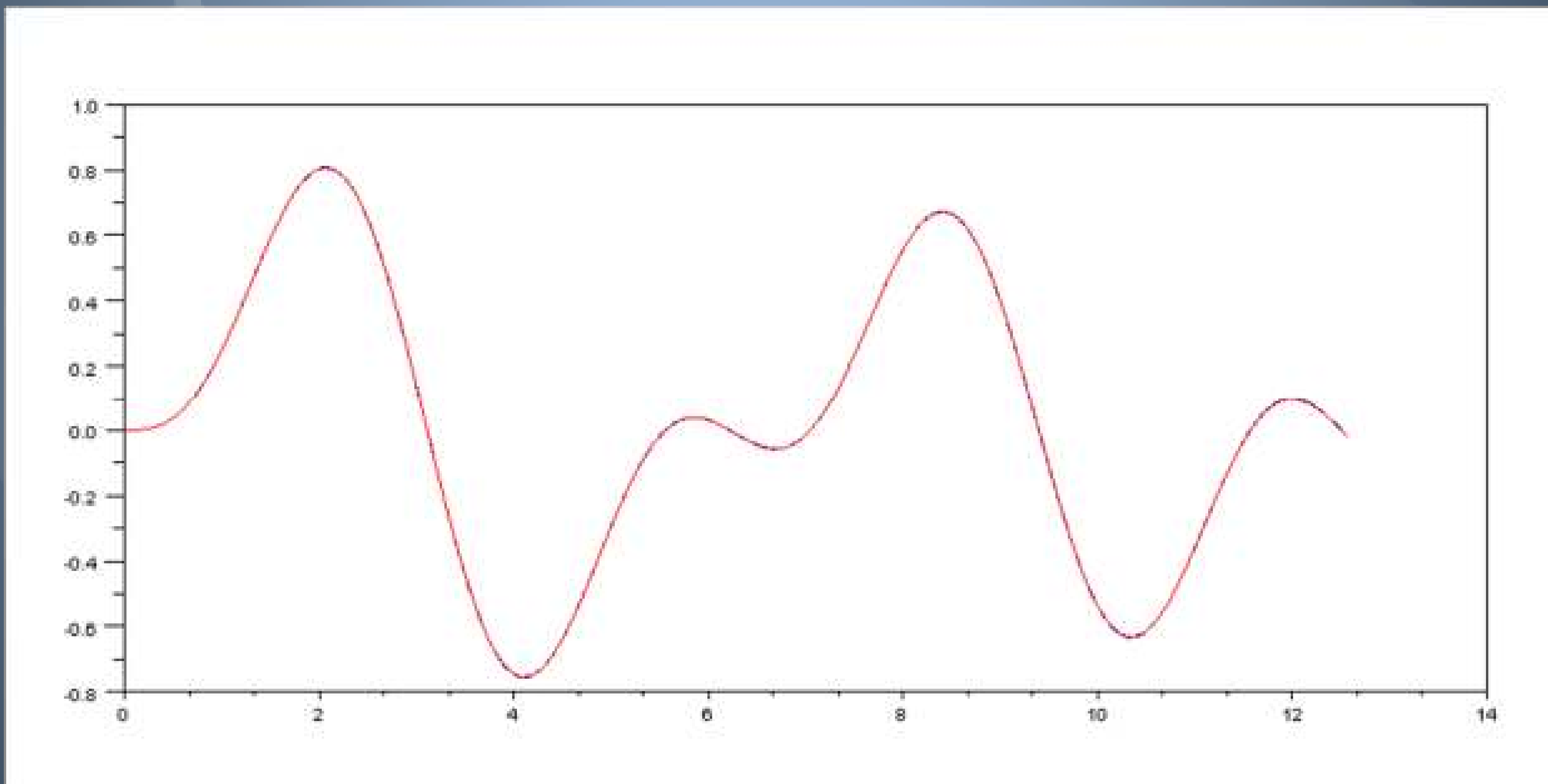
$$\begin{bmatrix} {R_{12}}^{(1)} & 0 & 0 \\ {R_{22}}^{(1)} + {R_{11}}^{(2)} & {R_{12}}^{(2)} & 0 \\ {R_{21}}^{(2)} & {R_{22}}^{(2)} & 1 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} g_1 + v_1 - {R_{11}}^{(1)} a_1 \\ g_2 - {R_{21}}^{(1)} a_1 \\ g_3 \end{Bmatrix}$$

One case (Linear Problem)

$$\delta = 0.1 \quad \gamma = 1 \quad \beta = 0 \quad a_0 = 1$$

$$f(t) = \sin 2t$$

Duffing Equation



Duffing Equation

Applying the initial conditions

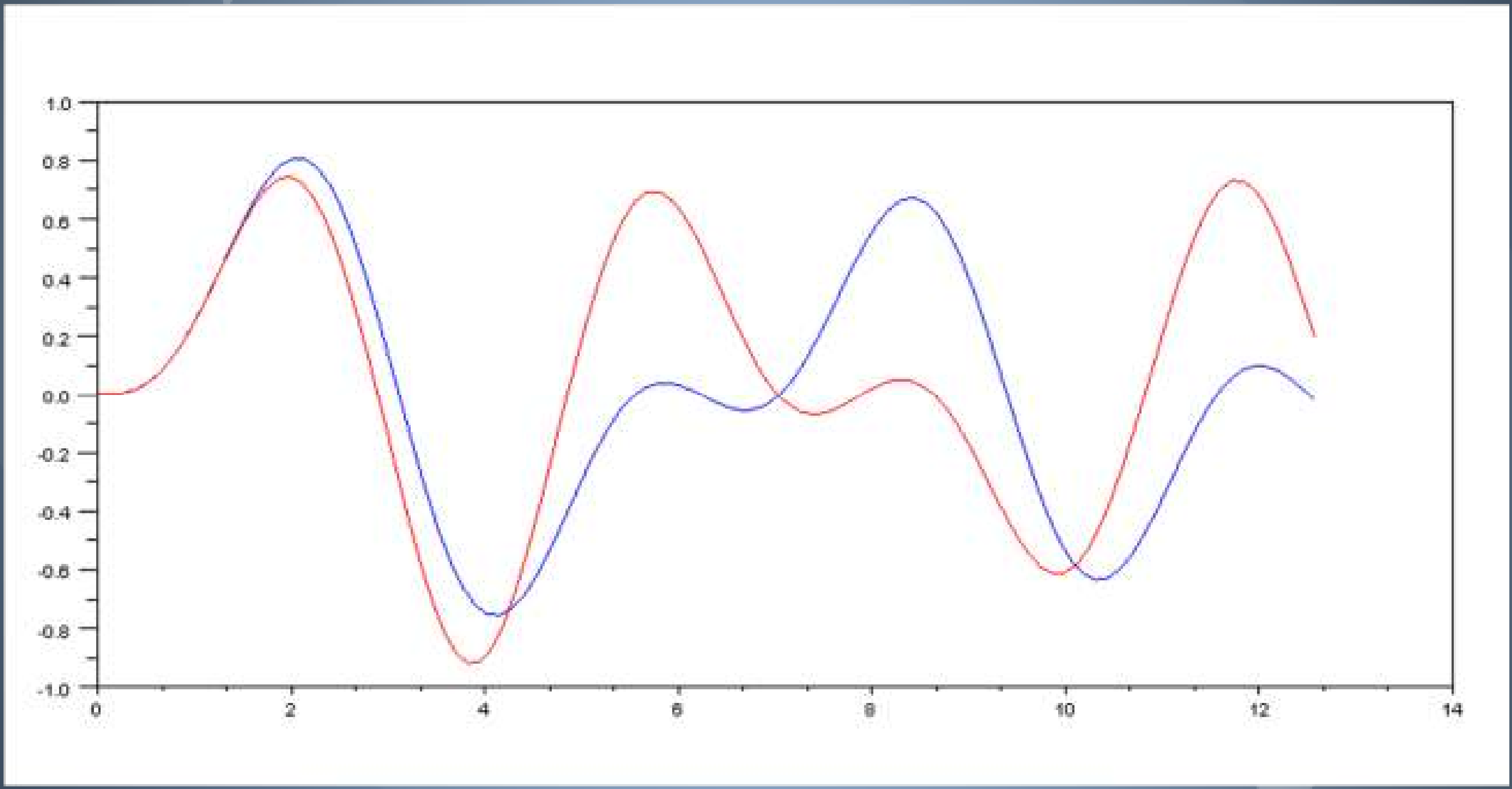
$$\begin{bmatrix} {R_{12}}^{(1)} & 0 & 0 \\ {R_{22}}^{(1)} + {R_{11}}^{(2)} & {R_{12}}^{(2)} & 0 \\ {R_{21}}^{(2)} & {R_{22}}^{(2)} & 1 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} g_1 + v_1 - {R_{11}}^{(1)} a_1 \\ g_2 - {R_{21}}^{(1)} a_1 \\ g_3 \end{Bmatrix}$$

Another case

$\delta = 0.1 \quad \gamma = 1 \quad \beta = 1 \qquad a_0 = 1$

$$f(t) = \sin 2t$$

Duffing Equation



Duffing Equation

Applying the initial conditions

$$\begin{bmatrix} {R_{12}}^{(1)} & 0 & 0 \\ {R_{22}}^{(1)} + {R_{11}}^{(2)} & {R_{12}}^{(2)} & 0 \\ {R_{21}}^{(2)} & {R_{22}}^{(2)} & 1 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} g_1 + v_1 - {R_{11}}^{(1)} a_1 \\ g_2 - {R_{21}}^{(1)} a_1 \\ g_3 \end{Bmatrix}$$

Another case

$$\delta = 0.1 \quad \gamma = 1 \quad \beta = 2 \quad a_0 = 1$$

$$f(t) = \sin 2t$$

Duffing Equation