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Factorization of self-adjoint ordinary differential equations

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ABSTRACT

This paper deals with the factorization of self-adjoint differential operators $L_{(2n)} = \frac{1}{\rho}\frac{d^n}{dx^n}\left(\rho\beta^n \frac{d^n}{dx^n}\right)$, and their spectral type differential equations. Sufficient conditions of factorization are reported. A large class of differential operators and equations that can be factorized is obtained. The factorizations of fourth- and sixth-order operators and equations are explicitly given. A particular fourth-order spectral type differential equation in which $\rho(x) = (1-x)^p(1+x)^q$, $p \geqslant 1,\ q \geqslant 1$, is considered. Its general solution is obtained in terms of hypergeometric functions. As application, the natural frequencies and mode shapes of mechanical transverse vibrations of a nonuniform structure are found.

1. Introduction

The goal of this paper is to provide sufficient conditions of factorization of self-adjoint differential equations

$$\frac{1}{\rho}\frac{d^n}{dx^n}\left(\rho\beta^n\frac{d^n y}{dx^n}\right) - \mu y = 0,$$

where $\rho(x)$, $\beta(x)$ are scalar functions, and μ is a positive constant. This result is important for solving various boundary value problems in mathematics, mechanics, and physics.

Factorization method has been successfully used for instance to obtain solutions of Schrődinger equations, differential equations modeling transverse vibration, differential equations of wave propagation, and to decide the existence of liouvillian solutions, Singer [1]. This method consists of writing differential operators and/or equations as a product of lower order differential operators. Various algorithms and applications requiring this method can be found in the literature.

In mathematics, factorization continues to be a very successful method for solving second-, fourth-, and higher-order differential equations. Factorizations of second-order differential equations have been subject to several studies. Hounkonnou et al. [2] proposed a factorization of the confluent, double confluent, and biconfluent second-order Heun's differential equations and investigated their solutions. Ronveaux [3] considered Heun's equation, a Fuchsian second-order differential equation with four regular singularities at 0, 1, a, and ∞. This equation appears in situations relevant to Laplace, Helmholtz, and Schrődinger equations, therefore in many applications in electromagnetism, quantum physics, and cosmology. Ronveaux reported that Heun's operator can be factored into a pair of first order differential operators. This factorization is different than the well-known factorization methods of Infeld and Hull [4] for second-order differential equations reducible to the Riemann hypergeometric equation. Berkovich [5] presented methods for factorization, autonomization, and exact linearization of linear and nonlinear nonautonomous ordinary differential equations of second-order, reduced to linear equations with constant coefficients. Appropriate algorithms were used to search for transformations, factorizations, and Liouvillian solutions in the system computer algebra REDUCE in the program SOLDE. Lorente [6] used Rodrigues formula to present a general construction of raising and lowering operators for orthogonal polynomials of continuous variables. Using Infeld–Hull [4] factorization

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method, he generated from raising and lowering operators the second-order self-adjoint differential operator of hypergeometric type. Berkovich [7] based his research on a uniform theory of factorization and transformation of differential equations of order equal or greater than two. He developed this method of factorization of differential operators not only in a base field but also in its algebraic and transcendental extensions. The method was extended to nonlinear equations. He also proposed a new method of exact linearization. Hermann [8] related the Infeld–Hull theory of factorization of second-order differential operators to the classical Picard-Vessiot theory and the work by Gelfand, Kirillov, and Herman on Lie algebras in the field of algebraic extensions of the enveloping associative algebra of a Lie algebra.

Fourth-order differential equations were investigated as well. Everitt et al. [9] reported the factorization of fourth-order Bessel-type differential equation showing that a factorization of this equation into a pair of second-order differential operators existed. Foupouagnigni et al. [10] factorized the fourth-order differential equations satisfied by the Laguerre-Hahn orthogonal polynomials obtained from some perturbations of classical orthogonal polynomials (rth associated, general co-recursive, general co-dilated, and general co-modified). They found the four linearly independent solutions of the fourth-order differential equations, and extended from integers to reals the results obtained for the associated classical orthogonal polynomials with integer order of association. Dosly [11] investigated oscillation and spectral properties of self-adjoint differential operators, and devoted a particular attention to fourth-order differential operators with a middle term for which new nonoscillation criteria were derived. Lewanowicz [12] investigated so called associated polynomials of classical orthogonal polynomials. These associated polynomials, known to belong to the Hahn–Laguerre class of orthogonal polynomials, satisfied a fourth-order differential equation. He showed that the differential operator of the equation can be written as a sum of two differential operators, one fourth-order, and the other second-order. The fourth-order differential operator belonged to the first associated polynomials (or numerator polynomials), and it was factored into a pair of second-order differential operators by Ronveaux [3].

Kwon et al. [13] showed that if an orthogonal polynomial system satisfied a certain spectral type differential equation, then the orthogonal polynomials were Hermite polynomials. Moreover the spectral type differential equation had to be a linear combination of iterations of a second-order differential equation of type. Further results were reported by Kwon et al. [14]. The method of factorization has been also used by He and Ricci [15], who defined two sequences of differential operators for a sequence of polynomials. They constructed these operators for Appel polynomials, and determined their differential equations via factorization method introduced by Infeld and Hull [4]. Van Hoeij [16] investigated a fast method to compute the rational solutions of so-called mixed differential equations. His method can be applied to the factorization of completely reducible linear differential operators with rational function coefficients.

Physics served as a rich source of mathematical problems from the very beginning, and played a major role in mathematics' development. Several researchers used the method of factorization in their investigations. Kuru [17] studied the Benjamin–Bona–Mahony (BBM) equation with a fully nonlinear dispersive term by means of the factorization technique. This partial differential equation described the unidirectional propagation of small-amplitude long waves on the surface of water in a channel, hydromagnetics waves in cold plasma, and acoustic waves in anharmonic crystals. Assuming traveling wave solutions, the BBM partial differential equation was reduced to a nonlinear, second-order, ordinary differential equation which was then factorized. Traveling wave solutions of this equation were found in terms of the Weierstrass function *wp* and its degenerated trigonometric and hyperbolic forms. They also gave the Lagrangian and the Hamiltonian, linked to the factorization, for the nonlinear second-order ordinary differential equations associated to the traveling wave equations. Fahmy [18] used the factorization method to find traveling wave solutions for the following nonlinear second-order partial differential equations: generalized time-delayed Burgers–Huxley, time delayed convective Fishers, and generalized time-delayed Burgers–Fisher. Ferapontov and Veselov [19] investigated the factorization method for Schrődinger operators with magnetic fields on a two-dimensional surface M^2 with nontrivial metric, and brought a new look at classical problems such as Dirac magnetic monopole and Landau problem. Amirkhanov et al. [20] reported on the Schrődinger equation in the relativistic space for a relativistic function $\psi(r)$. This equation was an infinite-order differential equation with a small parameter at higher derivatives. They considered the fourth-order and sixth-order differential equations which corresponded to truncations of the higher-order derivatives. The differential equations were factorized in terms of second-order differential operators. Barut et al. [21] reported a new method of algebraisation of quantum mechanical eigenvalue equations. Infeld–Hull–Miller factorizations were used to obtain the ladder operators of the dynamical algebra. Weston [22] factorized the wave equation into a coupled system of wave components for a case where the field quantities were multivariate functions of spatial variables and the velocity was a function of only one variable.

Several applications in mechanics used the factorization method for solving second-order and fourth-order differential equations. Soh [23] obtained isospectral Euler–Bernoulli beams by factorization and Lie symmetry techniques. The Euler–Bernoulli operator has been factorized as a product of a second-order differential operator and its adjoint. The factorization was possible provided the coefficients of the factors satisfied a system of nonlinear ordinary differential equations. The system reduced to a single nonlinear third-order differential equation, called principal equation and analyzed using Lie group methods. Caruntu [24,25] reported the factorization of Euler–Bernoulli fourth-order differential operator describing transverse vibrations of nonuniform beams into a pair of commuting Sturm–Liouville second-order differential operators. He also reported orthogonal polynomials closed-form solutions for transverse vibrations [26,27], and self-adjoint differential equations of orthogonal polynomials [28]. Rosu and Reyes [29] employed the factorization of Newton's second-order differential equation of motion of free damped oscillator to determine the class of damped modes related to common free damping modes by supersymmetry. They obtained the Riccati parameter families of damping modes (directly related to Newtonian free damping) by using Witten's supersymmetric scheme and the general Riccati solution. Lokshin [30] considered a special asymptotic factorization of the second-order nonlinear wave equation describing the interaction of a short pulse and a single wave of finite amplitude moving in opposite directions in a nonlinear rod. Storti and Aboelnaga [31] factorized the fourth-order differential equation of bending vibrations of a class of variable cross-section rotating beams, found the solution in terms of hypergeometric functions, and presented a technique of computing natural frequencies and mode shapes as functions of setting angle and rotation rate.

This paper reports a large class of self-adjoint operators and differential equations that can be factorized. A recurrence relationship between these operators gives sufficient conditions of factorization. Conditions of factorization for fourth-order operators are different than those for sixth- and higher-order. Finding general solutions of self-adjoint differential equations depends on the ability of solving the second-order differential equations resulting from factorization. A particular fourth-order differential equation in which $\rho(x) = (1-x)^p(1+x)^q$, $p \geqslant 1,\ q \geqslant 1$, is presented. Its general solution is found in terms of hypergeometric functions. As application, transverse vibrations of nonuniform beams are investigated using the factorization method.

2. Recurrence of self-adjoint differential operators

A recurrence relationship between self-adjoint ordinary differential operators of order $2n$ and $2(n+2)$ is presented as follows.

Lemma. *If $\rho(x)$, $\beta(x)$, and $\alpha(x)$ are scalar functions satisfying the following equation*

$$\frac{1}{\rho}\frac{d\rho}{dx} = \frac{\alpha}{\beta}, \tag{1}$$

then the self-adjoint ordinary differential operators given by

$$L_{(2n)} = \frac{1}{\rho}\frac{d^n}{dx^n}\left(\rho\beta^n\frac{d^n}{dx^n}\right), \tag{2}$$

satisfy the following recurrence relationship

$$L_{(2n)}L_2 = L_{(2n+2)} + \frac{1}{\rho}\frac{d^n}{dx^n}\left\{\left[n\frac{d\alpha}{dx} + \frac{n(n+1)}{2}\frac{d^2\beta}{dx^2}\right]\rho\beta^n\frac{d^n}{dx^n}\right\} + \frac{1}{\rho}\frac{d^n}{dx^n}\left\{\rho\beta^n\sum_{k=2}^{n}\left[\binom{n}{k}\frac{d^k\alpha}{dx^k} + \binom{n+1}{k+1}\frac{d^{k+1}\beta}{dx^{k+1}}\right]\frac{d^{n-k+1}}{dx^{n-k+1}}\right\}, \tag{3}$$

where n is any natural number.

Proof. The product of the operators $L_{(2n)}$ and L_2 given by Eq. (2) can be written as

$$L_{(2n)}L_2 = \frac{1}{\rho}\frac{d^n}{dx^n}\left\{\rho\beta^n\frac{d^n}{dx^n}\left[\frac{1}{\rho}\frac{d}{dx}\left(\rho\beta\frac{d}{dx}\right)\right]\right\}. \qquad \square \tag{4}$$

Using Eq. (1) and the product rule for differentiation one obtains an $n+1$ order derivative term in Eq. (4) as follows

$$L_{(2n)}L_2 = \frac{1}{\rho}\frac{d^{n+1}}{dx^{n+1}}\left\{\rho\beta^n\frac{d^{n-1}}{dx^{n-1}}\left[\beta\frac{d^2}{dx^2} + \left(\alpha + \frac{d\beta}{dx}\right)\frac{d}{dx}\right]\right\} - \frac{1}{\rho}\frac{d^n}{dx^n}\left\{\rho\beta^{n-1}\left(\alpha + n\frac{d\beta}{dx}\right)\frac{d^{n-1}}{dx^{n-1}}\left[\beta\frac{d^2}{dx^2} + \left(\alpha + \frac{d\beta}{dx}\right)\frac{d}{dx}\right]\right\}. \tag{5}$$

Then, using Leibniz rule (generalized product rule) inside braces of Eq. (5), the following relationship results

$$\begin{aligned} L_{(2n)}L_2 = {} & \frac{1}{\rho}\frac{d^{n+1}}{dx^{n+1}}\left\{\rho\beta^n\left[\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{d^k\beta}{dx^k}\frac{d^{n-k+1}}{dx^{n-k+1}} + \sum_{k=0}^{n-1}\binom{n-1}{k}\left[\frac{d^k}{dx^k}\left(\alpha+\frac{d\beta}{dx}\right)\right]\frac{d^{n-k}}{dx^{n-k}}\right]\right\} \\ & - \frac{1}{\rho}\frac{d^n}{dx^n}\left\{\rho\beta^{n-1}\left(\alpha+n\frac{d\beta}{dx}\right)\left[\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{d^k\beta}{dx^k}\frac{d^{n-k+1}}{dx^{n-k+1}} + \sum_{k=0}^{n-1}\binom{n-1}{k}\left[\frac{d^k}{dx^k}\left(\alpha+\frac{d\beta}{dx}\right)\right]\frac{d^{n-k}}{dx^{n-k}}\right]\right\}. \end{aligned} \tag{6}$$

It can be noticed that the first sum of the right-hand side of Eq. (6) gives the operator $L_{(2n+2)}$ when $k=0$

$$L_{(2n+2)} = \frac{1}{\rho}\frac{d^{n+1}}{dx^{n+1}}\left(\rho\beta^{n+1}\frac{d^{n+1}}{dx^{n+1}}\right). \tag{7}$$

Next, the operator d^{n+1}/dx^{n+1} is reduced to d^n/dx^n by applying the operator d/dx to the remaining braced expression of the first term of the right-hand side of Eq. (6). Regrouping and canceling out terms, Eq. (6) becomes

$$L_{(2n)}L_2 = L_{(2n+2)} + \frac{1}{\rho}\frac{d^n}{dx^n}\left\{\rho\beta^n\left\{\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{d^{k+1}\beta}{dx^{k+1}}\frac{d^{n-k+1}}{dx^{n-k+1}} + \sum_{k=1}^{n-1}\binom{n-1}{k}\frac{d^k\beta}{dx^k}\frac{d^{n-k+2}}{dx^{n-k+2}}\right.\right.$$
$$\left.\left. + \sum_{k=0}^{n-1}\binom{n-1}{k}\left[\frac{d^{k+1}}{dx^{k+1}}\left(\alpha + \frac{d\beta}{dx}\right)\right]\frac{d^{n-k}}{dx^{n-k}} + \sum_{k=0}^{n-1}\binom{n-1}{k}\left[\frac{d^k}{dx^k}\left(\alpha + \frac{d\beta}{dx}\right)\right]\frac{d^{n-k+1}}{dx^{n-k+1}} - \left(\alpha + n\frac{d\beta}{dx}\right)\frac{d^{n+1}}{dx^{n+1}}\right\}\right\}. \tag{8}$$

Shifting the summation indexes of the second and third sums inside braces to have d^{n-k+1}/dx^{n-k+1} under sums, and canceling out terms, Eq. (8) results as

$$L_{(2n)}L_2 = L_{(2n+2)} + \frac{1}{\rho}\frac{d^n}{dx^n}\left\{\rho\beta^n\left\{\binom{n}{1}\left[\frac{d}{dx}\left(\alpha + \frac{d\beta}{dx}\right)\right] + \binom{n}{2}\frac{d^2\beta}{dx^2}\right\}\frac{d^n}{dx^n}\right\}$$
$$+ \frac{1}{\rho}\frac{d^n}{dx^n}\left\{\rho\beta^n\sum_{k=1}^{n-1}\left\{\binom{n}{k}\left[\frac{d^k}{dx^k}\left(\alpha + \frac{d\beta}{dx}\right)\right] + \binom{n}{k+1}\frac{d^{k+1}\beta}{dx^{k+1}}\right\}\frac{d^{n-k+1}}{dx^{n-k+1}}\right\}. \tag{9}$$

Rearranging Eq. (9), it becomes Eq. (3) in its condensed form

$$L_{(2n)}L_2 = L_{(2n+2)} + \frac{1}{\rho}\frac{d^n}{dx^n}\left\{\rho\beta^n\sum_{k=1}^{n}\left[\binom{n}{k}\frac{d^k\alpha}{dx^k} + \binom{n+1}{k+1}\frac{d^{k+1}\beta}{dx^{k+1}}\right]\frac{d^{n-k+1}}{dx^{n-k+1}}\right\}. \quad \square \tag{10}$$

3. Factorization of self-adjoint ordinary differential operators

Two propositions regarding sufficient conditions of factorization of self-adjoint differential ordinary operators will follow. These two propositions correspond to two cases (1) operators of fourth-order, and (2) operators of sixth-order and higher.

3.1. Factorization of fourth-order operators

Proposition 1. *If* Eq. (1) *is satisfied and*

$$\frac{d^2\alpha}{dx^2} + \frac{d^3\beta}{dx^3} = 0, \tag{11}$$

then the fourth-order operator L_4 given by Eq. (1) *for n = 2 can be factorized as follows*

$$L_4 = L_2(L_2 - \delta_2), \tag{12}$$

where δ_2 is given by

$$\delta_k = (k-1)\frac{d\alpha}{dx} + \frac{k(k-1)}{2}\frac{d^2\beta}{dx^2}, \tag{13}$$

for $k = 2$, and operators L_2 and L_4 are given by Eq. (2) *for $n = 1$ and $n = 2$, respectively.*

Proof. If $n = 1$ in Eq. (3), the third right-hand side term of this equation does not appear since the first value of the summation index k is 2. So, Eq. (3) becomes

$$L_2L_2 = L_4 + \frac{1}{\rho}\frac{d}{dx}\left[\left(\frac{d\alpha}{dx} + \frac{d^2\beta}{dx^2}\right)\rho\beta\frac{d}{dx}\right]. \tag{14}$$

$\square$

The second term of the right-hand side of Eq. (14) is the operator L_2 multiplied by a constant coefficient $\delta_2 = (d\alpha/dx + d^2\beta/dx^2)$ due to Eq. (11). This coefficient is given by Eq. (13) for $k = 2$. Solving Eq. (14) for L_4, one obtains Eq. (12).

Corollary 1. *The class of fourth-order operators of* Proposition 1 *is given by* Eq. (2) *for n* = 2 *and $\beta(x)$ as follows*

$$\beta = \frac{1}{\rho}\int\rho(b_0 + b_1x)dx, \tag{15}$$

where b_0 and b_1 are real constants, and $\rho(x)$ is scalar function.

Proof. The scalar function $\alpha(x)$ is eliminated from Eqs. (1) and (11). $\square$

3.2. Factorization of sixth- and higher-order operators

Proposition 2. *If* Eq. (1) *is satisfied and*

$$\frac{d^2\alpha}{dx^2} = \frac{d^3\beta}{dx^3} = 0, \tag{16}$$

then the operators $L_{(2n)}$ of sixth-order and higher, $n \geqslant 3$, can be factorized as follows

$$L_{(2n)} = \prod_{k=1}^{n}(L_2 - \delta_k), \tag{17}$$

where δ_k are constants given by Eq. (13)*, and k is any natural number less than or equal to n. It can be noticed that $\delta_1 = 0$.*

Proof. Due to Eq. (16), the third term of the right-hand side of Eq. (3) is zero. All coefficients inside brackets of this term are zero since they are derivatives of α and β same order or higher than Eq. (16); the summation index of this term's sum is $k = 2, 3, \ldots, n$. Also, the second term of the right-hand side of Eq. (3) reduces to the operator L_2 multiplied by a constant, due to Eq. (16), coefficient δ_{n+1}. This constant coefficient δ_{n+1} is given by Eq. (13) for $k = n+1$. Consequently, Eq. (3) becomes

$$L_{(2n)}L_2 = L_{(2n+2)} + \delta_{n+1}L_{(2n)}. \qquad \square \tag{18}$$

Solving Eq. (18) for $L_{(2n+2)}$ and then shifting the index, the recurrence relation can written as

$$L_{(2n)} = L_{(2n-2)}(L_2 - \delta_n). \tag{19}$$

Using repeated substitution for solving the recurrence relationship Eq. (19), one obtains Eq. (17).

Corollary 2. The *class of six- and higher-order operators of* Proposition 2 *is given by* Eq. (2) *for $n \geqslant 3$,* Eq. (1)*, and $\alpha(x) = \alpha_0 + \alpha_1 x$ and $\beta(x) = \beta_0 + \beta_1 x + \beta_2 x^2$ as resulting from* Eq. (15)*, where $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2$ are real constants.*

Remark. The class of the fourth-order differential operators that can be factorized, Corollary 1, is much larger than the class of sixth- and higher-order operators, Corollary 2. One can notice that any functions α and β that satisfy Eq. (16) will also satisfy Eq. (11). The converse is not true. In the case of sixth- and higher-order operators, the functions α and β are only first and second degree polynomials, respectively (function ρ is found using Eq. (1), while in the case of fourth-order operators, $\rho(x)$ is scalar function, Eq. (15).

4. Factorization of self-adjoint differential equations

In what follows it is showed that the spectral type equations associated to the self-adjoint operators given by Eq. (2) can be factorized into commuting second-order differential operators. Consequently, the general solution of such spectral type equations can be written as a sum of solutions of the second-order differential equations resulted from factorization.

4.1. Factorization of self-adjoint equations

Theorem 1. *If* Eqs. (1) and (11) *are satisfied for operators of fourth-order, and/or Eqs.* (1) and (16) *are satisfied for operators of sixth-order and higher, then their spectral type equations given by*

$$(L_{(2n)} - \mu)[y] = 0, \tag{20}$$

can be factorized as follows

$$\prod_{k=1}^{n}(L_2 - \lambda_k)[y] = 0, \tag{21}$$

where μ is a constant positive parameter, n is any natural number, and constants λ_k are given by the following system of algebraic equations

$$\begin{cases} \lambda_1 + \lambda_2 + \cdots + \lambda_n = \delta_2 + \cdots + \delta_n \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n = \delta_2\delta_3 + \delta_2\delta_4 + \cdots + \delta_{n-1}\delta_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_1\lambda_2\ldots\lambda_n = (-1)^{n-1}\mu \end{cases} \tag{22}$$

Constants δ_k, $k = 2, 3, \ldots, n$ are given by Eq. (13).

Proof. If Eqs. (1) and (11) are satisfied, then $L_{(2n)}$ can be factorized as in Eq. (12). If Eqs. (1) and (16) are satisfied, then the operator $L_{(2n)}$ can be factorized as in Eq. (17). Therefore, in both cases Eq. (20) can be written as

$$\left[\prod_{k=1}^{n}(L_2 - \delta_k) - \mu\right][y] = 0, \tag{23}$$

where the constants δ_k are given by Eq. (13). Expanding both Eq. (21) and Eq. (23), and equating the coefficients of like terms, one obtains the following system of algebraic equations

$$\begin{cases} \lambda_1 + \lambda_2 + \cdots + \lambda_n = \delta_1 + \delta_2 + \cdots + \delta_n \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n = \delta_1\delta_2 + \delta_1\delta_3 + \cdots + \delta_{n-1}\delta_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_1\lambda_2\ldots\lambda_n = \delta_1\delta_2, \ldots, \delta_n + (-1)^{n-1}\mu \end{cases} \qquad \square \tag{24}$$

Since $\delta_1 = 0$, the system given by Eq. (22) is obtained.

4.2. General solution

Theorem 2. *The general solution of the spectral type Eq. (20) is given by*

$$y = \sum_{k=1}^{n} y_k, \tag{25}$$

where y_k are the general solutions of the following second-order differential equations resulting from factorization

$$L_2[y_k] - \lambda_k y_k = 0, \quad k = 1, 2, \ldots, n. \tag{26}$$

The extended form of Eq. (25) is as follows

$$\beta(x)\frac{d^2y_k}{dx^2} + [\alpha(x) + \beta'(x)]\frac{dy_k}{dx} - \lambda_k y_k = 0 \quad k = 1, 2. \tag{27}$$

If λ_k is a multiple root, then the corresponding y_k functions will be modified accordingly.

Proof. Equation (19) can be factorized as given by Eq. (21). Since $(L_2 - \lambda_k)$, $k = 1, 2, \ldots, n$, are commuting operators, the general solution of Eq. (21) is the sum of the general solutions of the second-order differential Eqs. (26). $\square$

Fourth- and sixth-order differential equations arise in many engineering applications. In the next two sections the factorization of these operators and their spectral type equations are explicitly given. The general solutions of their spectral type equations are obtained solving the second-order differential equations resulting from factorization.

5. Factorization of fourth-order differential operators and equations

5.1. Factorization of fourth-order differential operator

The fourth-order differential operator in its self-adjoint form is given by Eq. (2) for $n = 2$, or in its extended form, provided Eq. (1) is satisfied, by

$$L_4 = \beta^2\frac{d^4}{dx^4} + 2\beta(\alpha + 2\beta')\frac{d^3}{dx^3} + \left\{[\beta(\alpha + 2\beta')]' + \alpha(\alpha + 2\beta')\right\}\frac{d^2}{dx^2}, \tag{28}$$

where ρ, β, and α are functions of x. If Eqs. (1) and (11) are satisfied, then the fourth-order differential operator L_4 can be factorized into a pair of second-order commuting operators as given by Eq. (12). The factorization of the operator in its extended form is as follows

$$\frac{1}{\rho}\frac{d^2}{dx^2}\left(\rho\beta^2\frac{d^2}{dx^2}\right) = \left[\beta\frac{d^2}{dx^2} + (\alpha + \beta')\frac{d}{dx}\right] \cdot \left[\beta\frac{d^2}{dx^2} + (\alpha + \beta')\frac{d}{dx} - (\alpha' + \beta')\right], \tag{29}$$

where $\delta_2 = \alpha' + \beta''$ is a constant as resulting from Eq. (11).

5.2. Factorization of fourth-order spectral type differential equation

The spectral type differential equation of fourth-order differential operator L_4 and its factorization are given by Eqs. (20) and (21) for $n = 2$, respectively. This factorization in its extended form is as follows

$$\left[\beta\frac{d^2}{dx^2}+(\alpha+\beta')\frac{d}{dx}-\lambda_1\right]\cdot\left[\beta\frac{d^2}{dx^2}+(\alpha+\beta')\frac{d}{dx}-\lambda_2\right]=0, \tag{30}$$

where λ_1 and λ_2 are given by

$$\lambda_1=\frac{\delta_2+\sqrt{\delta_2^2+4\mu}}{2},\quad \lambda_2=\frac{\delta_2-\sqrt{\delta_2^2+4\mu}}{2}, \tag{31}$$

and constant δ_2 by Eq. (13).

6. Factorization of sixth-order differential operators and equations

6.1. Factorization of sixth-order differential operator

The sixth-order differential operator is given by Eq. (2) for $n = 3$, or in extended form, provided Eq. (1) is satisfied, by

$$\begin{aligned}L_6=\beta^3\frac{d^6}{dx^6}&+3\beta^2(\alpha+3\beta')\frac{d^5}{dx^5}+3\{[\beta^2(\alpha+3\beta'')]'+\alpha\beta(\alpha+3\beta')\}\frac{d^4}{dx^4}\\&+\left\{[\beta^2(\alpha+3\beta'')]'+2\alpha\beta(\alpha+3\beta')'+\alpha'\beta(\alpha+3\beta')+\alpha(\alpha+3\beta')^2\right\}\frac{d^3}{dx^3},\end{aligned} \tag{32}$$

where ρ, β, and α are functions of x. If Eqs. (1) and (16) are satisfied, then the sixth-order differential operator L_6 can be factored into three second-order commuting operators as given by Eq. (17) for $n = 3$, where δ_1, δ_2, δ_3 are given by Eq. (13). The extended form of the factorization of the operator L_6 is as follows

$$\frac{1}{\rho}\frac{d^3}{dx^3}\left(\rho\beta^3\frac{d^3}{dx^3}\right)=\left[\beta\frac{d^2}{dx^2}+(\alpha+\beta')\frac{d}{dx}\right]\cdot\left[\beta\frac{d^2}{dx^2}+(\alpha+\beta')\frac{d}{dx}-(\alpha'+\beta'')\right]\cdot\left[\beta\frac{d^2}{dx^2}+(\alpha+\beta')\frac{d}{dx}-(2\alpha'+3\beta'')\right]. \tag{33}$$

6.2. Factorization of sixth-order differential equation

The spectral type equation of the sixth-order differential operator L_6 and its factorization are given by Eqs. (20) and (21) for $n = 3$, respectively. The factorization in its extended form is as follows

$$\left[\beta\frac{d^2}{dx^2}+(\alpha+\beta')\frac{d}{dx}-\lambda_1\right]\cdot\left[\beta\frac{d^2}{dx^2}+(\alpha+\beta')\frac{d}{dx}-\lambda_2\right]\cdot\left[\beta\frac{d^2}{dx^2}+(\alpha+\beta')\frac{d}{dx}-\lambda_3\right][y]=0, \tag{34}$$

where λ_1, λ_2, and λ_3 are given by Eq. (22).

7. Factorization of fourth-order differential equation with $\rho(x)=(1-x)^p(1+x)^q$, $p\geqslant 1$, $q\geqslant 1$

In this section, a particular case of fourth-order spectral type differential equation, $\rho(x)=(1-x)^p(1+x)^q$, $p\geqslant 1$, $q\geqslant 1$, is considered. One can notice that the function $\rho(x)$ is the weight function of Jacobi orthogonal polynomials. Using the factorization method the general solution of the fourth-order differential equation is found in terms of hypergeometric functions.

Proposition 3. *If* Eq. (1) *is satisfied, and*

$$\rho(x)=(1-x)^p(1+x)^q,\quad \text{where } p\geqslant 1,\quad q\geqslant 1, \tag{35}$$

then the spectral type fourth-order self-adjoint differential equation (20) *for n = 2 is given by*

$$\frac{1}{(1-x)^p(1+x)^q}\frac{d^2}{dx^2}\left[(1-x)^{p+2}(1+x)^{q+2}\frac{d^2y}{dx^2}\right]-\mu y=0, \tag{36}$$

or in its extended form

$$\begin{aligned}(1-x^2)^2\frac{d^4y}{dx^4}&+2(1-x^2)[-(p+q+4)x+q-p]\frac{d^3y}{dx^3}+\Big[(p+q+3)(p+q+4)x^2+2(p-q)(p+q+3)x+(q-p)^2\\&-p-q-4\Big]\frac{d^2y}{dx^2}-\mu y=0,\end{aligned} \tag{37}$$

The general solution of Eq. (36) *and/or Eq.* (37) *is as follows*

$$y(x) = A_1 \cdot {}_2F_1\left(a_1, b_1, c_1; \frac{1-x}{2}\right) + A_2 \cdot {}_2F_1\left(a_2, b_2, c_2; \frac{1-x}{2}\right) + B_1 \cdot w\left(a_1, b_1, c_1; \frac{1-x}{2}\right) + B_2 \cdot w\left(a_2, b_2, c_2; \frac{1-x}{2}\right), \quad (38)$$

where μ is real and positive, and a_i, b_i, c_i are the constant parameters of the canonical form of Gauss equations, Abramovitz and Stegun [32]

$$x(1-x)\frac{d^2y}{dx^2} + [c_i - (a_i + b_i + 1)x]\frac{dy}{dx} - a_i b_i x = 0 \quad (39)$$

and they are given by

$$\begin{cases} c_i = p + 1 \\ a_i + b_i = p + q + 1, \quad i = 1, 2 \\ a_i b_i = \lambda_i \end{cases} \quad (40)$$

The hypergeometric function ${}_2F_1(a, b, c; x)$ is given by

$${}_2F_1(a, b, c; x) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}. \quad (41)$$

The function $w(a, b, c; x)$ is as follows, Abramovitz and Stegun [32]

$$w(a, b, c; x) = x^{1-c} \cdot {}_2F_1(a - c + 1,\ b - c + 1, 2 - c;\ x), \quad \text{if} \quad c > 0,\ c \notin N, \quad (42)$$

$$w(a, b, c; x) = {}_2F_1(a, b, 1; x) \ln x + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} x^n \Psi(a, b, n, 0), \quad \text{if} \quad |x| < 1, \quad c = 1, \quad (43)$$

$$w(a, b, c; x) = {}_2F_1(a, b, m+1; x) \ln x + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(n!)(m+1)_n} x^n \Psi(a, b, n, m) - \sum_{n=1}^{m} \frac{(n-1)!(-m)_n}{(1-a)_n (1-b)_n} x^{-n}, \quad \text{if} \quad c = m + 1 \in N \quad , \quad (44)$$

where

$$\Psi(a, b, n, m) = \psi(a+n) - \psi(a) + \psi(b+n) - \psi(b) - \psi(m+n+1) + \psi(m+1) - \psi(n+1) + \psi(1) \quad (45)$$

and ψ is the logarithmic derivative of Γ function. The Pochhammer symbol or rising factorial $(a)_n$ is given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1), \quad (46)$$

where $n = 1, 2, 3, \ldots$

Proof. If Eqs. (1) and (35) are satisfied, then β and α are polynomial functions given by

$$\beta(x) = 1 - x^2, \quad \alpha(x) = -(p+q)x + q - p. \qquad \square \quad (47)$$

In this case Eq. (11) is satisfied. Therefore the factorization method can be used, Proposition 3. Using Eqs. (35) and (47), the differential Eq. (20) becomes Eq. (36). Using Eq. (30), the differential Eq. (36) is factorized into a pair of second-order commuting operators as follows

$$\prod_{i=1}^{2} \left\{ (1 - x^2)\frac{d^2}{dx^2} + [p - q + (p + q - 2)x]\frac{d}{dx} - \lambda_i \right\} [y] = 0, \quad (48)$$

Where $i = 1, 2$, λ_1 and λ_2 are given by Eq. (31), and the constant δ_2 results from Eqs. (13) and (47) as follows

$$\delta_2 = -p - q - 2. \quad (49)$$

The general solution of Eq. (36) can be written as the sum of general solutions of the two second-order differential equations resulting from Eq. (48), Theorem 2. Since p and q are assumed greater than or equal to 1, Eq. (35), one can see from Eq. (49) that $\delta_2 < 0$. Also, because $\mu > 0$ from Eqs. (31), λ_1 and λ_2 must have different signs

$$\lambda_1 > 0, \quad \lambda_2 < 0. \quad (50)$$

To solve the two second-order differential Eqs. (48), the following variable changing is used

$$x = 1 - 2z. \quad (51)$$

Therefore Eq. (48) become Gauss equations as follows

$$z(1-z)\frac{d^2y_i}{dz^2} + [p + 1 - (p + q + 2)z]\frac{dy_i}{dz} - \lambda_i y_i = 0, \quad i = 1, 2. \quad (52)$$

The canonical form of a Gauss equation is given by Eq. (39). Using Eqs. (52) and (39), the constant parameters a_i, b_i, c_i of the canonical form of Gauss equation result as in Eq. (40). Therefore, the general solutions of Eq. (52) are given by, Abramovitz and Stegun [32].

$$y_i(z) = A_i \cdot {}_2F_1(a_i, b_i, c_i, z) + B_i \cdot w(a_i, b_i, c_i, z), \quad i = 1, 2, \tag{53}$$

where ${}_2F_1(a, b, c; x)$ and $w(a, b, c; x)$ are given by Eqs. (41)–(46). The general solution of Eq. (36) is the sum of the general solutions given by Eqs. (53) and (51). In conclusion, the general solution Eq. (38) of the differential Eq. (36) is obtained.

Remark. The eigenvalue problems associated to spectral type Eq. (36) can be either regular $-1 < x_1 \leqslant x \leqslant x_2 < 1$, singular at one end such as $-1 < x_1 \leqslant x < 1$, or singular at both ends $-1 < x < 1$. In the first two cases, eigenfunctions and eigenvalue equations are written in terms of hypergeometric functions.

8. Application: free bending vibrations of nonuniform beams

The above concepts can be used to study bending vibrations of nonuniform beams. The beams considered in this section are cantilevers with rectangular cross-section, parabolic thickness variation with the longitudinal coordinate, constant width, and one sharp end. The boundary conditions are fixed-free. The sharp end is free since it cannot sustain any bending moment or shear force. The natural frequencies and mode shapes of transverse vibrations of this beam are to be found.

8.1. Differential equation of transverse vibrations

The dimensionless differential equation of transverse vibrations of Euler–Bernoulli beams is as follows, Caruntu [24]

$$\frac{d^2}{dx^2}\left[I(x)\frac{d^2y(x)}{dx^2}\right] - \frac{\rho_0\ell^4\omega^2}{E}A(x)y(x) = 0, \tag{54}$$

where x is the dimensionless longitudinal coordinate of the beam, taken as the dimensional longitudinal coordinate divided by ℓ which is a reference length, see Fig. 1 in Ref. [24]. The mode shape of vibration is $y(x)$, Young modulus E, cross-section moment of inertia $I(x)$, cross-section area $A(x)$, density ρ_0, and frequency ω. Consider a cantilever beam of constant width w_0 and thickness given by

$$h(x) = h_0(1 - x^2), \quad 1 < x_1 < x < 1, \tag{55}$$

where h_0 is the reference radius, and x_1 is the dimensionless longitudinal coordinate of the fixed end. The cross-section area $A(x)$ and moment of inertia $I(x)$ are

$$A(x) = A_0(1 - x^2), \quad I(x) = I_0(1 - x^2)^3, \tag{56}$$

where A_0 and I_0 are the reference corresponding cross-section quantities

$$A_0 = w_0h_0 \quad I_0 = \frac{w_0h_0^3}{12}. \tag{57}$$

Replacing Eq. (56) into Eq. (54), the following equation, which is Eq. (36) for $p = 1$ and $q = 1$, results

$$\frac{1}{(1 - x^2)}\frac{d^2}{dx^2}\left[(1 - x^2)^3\frac{d^2y(x)}{dx^2}\right] - \mu \cdot y(x) = 0, \tag{58}$$

where

$$\mu = \frac{\rho_0A_0\ell^4}{EI_0}\omega^2. \tag{59}$$

8.2. Natural frequencies and mode shapes

Using Eqs. (1), (35), (36), and Eq. (58), one obtains

$$\rho = (1 - x^2), \quad \beta = 1 - x^2, \quad \alpha = -2x. \tag{60}$$

The fixed-free boundary conditions of the cantilever beam consist of zero deflection and zero slope at the fixed end, and finite deflection at the free end (see Caruntu [24]), and they are as follows

$$y(x_1) = \frac{dy}{dx}(x_1) = 0, \quad y(1) \textit{ finite} \tag{61}$$

According to Proposition 3, the general solution of Eq. (58) is given by Eq. (58) which includes functions w given by Eqs. (42)–(44). Since the functions w are not finite at the sharp end ($x = 1$ in Eq. (38)) and $y(1)$ must be finite, the coefficients of the functions w must be zero

$$B_1 = B_2 = 0. \tag{62}$$

Using the general solution and the two boundary conditions at $x = x_1$ given by Eq. (61) one obtains the natural frequency equation, Caruntu [24], from which the natural frequencies and consequently the mode shapes can be found. Numerical simulations in this case of nonuniform cantilever have been conducted in Ref. [24]. For instance in the case of $x_1 = 0$, the second dimensionless natural frequency $\bar{\omega} = \omega\ell^2\sqrt{(\rho_0 A_0)/(EI_0)}$ and the corresponding mode shape are as follows

$$\bar{\omega}_2 = 21.183, \tag{63}$$

$$y_2(\xi) = {}_2F_1\left(6.5525, -3.5525, 2, \frac{1-\xi}{2}\right) - 17.779 \cdot 10^{-4} \cdot {}_2F_1\left(1.5 + 4.12644i, 1.5 - 4.12644i, 2, \frac{1-\xi}{2}\right). \tag{64}$$

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