

The effect of asymmetric large-scale dissipation on energy and potential enstrophy injection in two-layer quasi-geostrophic turbulence

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Publications

1. K.K. Tung and W.T. Welch (2001), *J. Atmos. Sci.* **58**, 2009-2012.
2. K.K. Tung and W.W. Orlando (2003a), *J. Atmos. Sci.* **60**, 824-835.
3. K.K. Tung and W.W. Orlando (2003b), *Discrete Contin. Dyn. Syst. Ser. B*, **3**, 145-162.
4. K.K. Tung (2004), *J. Atmos. Sci.*, **61**, 943-948.
5. E. Gkioulekas and K.K. Tung (2005), *Discrete Contin. Dyn. Syst. Ser. B*, **5**, 79-102
6. E. Gkioulekas and K.K. Tung (2005), *Discrete Contin. Dyn. Syst. Ser. B*, **5**, 103-124.
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9. E. Gkioulekas and K.K. Tung (2007), *Discrete Contin. Dyn. Syst. Ser. B*, **7**, 293-314
10. E. Gkioulekas (2010), *J. Fluid Mech.*, submitted. [arXiv:1011.3163v1 [nlin.CD]]

Outline

- Review of 2D turbulence.
- The Nastrom-Gage spectrum
- The Tung-Orlando theory of double cascade.
- The two-layer quasi-geostrophic model.

2D Navier-Stokes equations

- In 2D turbulence, the scalar vorticity $\zeta(x, y, t)$ is governed by

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = d + f, \quad (1)$$

where $\psi(x, y, t)$ is the streamfunction, and $\zeta(x, y, t) = -\nabla^2 \psi(x, y, t)$, and

$$d = -[\nu(-\Delta)^\kappa + \nu_1(-\Delta)^{-m}] \zeta \quad (2)$$

- The Jacobian term $J(\psi, \zeta)$ describes the advection of ζ by ψ , and is defined as

$$J(a, b) = \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial b}{\partial x} \frac{\partial a}{\partial y}. \quad (3)$$

Energy and enstrophy spectrum. I

- Two conserved quadratic invariants: energy E and enstrophy G defined as

$$E(t) = -\frac{1}{2} \int \psi(x, y, t) \zeta(x, y, t) dx dy \quad G(t) = \frac{1}{2} \int \zeta^2(x, y, t) dx dy. \quad (4)$$

- Let $a^{<k}(\mathbf{x})$ be the field obtained from $a(\mathbf{x})$ by setting to zero, in Fourier space, the components corresponding to wavenumbers with norm greater than k :

$$a^{<k}(\mathbf{x}) = \int d\mathbf{y} P(k|\mathbf{x} - \mathbf{y}) a(\mathbf{y}) \quad (5)$$

$$= \int_{\mathbb{R}^2} d\mathbf{x}_0 \int_{\mathbb{R}^2} d\mathbf{k}_0 \frac{H(k - \|\mathbf{k}_0\|)}{4\pi^2} \exp(i\mathbf{k}_0 \cdot (\mathbf{x} - \mathbf{x}_0)) a(\mathbf{x}_0) \quad (6)$$

- Filtered inner product:

$$\langle a, b \rangle_k = \frac{d}{dk} \int_{\mathbb{R}^2} d\mathbf{x} a^{<k}(\mathbf{x}) b^{<k}(\mathbf{x}) \quad (7)$$

Energy and enstrophy spectrum. II

- Energy spectrum: $E(k) = \langle \psi, \zeta \rangle_k$
- Enstrophy spectrum $G(k) = \langle \zeta, \zeta \rangle_k$
- Consider the conservation laws for $E(k)$ and $G(k)$:

$$\frac{\partial E(k)}{\partial t} + \frac{\partial \Pi_E(k)}{\partial k} = D_E(k) + F_E(k) \quad (8)$$

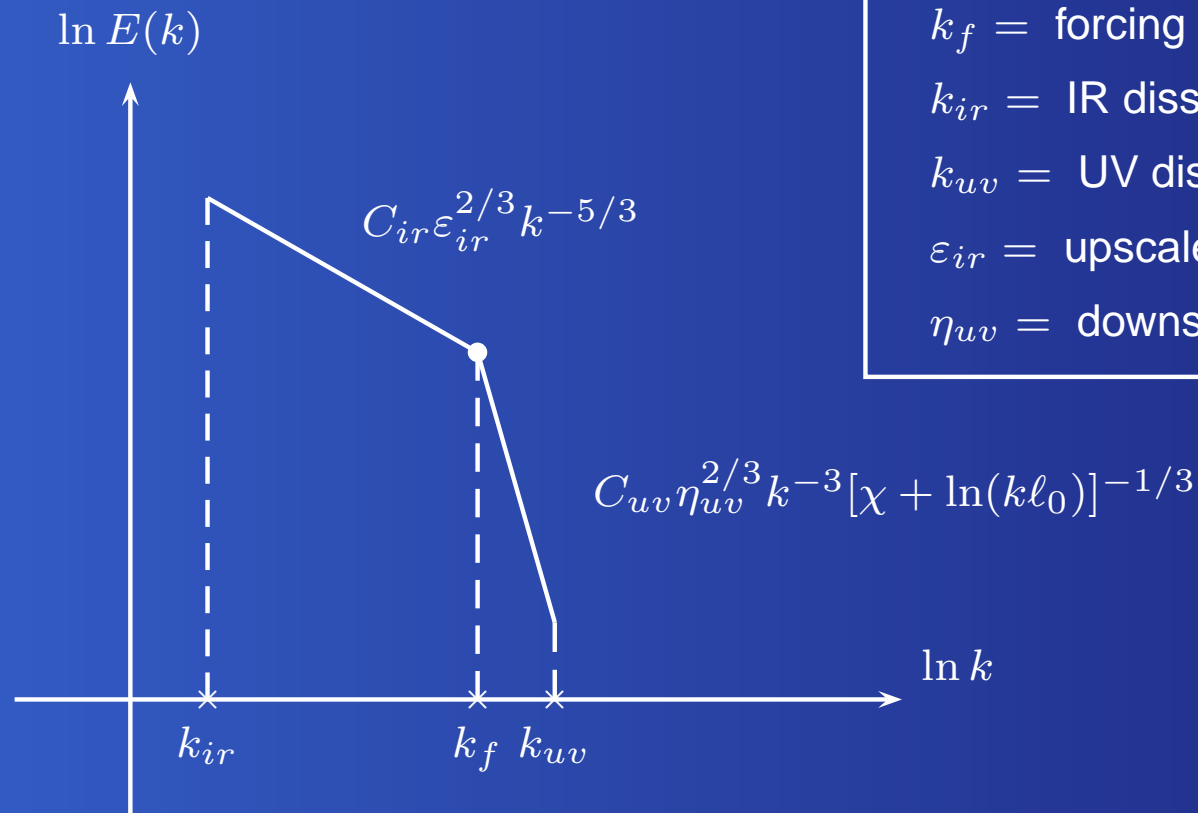
$$\frac{\partial G(k)}{\partial t} + \frac{\partial \Pi_G(k)}{\partial k} = D_G(k) + F_G(k) \quad (9)$$

- In two-dimensional turbulence, the energy flux $\Pi_E(k)$ and the enstrophy flux $\Pi_G(k)$ are constrained by

$$k^2 \Pi_E(k) - \Pi_G(k) < 0 \quad (10)$$

for all k not in the forcing range.

KLB theory.



k_f = forcing wavenumber

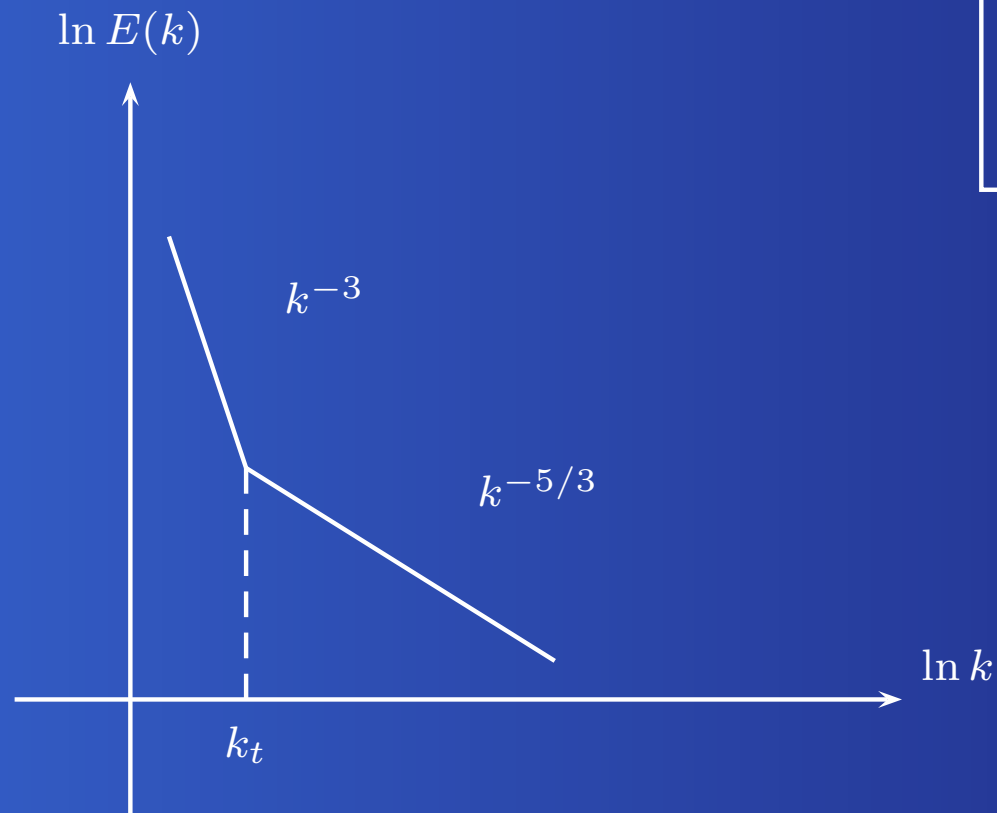
k_{ir} = IR dissipation wavenumber

k_{uv} = UV dissipation wavenumber

ε_{ir} = upscale energy flux

η_{uv} = downscale enstropt flux

Nastrom-Gage spectrum schematic

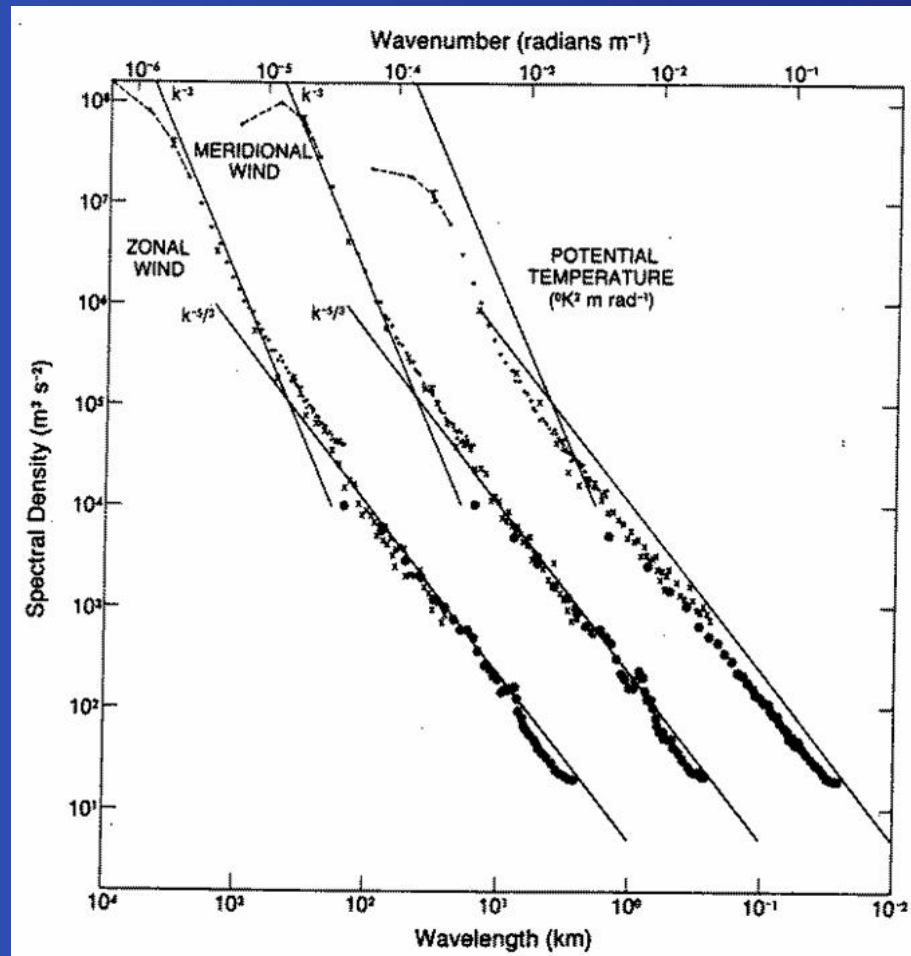


$$k^{-3} \rightarrow 3000\text{km} - 800\text{km}$$

$$k^{-5/3} \rightarrow 600\text{km} - \ll 1\text{km}$$

$$k_t \approx 700\text{km} \approx k_R$$

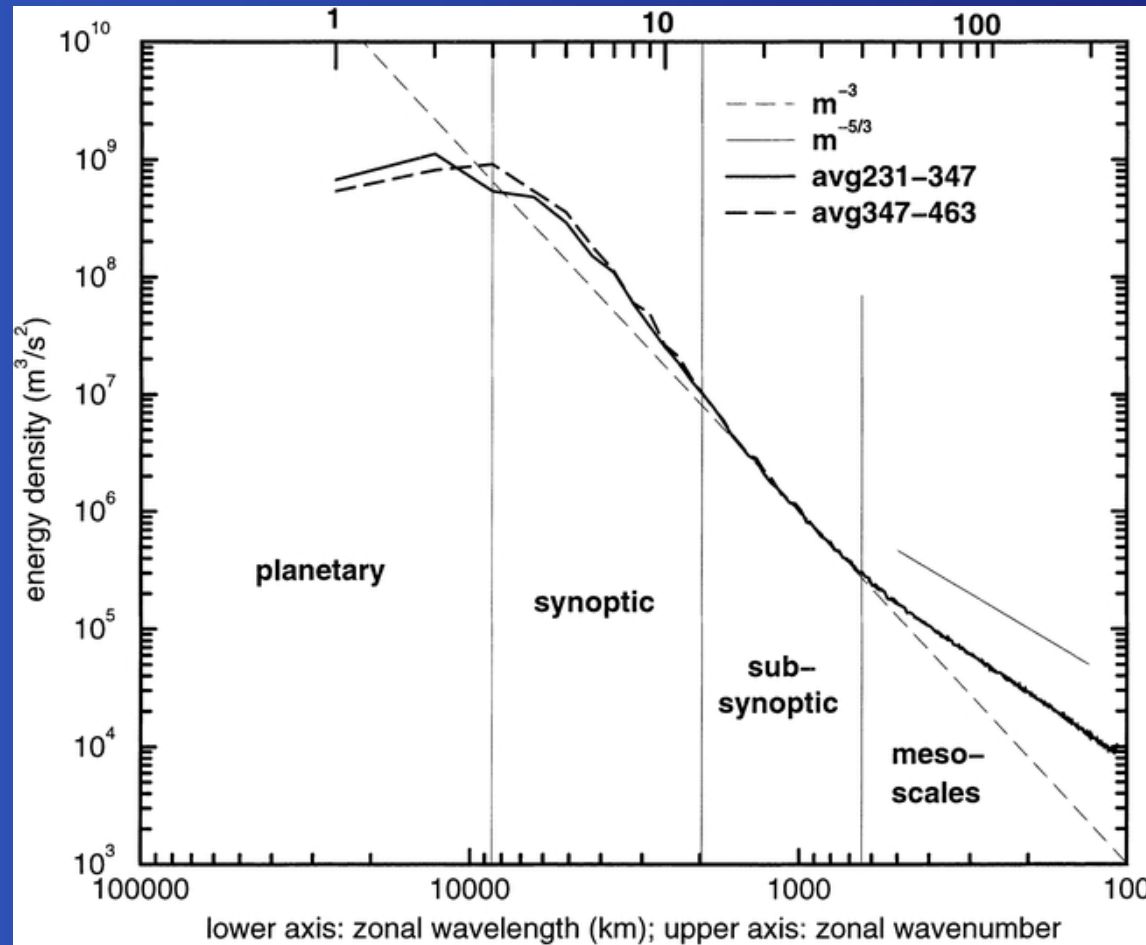
Nastrom-Gage spectrum



Interpretation of NG spectrum.

- The k^{-3} range is interpreted as downscale enstrophy cascade.
- The $k^{-5/3}$ range used to be interpreted as an 2D inverse energy cascade forced at small scales by thunderstorms.
- This tortured interpretation followed from the perceived need to explain the Nastrom-Gage spectrum in terms of 2D turbulence.
- Tung and Orlando (formerly: Welch) challenged the Charney QG-2D equivalence:
 - K.K. Tung and W.T. Welch (2001), *J. Atmos. Sci.* **58**, 2009-2012.
 - K.K. Tung and W.W. Orlando (2003a), *J. Atmos. Sci.* **60**, 824-835.
 - K.K. Tung and W.W. Orlando (2003b), *Discrete Contin. Dyn. Syst. Ser. B*, **3**, 145-162.
- New interpretation: A double downscale cascade of enstrophy and energy with enstrophy flux η_{uv} and energy flux ε_{uv} and transition from k^{-3} scaling to $k^{-5/3}$ scaling at the transition wavenumber $k_t \sim \sqrt{\eta_{uv}/\varepsilon_{uv}}$.

Tung and Orlando spectrum



More on Tung-Orlando theory

- Cho-Lindborg: Confirm downscale energy cascade.
 - J.Y.N. Cho and E. Lindborg (2001), *J. Geophys. Res.* **106 D10**, 10,223-10,232.
 - J.Y.N. Cho and E. Lindborg (2001), *J. Geophys. Res.* **106 D10**, 10,232-10,241.
- Smith-Tung debate: Transition not possible in 2D turbulence
 - K.S. Smith (2004), *J. Atmos. Sci.* **61**, 937-942
 - K.K. Tung (2004), *J. Atmos. Sci.*, **61**, 943-948.
- Gkioulekas-Tung superposition principle
 - E. Gkioulekas and K.K. Tung (2005), *Discr. Cont. Dyn. Syst. Ser. B*, **5**, 79-102
 - E. Gkioulekas and K.K. Tung (2005), *Discr. Cont. Dyn. Syst. Ser. B*, **5**, 103-124.
- Transition may be possible in 2-layer QG with asymmetric dissipation – still open question.
 - E. Gkioulekas and K.K. Tung (2007), *Discr. Cont. Dyn. Syst. Ser. B*, **7**, 293-314

The two-layer model. I

The governing equations for the two-layer quasi-geostrophic model are

$$\frac{\partial \zeta_1}{\partial t} + J(\psi_1, \zeta_1 + f) = -\frac{2f}{h}\omega + d_1 \quad (11)$$

$$\frac{\partial \zeta_2}{\partial t} + J(\psi_2, \zeta_2 + f) = +\frac{2f}{h}\omega + d_2 + 2e_2 \quad (12)$$

$$\frac{\partial T}{\partial t} + \frac{1}{2}[J(\psi_1, T) + J(\psi_2, T)] = -\frac{N^2}{f}\omega + Q_0 \quad (13)$$

where $\zeta_1 = \nabla^2 \psi_1$; $\zeta_2 = \nabla^2 \psi_2$; $T = (2/h)(\psi_1 - \psi_2)$. f is the Coriolis term; N the Brunt-Väisälä frequency; Q_0 is the thermal forcing on the temperature equation; d_1 , d_2 , e_2 the dissipation terms:

$$d_1 = (-1)^{\kappa+1} \nu \nabla^{2\kappa} \zeta_1 \quad (14)$$

$$d_2 = (-1)^{\kappa+1} \nu \nabla^{2\kappa} \zeta_2 \quad (15)$$

$$e_2 = -\nu_E \zeta_2 \quad (16)$$

The two-layer model. II

- The potential vorticity is defined as

$$q_1 = \nabla^2 \psi_1 + f + \frac{k_R^2}{2} (\psi_2 - \psi_1) \quad (17)$$

$$q_2 = \nabla^2 \psi_2 + f - \frac{k_R^2}{2} (\psi_2 - \psi_1) \quad (18)$$

with $k_R \equiv 2\sqrt{2}f/(hN)$ and it satisfies

$$\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = f_1 + d_1 \quad (19)$$

$$\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = f_2 + d_2 + e_2 \quad (20)$$

with $f_1 = (1/4)k_R^2 hQ_0$ and $f_2 = (1/4)k_R^2 hQ_0$.

The two-layer model. III

- The two-layer model conserves total energy E and potential layer enstrophies G_1 and G_2 :

$$E = \int [-\psi_1(x, y, t)q_1(x, y, t) - \psi_2(x, y, t)q_2(x, y, t)] dx dy \quad (21)$$

$$G_1 = \int q_1^2(x, y, t) dx dy \quad (22)$$

$$G_2 = \int q_2^2(x, y, t) dx dy \quad (23)$$

- We define:

- The energy spectrum $E(k) = \langle \psi_1, q_1 \rangle_k + \langle \psi_2, q_2 \rangle_k$
- Top-layer potential enstrophy spectrum $G_1(k) = \langle q_1, q_1 \rangle_k$
- Bottom-layer potential enstrophy spectrum $G_2(k) = \langle q_2, q_2 \rangle_k$
- Total potential enstrophy spectrum $G(k) = G_1(k) + G_2(k)$

The two-layer model. IV

- Consider the generalized form of an n-layer model:

$$\frac{\partial q_\alpha}{\partial t} + J(\psi_\alpha, q_\alpha) = \sum_{\beta} \mathcal{D}_{\alpha\beta} \psi_\beta + f_\alpha \quad (24)$$

$$\hat{q}_\alpha(\mathbf{k}, t) = \sum_{\beta} L_{\alpha\beta}(\|\mathbf{k}\|) \hat{\psi}_\beta(\mathbf{k}, t) \quad (25)$$

- The energy spectrum $E(k)$ and the potential enstrophy spectrum $G(k)$ are given by:

$$E(k) = \sum_{\alpha} \langle \psi_\alpha, q_\alpha \rangle_k = \sum_{\alpha\beta} L_{\alpha\beta}(k) C_{\alpha\beta}(k) \quad (26)$$

$$G(k) = \sum_{\alpha} \langle q_\alpha, q_\alpha \rangle_k = \sum_{\alpha\beta\gamma} L_{\alpha\beta}(k) L_{\alpha\gamma}(k) C_{\beta\gamma}(k) \quad (27)$$

with $C_{\alpha\beta}(k) = \langle \psi_\alpha, \psi_\beta \rangle_k$.

The two-layer model. V

- Let $\phi_{\alpha\beta}(k) = \langle f_\alpha, \psi_\beta \rangle_k$, and $D_{\alpha\beta}(k)$ be the spectrum of the operator $\mathcal{D}_{\alpha\beta}$
- The energy forcing spectrum $F_E(k)$ and the potential enstrophy forcing spectrum $F_G(k)$ are:

$$F_E(k) = 2 \sum_{\alpha} \phi_{\alpha\alpha}(k) \quad (28)$$

$$F_G(k) = 2 \sum_{\alpha\beta} L_{\alpha\beta}(k) \phi_{\alpha\beta}(k) \quad (29)$$

- The energy dissipation spectrum $D_E(k)$ and the potential dissipation enstrophy spectrum $D_G(k)$ are related with $C_{\alpha\beta}(k) = \langle \psi_\alpha, \psi_\beta \rangle_k$ according to:

$$D_E(k) = 2 \sum_{\alpha\beta} D_{\alpha\beta}(k) C_{\alpha\beta}(k) \quad (30)$$

$$D_G(k) = 2 \sum_{\alpha\beta\gamma} L_{\alpha\beta}(k) D_{\alpha\gamma}(k) C_{\beta\gamma}(k) \quad (31)$$

Forcing spectrum. I.

- For antisymmetric forcing $f_1 = f$ and $f_2 = -f$:

$$F_E(k) = 2(\Phi_1(k) - \Phi_2(k)) \quad (32)$$

$$F_G(k) = (k^2 + k_R^2)F_E(k) \quad (33)$$

with $\Phi_1(k) = \langle f, \psi_1 \rangle_k$ and $\Phi_2(k) = \langle f, \psi_2 \rangle_k$.

- It follows that for $k_f \ll k_R$: $(\eta/\varepsilon) \sim k_R^2$.
- If both are dissipated at small scales, then $k_t \sim k_R$.
- In 2D turbulence Ekman damping dissipates most of injected energy and some of injected enstrophy.
- Not true in two-layer QG model because the Ekman term appears only on bottom layer.

Forcing spectrum. II.

- For Ekman-damped forcing: $f_1 = f$ and $f_2 = -f - \nu_E \Delta \psi_2$, incorporating the damping effect to the forcing spectrum gives:

$$F_E(k) = 2(\Phi_1(k) - \Phi_2(k)) + 2\nu_E k^2 U_2(k) \quad (34)$$

$$F_G(k) = (k^2 + k_R^2)F_E(k) - \nu_E k^2 k_R^2 (U_2(k) + C_{12}(k)) \quad (35)$$

with $U_1(k) = \langle \psi_1, \psi_1 \rangle_k$, $U_2(k) = \langle \psi_2, \psi_2 \rangle_k$, and $C_{12}(k) = \langle \psi_1, \psi_2 \rangle_k$.

- Note the $F_E(k)$ increases, because $U_2(k) > 0$.
- If $E_K(k) \geq E_P(k) \implies C_{12}(k) \geq 0 \implies F_G(k)$ decreases.
- Thus, the tendency is to decrease k_t .
- Layer interaction makes it non-obvious whether $F_G(k)$ increases or decreases.
- In SQG: $E_K(k)/E_P(k) = 1$.
- In stratified 3D: $E_K(k)/E_P(k) \approx 3$. (Lindborg (2009))
- In full QG: $E_K(k)/E_P(k) \approx 2$. (Charney theory – Vallgren and Lindborg (2010))

Forcing spectrum. IV.

- Claim: Suppressing bottom-layer forcing tends to decrease k_t .
- Let $f_1 = \varphi$ and $f_2 = -\mu\varphi$ with $\mu \in (0, 1)$ (suppression factor).
- Assume φ is random Gaussian with

$$\langle \varphi(\mathbf{x}_1, t_1) \varphi(\mathbf{x}_2, t_2) \rangle = 2Q(\mathbf{x}_1, \mathbf{x}_2) \delta(t_1 - t_2) \quad (36)$$

$$\langle f_\alpha(\mathbf{x}_1, t_1) f_\beta(\mathbf{x}_2, t_2) \rangle = 2Q_{\alpha\beta}(\mathbf{x}_1, \mathbf{x}_2) \delta(t_1 - t_2) \quad (37)$$

- Define the forcing spectrum:

$$\mathcal{Q}(k) = \frac{d}{dk} \int d\mathbf{x} d\mathbf{y} d\mathbf{z} P(k|\mathbf{x} - \mathbf{y}) P(k|\mathbf{x} - \mathbf{z}) Q(\mathbf{y}, \mathbf{z}) \quad (38)$$

$$Q_{\alpha\beta}(k) = \frac{d}{dk} \int d\mathbf{x} d\mathbf{y} d\mathbf{z} P(k|\mathbf{x} - \mathbf{y}) P(k|\mathbf{x} - \mathbf{z}) Q_{\alpha\beta}(\mathbf{y}, \mathbf{z}) \quad (39)$$

Forcing spectrum. V.

- It follows that the streamfunction-forcing spectrum reads:

$$\varphi_{\alpha\beta}(k) = \langle f_{\alpha}, \psi_{\beta} \rangle_k = \sum_{\gamma} L_{\beta\gamma}^{-1}(k) \mathcal{Q}_{\alpha\gamma}(k) \quad (40)$$

- Energy and enstrophy forcing spectrum:

$$F_E(k) = 2 \sum_{\alpha} \varphi_{\alpha\alpha}(k) = \frac{2\mathcal{Q}(k)[2(1 + \mu^2)k^2 + (1 - \mu)^2 k_R^2]}{2k^2(k^2 + k_R^2)} \quad (41)$$

$$F_G(k) = 2 \sum_{\alpha} L_{\alpha\beta}(k) \varphi_{\alpha\beta}(k) = 2(1 + \mu^2) \mathcal{Q}(k) \quad (42)$$

- Consider the forcing range limit: $k \ll k_R$.
- For $\mu = 1$: $F_G(k) \sim k_R^2 F_E(k) \implies k_t \sim k_R$
- For $\mu = 0$: $F_G(k) \sim 2k^2 F_E(k) \implies k_t \sim 2k_f$
- Thus: Baroclinically damped forcing contributes to inertial range transition.

Conclusion

- The energy flux constraint prevents a transition from k^{-3} scaling to $k^{-5/3}$ scaling in 2D turbulence.
- The energy flux constraint can be broken in two-layer QG turbulence under asymmetric dissipation.
- In the two-layer QG model, the rates of enstrophy over energy injection satisfy:
$$(\eta/\varepsilon) = k_R^2$$
- Asymmetric Ekman dissipation increases ε and may decrease η ,
- Suppressing the bottom layer forcing directly always decreases the ratio (η/ε) .
- Open question: Can the injected energy and enstrophy be dissipated?